

Supplemental Material for “Parameter Estimation Robust to Low-Frequency Contamination”

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This supplemental appendix provides technical proofs for the main results of the paper. These proofs make use of several supporting lemmas which are also stated and proved here.

Recall the assumptions.

Assumption 1. *The process $\{x_t\}$ is generated by the following data-generating process (DGP): $x_t = \mu + v_t + u_t$, where μ is a finite constant and v_t and u_t are independent at all leads and lags.*

(a) *$\{v_t\}$ is strictly stationary and ergodic and $v_t = \sum_{j=0}^{\infty} c(j; \theta_0) e_{t-j}$, where $c(0; \theta_0) = 1$, $E[e_t] = 0$, $E[e_t e_s] = I(t = s) \sigma^2(\theta_0)$ and $\sum_{j=0}^{\infty} c(j; \theta_0)^2 < \infty$ so that the spectrum of $\{v_t\}$ takes the form*

$$f(\lambda; \theta_0) = \frac{\sigma^2(\theta_0)}{2\pi} g(\lambda; \theta_0) \text{ with } g(\lambda; \theta_0) \equiv \left| \sum_{j=0}^{\infty} c(j; \theta_0) e^{-i\lambda j} \right|^2.$$

(b) *$\Theta \subset \mathbb{R}^s$ is compact and the following properties hold over Θ : (i) $g(\lambda; \theta)$ is continuous in $(\lambda, \theta) \in [-\pi, \pi] \times \Theta$; (ii) $\sigma^2(\theta)$ is continuous and strictly greater than zero over Θ ; (iii) $g(\lambda, \theta) > 0$ for all $(\lambda, \theta) \in [-\pi, \pi] \times \Theta$; (iv) If $\theta_0 \neq \theta \in \Theta$, $g(\lambda; \theta) \neq g(\lambda; \theta_0)$. Furthermore, for $\theta_1, \theta_2 \in \Theta$, if $\theta_1 \neq \theta_2$, then $f(\lambda; \theta_1) \neq f(\lambda; \theta_2)$ on a subset of $[-\pi, \pi]$ that is of positive Lebesgue measure; (v) $\theta_0 \in \Theta$.*

(c) *$E[I_u(\lambda_j)] = O(T/j^2)$ for $j \in \mathcal{F}_1$.*

Assumption 2. *$l/T + \log^4 T/l \rightarrow 0$.*

Assumption 3. *For the process $\{v_t\}$ in Assumption 1, the following hold for its spectrum and innovations:*

(a) (i) *$f(\lambda; \theta)$ is twice continuously differentiable in $\theta \in \Theta$. For $\theta \in \Theta$, $\partial f(\lambda; \theta)/\partial \theta$ and $\partial^2 f(\lambda; \theta)/\partial \theta \partial \theta'$ are continuous in $\lambda \in [-\pi, \pi]$; (ii) $\theta_0 \in \text{interior}(\Theta)$.*

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(b) $f(\lambda; \theta_0)$ is Hölder continuous of maximal degree $\alpha \in (1/2, 1]$ in λ , i.e., there is a constant H such that $|f(\lambda; \theta_0) - f(\omega; \theta_0)| \leq H|\lambda - \omega|^\alpha$ for all $\lambda, \omega \in [-\pi, \pi]$.

(c) Let \mathcal{I}_t denote the σ -field generated by e_s , $s \leq t$. (i) $E[e_t | \mathcal{I}_{t-1}] = 0$ a.s.; (ii) $E[e_t^2 | \mathcal{I}_{t-1}] = \sigma^2(\theta_0)$ a.s.; (iii) $E[e_t^3 | \mathcal{I}_{t-1}] = \mu_3$ a.s.; (iv) $E[e_t^4] = \mu_4 < \infty$.

(d) $\sum_{j=0}^{\infty} |c(j; \theta_0)| < \infty$.

Assumption 4. $l/T + T^{1/2}/l \rightarrow 0$.

The consistency proof of Theorem 1 uses the following lemma which helps to describe the trimmed FDQML objective function of the periodogram of the observed process in terms of the untrimmed objective function of the periodogram of the latent contaminated process. Throughout $K > 0$ is a finite constant, the value of which may change from line to line.

Lemma 1. Let $\eta(\cdot; \cdot) : [-\pi, \pi] \times \Theta \rightarrow \mathbb{R}$ be any mapping that satisfies $|\eta(\lambda; \theta)| \leq K$ for all $(\lambda, \theta) \in [-\pi, \pi] \times \Theta$. Let v_t be covariance stationary with periodogram $I_v(\lambda)$ that satisfies $\sup_{\lambda \in [-\pi, \pi]} E[I_v(\lambda)] \leq K < \infty$, and let u_t satisfy Assumption 1(c). Then under Assumption 2:

$$\sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{j \in \mathcal{F}_l} I_x(\lambda_j) \eta(\lambda_j; \theta) - \frac{1}{T} \sum_{j \in \mathcal{F}_1} I_v(\lambda_j) \eta(\lambda_j; \theta) \right| = o_p(1).$$

Proof: Recall $w_y(\lambda) \equiv 1/\sqrt{2\pi T} \sum_{t=1}^T y_t e^{-i\lambda t}$ and note the decomposition

$$I_x(\lambda) = I_v(\lambda) + I_u(\lambda) + w_v(\lambda)w_u(-\lambda) + w_v(-\lambda)w_u(\lambda).$$

Hence:

$$\begin{aligned} & \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{j \in \mathcal{F}_l} I_x(\lambda_j) \eta(\lambda_j; \theta) - \frac{1}{T} \sum_{j \in \mathcal{F}_1} I_v(\lambda_j) \eta(\lambda_j; \theta) \right| \\ & \leq \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{j \in \mathcal{F}_l} I_v(\lambda_j) \eta(\lambda_j; \theta) - \frac{1}{T} \sum_{j \in \mathcal{F}_1} I_v(\lambda_j) \eta(\lambda_j; \theta) \right| + \left| \frac{1}{T} \sum_{j \in \mathcal{F}_l} I_u(\lambda_j) \eta(\lambda_j; \theta) \right| \\ & \quad + \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{j \in \mathcal{F}_l} w_v(\lambda_j) w_u(-\lambda_j) \eta(\lambda_j; \theta) \right| + \left| \frac{1}{T} \sum_{j \in \mathcal{F}_l} w_v(-\lambda_j) w_u(\lambda_j) \eta(\lambda_j; \theta) \right| \\ & \leq K \left| \frac{1}{T} \sum_{j=-l, j \neq 0}^l I_v(\lambda_j) \right| + K \left| \frac{1}{T} \sum_{j \in \mathcal{F}_l} I_u(\lambda_j) \right| + K \frac{1}{T} \sum_{j \in \mathcal{F}_l} |w_v(\lambda_j) w_u(-\lambda_j)| \end{aligned}$$

$$\begin{aligned}
&= O_p\left(\frac{l}{T}\right) + O_p\left(\frac{1}{T} \sum_{j=l}^T \frac{T}{j^2}\right) + O_p\left(\frac{1}{T} \sum_{j=l}^T \frac{T^{1/2}}{j}\right) \\
&= O_p\left(\frac{l}{T}\right) + O_p\left(\frac{\log T}{l}\right) + O_p\left(\frac{\log T}{T^{1/2}}\right) = o_p(1).
\end{aligned}$$

The first equality follows from $\sup_{\lambda \in [\pi-, \pi]} E[I_v(\lambda)] \leq K < \infty$, the Cauchy-Schwartz inequality and Assumption 1(c): $E[|w_v(\lambda_j)w_u(-\lambda_j)|] \leq (E[I_v(\lambda_j)])^{1/2}(E[I_u(\lambda_j)])^{1/2} = O(T/j^2)^{1/2}$. The last equality uses Assumption 2 $l/T + (\log^4 T)/l \rightarrow 0$. ■

Proof of Theorem 1: Note that by Assumption 1(b) and the compactness of $[-\pi, \pi] \times \Theta$, $f(\lambda; \theta)$ is both bounded and bounded away from zero for all $(\lambda, \theta) \in [-\pi, \pi] \times \Theta$ so that $\log f(\lambda; \theta)$ is bounded for all $(\lambda, \theta) \in [-\pi, \pi] \times \Theta$. This implies

$$\frac{1}{T} \sum_{j \in \mathcal{F}_l} \log f(\lambda_j; \theta) = \frac{1}{T} \sum_{j \in \mathcal{F}_1} \log f(\lambda_j; \theta) + O(l/T) \quad (1)$$

uniformly in $\theta \in \Theta$. Similarly, $f(\lambda; \theta)^{-1} \leq C$ for some $C < \infty$ so that by Lemma 1,

$$\frac{1}{T} \sum_{j \in \mathcal{F}_l} \frac{I_x(\lambda_j)}{f(\lambda_j; \theta)} = \frac{1}{T} \sum_{j \in \mathcal{F}_1} \frac{I_v(\lambda_j)}{f(\lambda_j; \theta)} + o_p(1) \quad (2)$$

uniformly in $\theta \in \Theta$. Since $f(\lambda; \theta)$ is continuous, bounded and bounded away from zero for all $(\lambda, \theta) \in [-\pi, \pi] \times \Theta$,

$$\frac{1}{T} \sum_{j \in \mathcal{F}_1} \log f(\lambda_j; \theta) \rightarrow (2\pi)^{-1} \int_{-\pi}^{\pi} \log f(\lambda; \theta) d\lambda = \log \sigma^2(\theta) - \log 2\pi \quad (3)$$

uniformly in $\theta \in \Theta$. A simple adaptation of the uniform law of large numbers Lemma 1 of Hannan (1973) to our parameter space Θ also provides that

$$\sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{j \in \mathcal{F}_1} \frac{I_v(\lambda_j)}{f(\lambda_j; \theta)} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\lambda; \theta_0)}{f(\lambda; \theta)} d\lambda \right| \rightarrow 0 \quad \text{a.s.} \quad (4)$$

uniformly in $\theta \in \Theta$. Now, by Assumption 2,

$$\begin{aligned}
\text{plimsup}_{T \rightarrow \infty} L_{T,l}(\hat{\theta}_T) &\leq \inf_{\theta \in \Theta} \text{plimsup}_{T \rightarrow \infty} L_{T,l}(\theta) \\
&= \inf_{\theta \in \Theta} \text{plimsup}_{T \rightarrow \infty} \left\{ \frac{1}{T} \sum_{j \in \mathcal{F}_1} \log f(\lambda_j; \theta) + \frac{1}{T} \sum_{j \in \mathcal{F}_1} \frac{I_v(\lambda_j)}{f(\lambda_j; \theta)} + o_p(1) \right\}
\end{aligned}$$

$$\begin{aligned}
&= \inf_{\theta \in \Theta} \left\{ \log \sigma^2(\theta) - \log 2\pi + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\lambda; \theta_0)}{f(\lambda; \theta)} d\lambda \right\} \\
&= \log \sigma^2(\theta_0) - \log 2\pi + 1,
\end{aligned} \tag{5}$$

where the first equality follows from (1) and (2) (given uniformity in $\theta \in \Theta$), the second equality follows from (3) and (4) and the final equality follows from Lemma 3.1(c) of Hosoya and Taniguchi (1982). The reason that Lemma 3.1(c) of Hosoya and Taniguchi (1982) implies the final equality is because when the spectrum $f(\lambda; \theta)$ has an integrable logarithm (which is implied by Assumption 1(b)),

$$\sigma^2(\theta) = 2\pi \exp((2\pi)^{-1} \int_{-\pi}^{\pi} \log f(\lambda; \theta) d\lambda)$$

(see, e.g., Theorem 4.3 on p. 577 of Doob, 1953), so that minimizing

$$\log \sigma^2(\theta) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\lambda; \theta_0)}{f(\lambda; \theta)} d\lambda$$

with respect to θ is equivalent to minimizing

$$\int_{-\pi}^{\pi} [\log f(\lambda; \theta) + \frac{f(\lambda; \theta_0)}{f(\lambda; \theta)}] d\lambda.$$

By way of contradiction, suppose $\hat{\theta}_T \xrightarrow{p} \theta_0$. Then, by compactness of Θ , there is a subsequence $\hat{\theta}_{M(T)}$ such that $\hat{\theta}_{M(T)} \xrightarrow{p} \theta' \in \Theta$ with $\theta' \neq \theta_0$. Note that Assumption 2 implies that l grows as a function of T so that we can write $l(T)$ and Assumption 2 must also hold for $l(M(T))$, i.e., $l(M(T))/M(T) + (\log^4 M(T))/l(M(T)) \rightarrow 0$ as $M(T) \rightarrow \infty$. Suppressing dependence of M on T and l on M , if we define the Fourier frequencies $\tilde{\lambda}_j \equiv 2\pi j/M$ and the sets $\tilde{\mathcal{F}}_1 \equiv (-M/2, M/2] \cap \mathbb{Z} \setminus \{0\}$ and $\tilde{\mathcal{F}}_l \equiv (-M/2, M/2] \cap \mathbb{Z} \setminus [-l+1, l-1]$, we then have

$$L_{M,l}(\theta) = M^{-1} \sum_{j \in \tilde{\mathcal{F}}_l} \left\{ \log f(\tilde{\lambda}_j; \theta) + \frac{I_x(\tilde{\lambda}_j)}{f(\tilde{\lambda}_j; \theta)} \right\}.$$

Then (1) and (2) imply

$$\begin{aligned}
\text{plim}_{M \rightarrow \infty} L_{M,l}(\hat{\theta}_M) &= \text{plim}_{M \rightarrow \infty} \frac{1}{M} \sum_{j \in \tilde{\mathcal{F}}_1} \left\{ \log f(\tilde{\lambda}_j; \hat{\theta}_M) + \frac{I_v(\tilde{\lambda}_j)}{f(\tilde{\lambda}_j; \hat{\theta}_M)} \right\} \\
&= \log \sigma^2(\theta') - \log 2\pi + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\lambda; \theta_0)}{f(\lambda; \theta')} d\lambda > \log \sigma^2(\theta_0) - \log 2\pi + 1
\end{aligned}$$

where the second equality follows from the continuous mapping theorem, (3) and (4) and the inequality results from Lemma 3.1(c) of Hosoya and Taniguchi (1982) and Assumption 1(b)(iv). This is in direct

contradiction with (5). ■

Lemmas 2-5 support asymptotic normality Theorem 2. The next lemma bounds the influence of the contaminating component in a trimmed weighted average of the observed periodogram, a quantity present in the expansion of $\hat{\theta}_T$.

Lemma 2. *Let $\eta(\cdot) : [-\pi, \pi] \rightarrow \mathbb{R}$ be any mapping that satisfies $|\eta(\lambda)| \leq K$ for all $\lambda \in [-\pi, \pi]$. Let v_t be covariance stationary with $\sum_{k=0}^{\infty} |\text{Cov}(v_0, v_k)| < \infty$ and periodogram $I_v(\lambda)$ that satisfies $\sup_{\lambda \in [-\pi, \pi]} E[I_v(\lambda)] \leq K < \infty$, and let u_t satisfy Assumption 1(c). If $(\log^4 T)/l \rightarrow 0$ then*

$$\frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_l} I_x(\lambda_j) \eta(\lambda_j) = \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_l} I_v(\lambda_j) \eta(\lambda_j) + O_p \left(\frac{T^{1/2}}{l} + \frac{1}{T^{1/2}} \right) + o_p(1).$$

Proof: Using the same decomposition as in Lemma 1, we have

$$\begin{aligned} \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_l} I_x(\lambda_j) \eta(\lambda_j) &= \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_l} I_v(\lambda_j) \eta(\lambda_j) + \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_l} I_u(\lambda_j) \eta(\lambda_j) \\ &\quad + \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_l} w_v(\lambda_j) w_u(-\lambda_j) \eta(\lambda_j) + \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_l} w_v(-\lambda_j) w_u(\lambda_j) \eta(\lambda_j). \end{aligned}$$

By Assumption 1(c) $E[I_u(\lambda_j)] = O(T/j^2)$, hence by Markov's inequality the second term satisfies:

$$\left| \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_l} I_u(\lambda_j) \eta(\lambda_j) \right| \leq \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_l} I_u(\lambda_j) |\eta(\lambda_j)| = O_p \left(T^{1/2} \sum_{j=l}^T \frac{1}{j^2} \right).$$

Since

$$T^{1/2} \sum_{j=l}^T \frac{1}{j^2} = T^{1/2} \sum_{j=l}^{\infty} \frac{1}{j^2} - T^{1/2} \sum_{j=T+1}^{\infty} \frac{1}{j^2} = T^{1/2} \times O(l^{-1}) - T^{1/2} \times O(T^{-1}) = O(T^{1/2}/l + T^{-1/2}),$$

it follows

$$\left| \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_l} I_u(\lambda_j) \eta(\lambda_j) \right| = O_p \left(\frac{T^{1/2}}{l} + \frac{1}{T^{1/2}} \right).$$

For the third and fourth terms, note by independence of $\{v_t\}$ and $\{u_t\}$:

$$\begin{aligned} E \left| \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_l} w_v(\lambda_j) w_u(-\lambda_j) \eta(\lambda_j) \right|^2 \\ = \frac{1}{T} \sum_{j \in \mathcal{F}_l} \sum_{k \in \mathcal{F}_l} E[w_v(\lambda_j) w_v(-\lambda_k)] E[w_u(-\lambda_j) w_u(\lambda_k)] \eta(\lambda_j) \eta(\lambda_k) \equiv \mathcal{E}_T, \end{aligned}$$

say. In order to find the asymptotic order of \mathcal{E}_T , suppose for now that $j \neq k$ and note that

$$\begin{aligned} w_v(\lambda_j)w_v(-\lambda_k) &= \frac{1}{2\pi T} \sum_{t=1}^T \sum_{s=1}^T v_t v_s e^{-i\lambda_j t + i\lambda_k s} = \frac{1}{2\pi T} \sum_{t=1}^T \sum_{s=1}^T v_t v_s e^{-i\lambda_j(t-s) + i(\lambda_k - \lambda_j)s} \\ &= \frac{1}{2\pi T} \sum_{n=1-T}^{T-1} \sum_{t=1}^{T-|n|} v_t v_{t+|n|} e^{i(\lambda_k - \lambda_j)(t+|n|)} e^{-i\lambda_j n} \end{aligned}$$

so that

$$\begin{aligned} E[w_v(\lambda_j)w_v(-\lambda_k)] &= \frac{1}{2\pi T} \sum_{n=1-T}^{T-1} \sum_{t=1}^{T-|n|} \gamma_n e^{i(\lambda_k - \lambda_j)(t+|n|)} e^{-i\lambda_j n} \\ &= \frac{1}{2\pi T} \sum_{n=1-T}^{T-1} \gamma_n e^{-i(\lambda_j n - (\lambda_k - \lambda_j)|n|)} \sum_{t=1}^{T-|n|} e^{i(\lambda_k - \lambda_j)t} = O\left(\frac{1}{k-j}\right) \end{aligned}$$

since $|\sum_{n=-\infty}^{\infty} \gamma_n e^{-i(\lambda_j n - (\lambda_k - \lambda_j)|n|)}| \leq \sum_{n=-\infty}^{\infty} |\gamma_n| < \infty$, and for any $r \in [0, 1]$

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^{T-\lfloor Tr \rfloor} e^{i(\lambda_k - \lambda_j)t} &= \frac{1}{T} \sum_{t=1}^{T-\lfloor Tr \rfloor} \left\{ \cos\left(\frac{2\pi(k-j)t}{T}\right) + i \sin\left(\frac{2\pi(k-j)t}{T}\right) \right\} \\ &\rightarrow \int_0^{1-r} \{\cos(2\pi(k-j)s) + i \sin(2\pi(k-j)s)\} ds \\ &= \frac{1}{2\pi(k-j)} [\sin(2\pi(k-j)(1-r)) - i \cos(2\pi(k-j)(1-r)) + i]. \end{aligned}$$

Using the Cauchy-Schwartz inequality, we also have $E[|w_u(-\lambda_j)||w_u(\lambda_k)|] \leq E[I_u(\lambda_j)]^{1/2} E[I_u(\lambda_k)]^{1/2} = O(T/jk)$, while $(\log^4 T)/l \rightarrow 0$ by supposition. Hence \mathcal{E}_T can be given the following order:

$$\begin{aligned} &O\left(\frac{1}{T} \sum_{j \in \mathcal{F}_l} E[I_v(\lambda_j)] E[I_u(\lambda_j)]\right) + O\left(\frac{1}{T} \sum_{j \in \mathcal{F}_l} \sum_{k \in \mathcal{F}_l, k \neq j} \frac{1}{j-k} \frac{T}{jk}\right) \\ &= O\left(\frac{1}{T} \sum_{j=l}^T \frac{T}{j^2}\right) + O\left(\sum_{k=l}^T \sum_{j=k+1}^T \frac{1}{jk} \frac{1}{j-k}\right) \end{aligned}$$

$$\begin{aligned}
&= O\left(\frac{\log T}{l}\right) + O\left(\sum_{k=l}^T \sum_{j=k+1}^{k+\lfloor \log^3 T \rfloor} \frac{1}{jk} \frac{1}{j-k}\right) + O\left(\sum_{k=l}^T \sum_{j=k+\lfloor \log^3 T \rfloor+1}^T \frac{1}{jk} \frac{1}{j-k}\right) \\
&= O\left(\frac{\log T}{l}\right) + O\left(\log^3 T \sum_{k=l}^T \frac{1}{k^2}\right) + O\left(\frac{1}{\log^2 T} \sum_{k=l}^T \frac{1}{k}\right) = O\left(\frac{\log^4 T}{l}\right) + O\left(\frac{1}{\log T}\right) = o(1). \blacksquare
\end{aligned}$$

The next lemma helps to determine the asymptotic behavior of the Hessian matrix.

Lemma 3. *Under Assumptions 1, 2 and 3(a), if $\tilde{\theta}_T \xrightarrow{p} \theta_0$, then $(\partial^2 / \partial \theta \partial \theta') L_{T,l}(\tilde{\theta}_T) \xrightarrow{p} \Omega$ where*

$$\Omega \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial \log f(\lambda; \theta_0)}{\partial \theta} \frac{\partial \log f(\lambda; \theta_0)}{\partial \theta'} d\lambda.$$

Proof: We have

$$\begin{aligned}
&\left\| \frac{\partial^2}{\partial \theta \partial \theta'} L_{T,l}(\tilde{\theta}_T) - \Omega \right\| \\
&\leq \left\| \frac{1}{T} \sum_{j \in \mathcal{F}_l} \frac{\partial}{\partial \theta} \ln f(\lambda_j; \theta_0) \frac{\partial}{\partial \theta'} \ln f(\lambda_j; \theta_0) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial \log f(\lambda; \theta_0)}{\partial \theta} \frac{\partial \log f(\lambda; \theta_0)}{\partial \theta'} d\lambda \right\| \\
&\quad + \left\| \frac{1}{T} \sum_{j \in \mathcal{F}_l} \left\{ I_x(\lambda_j) - f(\lambda_j; \tilde{\theta}_T) \right\} \frac{\partial}{\partial \theta} \varpi(\lambda_j; \tilde{\theta}_T) \right\| \\
&\quad + \left\| \frac{1}{T} \sum_{j \in \mathcal{F}_l} \frac{\partial}{\partial \theta} \ln f(\lambda_j; \tilde{\theta}_T) \frac{\partial}{\partial \theta'} \ln f(\lambda_j; \tilde{\theta}_T) - \frac{1}{T} \sum_{j \in \mathcal{F}_l} \frac{\partial}{\partial \theta} \ln f(\lambda_j; \theta_0) \frac{\partial}{\partial \theta'} \ln f(\lambda_j; \theta_0) \right\| \\
&= \mathcal{A}_T + \mathcal{B}_T + \mathcal{C}_T.
\end{aligned}$$

By construction of the Fourier frequencies $\mathcal{A}_T \rightarrow 0$ (e.g., Hannan, 1973: p. 133-134.).

By Lemma 1 and boundedness properties of $(\partial / \partial \theta)^i f(\lambda; \theta)$ for $i = 0, 1, 2$ under Assumptions 1(a)-(b) and 3(a)

$$\mathcal{B}_T = \left\| \frac{1}{T} \sum_{j \in \mathcal{F}_l} \left\{ I_v(\lambda_j) - f(\lambda_j; \tilde{\theta}_T) \right\} \frac{\partial}{\partial \theta} \varpi(\lambda_j; \tilde{\theta}_T) \right\| + o_p(1).$$

Hence by boundedness of $(\partial / \partial \theta)^i f(\lambda; \theta)$ and the mean-value-theorem:

$$\mathcal{B}_T = \left\| \frac{1}{T} \sum_{j \in \mathcal{F}_l} \left\{ I_v(\lambda_j) - f(\lambda_j; \theta_0) \right\} \frac{\partial}{\partial \theta} \varpi(\lambda_j; \tilde{\theta}_T) \right\| + K \times \|\tilde{\theta}_T - \theta_0\| + o_p(1).$$

The uniform law of large numbers Lemma 1 in Hannan (1973) can be easily generalized to hold for the first term since $(\partial/\partial\theta)\varpi(\lambda; \theta)$ is bounded. Hence $\sup_{\theta \in \Theta} \|1/T \sum_{j \in \mathcal{F}_1} \{I_v(\lambda_j) - f(\lambda_j; \theta_0)\}(\partial/\partial\theta)\varpi(\lambda_j; \theta)\| \xrightarrow{p} 0$, and therefore $\mathcal{B}_T \xrightarrow{p} 0$ given $\|\tilde{\theta}_T - \theta_0\| \xrightarrow{p} 0$.

The remaining term $\mathcal{C}_T \xrightarrow{p} 0$ by a similar argument. Simply note by the mean value theorem and boundedness $\mathcal{C}_T \leq K \times \|\tilde{\theta}_T - \theta_0\|$. ■

The following lemma allows us to replace the spectral density function by the expectation of the periodogram when summing over Fourier frequencies.

Lemma 4. *Let $\eta(\cdot) : [-\pi, \pi] \rightarrow \mathbb{R}$ be any mapping that satisfies $|\eta(\lambda)| \leq K$ for all $\lambda \in [-\pi, \pi]$. Under Assumptions 1(a) and 3(b):*

$$\sum_{j \in \mathcal{F}_l} \{E[I_v(\lambda_j)] - f(\lambda_j; \theta_0)\} \eta(\lambda_j) = O(T^{1-\alpha} \log T).$$

Proof: Letting γ_k denote the k^{th} order autocovariance of $\{v_t\}$, begin by noting that $E[I_v(\lambda)] = (1/2\pi) \sum_{k=-T+1}^{T-1} \gamma_k e^{ik\lambda} = f_{T-1}(\lambda; \theta_0)$, the $(T-1)^{th}$ Fourier expansion of $f(\lambda; \theta_0)$. Since $f(\lambda; \theta_0)$ is Hölder continuous of degree $\alpha \in (1/2, 1]$, Jackson (1930) has shown that $\sup_{\lambda} |f(\lambda; \theta_0) - f_T(\lambda; \theta_0)| = O(T^{-\alpha} \log T)$. Hence, $|\sum_{j \in \mathcal{F}_l} \{E[I_v(\lambda_j)] - f(\lambda_j; \theta_0)\} \eta(\lambda_j)| \leq K \sum_{j \in \mathcal{F}_l} |E[I_v(\lambda_j)] - f(\lambda_j; \theta_0)| = O(T^{1-\alpha} \log T)$. ■

Lemma 5. *Let $\eta(\cdot) : [-\pi, \pi] \rightarrow \mathbb{R}$ be any mapping that satisfies $|\eta(\lambda)| \leq K$ for all $\lambda \in [-\pi, \pi]$. Under Assumptions 1(a) and 3(c)-(d):*

$$\sum_{j=-l}^l \{I_v(\lambda_j) - E[I_v(\lambda_j)]\} \eta(\lambda_j) = O_p(l^{1/2}).$$

Proof: Assumption 3(c)-(d) implies

$$\begin{aligned} E[v_t^4] &= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \sum_{j=0}^{\infty} c(j; \theta_0) c(h; \theta_0) c(k; \theta_0) c(m; \theta_0) E[e_{t-j} e_{t-h} e_{t-k} e_{t-m}] \\ &= \mu_4 \sum_{j=0}^{\infty} c(j; \theta_0)^4 + 3\sigma^4(\theta_0) \sum_{j=0}^{\infty} \sum_{k=0, k \neq j}^{\infty} c(j; \theta_0)^2 c(k; \theta_0)^2 < \infty \end{aligned}$$

since $\sum_{j=0}^{\infty} c(j; \theta_0)^4 + \sum_{j=0}^{\infty} \sum_{k=0, k \neq j}^{\infty} c(j; \theta_0)^2 c(k; \theta_0)^2$ is bounded from above by

$$\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \sum_{j=0}^{\infty} |c(j; \theta_0)| |c(h; \theta_0)| |c(k; \theta_0)| |c(m; \theta_0)| = \left(\sum_{j=0}^{\infty} |c_j| \right)^4 < \infty$$

by Assumption 3(d). It is also clear that $\{v_t\}$ is fourth-order stationary by Assumption 3(c)-(d). Define $\tilde{c}_j = c(j; \theta_0)I(j \geq 0)$ so that $v_t = \sum_{j=-\infty}^{\infty} \tilde{c}_j e_{t-j}$. For any integers m, n, p , define the fourth order joint cumulant of v_t, v_{t+m}, v_{t+n} and v_{t+p} :

$$\kappa(0, m, n, p) \equiv E[v_t v_{t+m} v_{t+n} v_{t+p}] - E[v_t v_{t+m}]E[v_{t+n} v_{t+p}] - E[v_t v_{t+n}]E[v_{t+m} v_{t+p}] - E[v_t v_{t+p}]E[v_{t+m} v_{t+n}].$$

Hence

$$\begin{aligned} \kappa(0, m, n, p) &= \mu_4 \sum_{j=-\infty}^{\infty} \tilde{c}_j \tilde{c}_{j+m} \tilde{c}_{j+n} \tilde{c}_{j+p} + \sigma^4(\theta_0) \sum_{j=-\infty}^{\infty} \sum_{k=-\infty, k \neq j+n}^{\infty} \tilde{c}_j \tilde{c}_{j+m} \tilde{c}_k \tilde{c}_{k+p-n} \\ &\quad + \sigma^4(\theta_0) \sum_{j=-\infty}^{\infty} \sum_{k=-\infty, k \neq j+m}^{\infty} \tilde{c}_j \tilde{c}_{j+n} \tilde{c}_k \tilde{c}_{k+p-m} + \sigma^4(\theta_0) \sum_{j=-\infty}^{\infty} \sum_{k=-\infty, k \neq j+m}^{\infty} \tilde{c}_j \tilde{c}_{j+p} \tilde{c}_k \tilde{c}_{k+n-m} \\ &\quad - \left(\sigma^2(\theta_0) \sum_{j=-\infty}^{\infty} \tilde{c}_j \tilde{c}_{j+m} \right) \left(\sigma^2(\theta_0) \sum_{k=-\infty}^{\infty} \tilde{c}_k \tilde{c}_{k+p-n} \right) \\ &\quad - \left(\sigma^2(\theta_0) \sum_{j=-\infty}^{\infty} \tilde{c}_j \tilde{c}_{j+n} \right) \left(\sigma^2(\theta_0) \sum_{k=-\infty}^{\infty} \tilde{c}_k \tilde{c}_{k+p-m} \right) \\ &\quad - \left(\sigma^2(\theta_0) \sum_{j=-\infty}^{\infty} \tilde{c}_j \tilde{c}_{j+p} \right) \left(\sigma^2(\theta_0) \sum_{k=-\infty}^{\infty} \tilde{c}_k \tilde{c}_{k+n-m} \right) \\ &= \mu_4 \sum_{j=-\infty}^{\infty} \tilde{c}_j \tilde{c}_{j+m} \tilde{c}_{j+n} \tilde{c}_{j+p} - 3\sigma^4(\theta_0) \sum_{j=-\infty}^{\infty} \tilde{c}_j \tilde{c}_{j+m} \tilde{c}_{j+n} \tilde{c}_{j+p}, \end{aligned}$$

and

$$\begin{aligned} &\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} |\kappa(0, m, n, p)| \\ &\leq |\mu_4 - 3\sigma^4(\theta_0)| \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\tilde{c}_j| |\tilde{c}_{j+m}| |\tilde{c}_{j+n}| |\tilde{c}_{j+p}| \\ &= |\mu_4 - 3\sigma^4(\theta_0)| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} |c(j; \theta_0)j| |c(j+m; \theta_0)| |c(j+n; \theta_0)| |c(j+p; \theta_0)| \\ &< \infty \end{aligned}$$

by Assumption 3(d).

Hence, under Assumption 3(c)-(d), all of the conditions of Theorem 9, Chapter V. of Hannan

(1970) are satisfied so that $\text{Var}(\sum_{j=-l}^l \{I_v(\lambda_j) - E[I_v(\lambda_j)]\}) = O(l)$ hence $\sum_{j=-l}^l \{I_v(\lambda_j) - E[I_v(\lambda_j)]\} = O_p(l^{1/2})$ and thus the lemma's statement. ■

Proof of Theorem 2: Using a first order Taylor expansion,

$$T^{1/2}(\hat{\theta}_T - \theta_0) = - \left[\frac{\partial^2}{\partial \theta \partial \theta'} L_{T,l}(\tilde{\theta}_T) \right]^{-1} T^{1/2} \frac{\partial L_{T,l}(\theta_0)}{\partial \theta} \quad (6)$$

for some $\tilde{\theta}_T$ such that $\|\tilde{\theta}_T - \theta_0\| \leq \|\hat{\theta}_T - \theta_0\|$. Now, since Assumption 1(b) implies that $f(\lambda; \theta_0)^{-1}$ is bounded for all $\lambda \in [-\pi, \pi]$,

$$\begin{aligned} T^{1/2} \frac{\partial L_{T,l}(\theta_0)}{\partial \theta} &= \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_l} \{I_x(\lambda_j) - f(\lambda_j; \theta_0)\} \frac{\partial f(\lambda_j; \theta_0)^{-1}}{\partial \theta} \\ &= \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_l} \{I_v(\lambda_j) - f(\lambda_j; \theta_0)\} \frac{\partial f(\lambda_j; \theta_0)^{-1}}{\partial \theta} + O_p \left(\frac{T^{1/2}}{l} + \frac{1}{T^{1/2}} \right) + o_p(1) \\ &= \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_1} \{I_v(\lambda_j) - E[I_v(\lambda_j)]\} \frac{\partial f(\lambda_j; \theta_0)^{-1}}{\partial \theta} \\ &\quad - \frac{1}{T^{1/2}} \sum_{j=-l, j \neq 0}^l \{I_v(\lambda_j) - E[I_v(\lambda_j)]\} \frac{\partial f(\lambda_j; \theta_0)^{-1}}{\partial \theta} \\ &\quad + \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_l} \{E[I_v(\lambda_j)] - f(\lambda_j; \theta_0)\} \frac{\partial f(\lambda_j; \theta_0)^{-1}}{\partial \theta} + O_p \left(\frac{T^{1/2}}{l} + \frac{1}{T^{1/2}} \right) + o_p(1) \\ &= \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_1} \{I_v(\lambda_j) - E[I_v(\lambda_j)]\} \frac{\partial f(\lambda_j; \theta_0)^{-1}}{\partial \theta} + O_p \left(\frac{l^{1/2}}{T^{1/2}} \right) \\ &\quad + O(T^{1/2-\alpha} \log T) + O_p \left(\frac{T^{1/2}}{l} + \frac{1}{T^{1/2}} \right) + o_p(1), \end{aligned} \quad (7)$$

where the second equality follows from Lemma 2 and the fourth equality follows from Lemmas 4 and 5. Observe $a \in (1/2, 1]$ from the Hölder continuity condition Assumption 3(b). In view of Assumption 4 $l/T \rightarrow 0$ and $T^{1/2}/l \rightarrow 0$ it therefore follows

$$T^{1/2} \frac{\partial L_{T,l}(\theta_0)}{\partial \theta} = \frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_1} \{I_v(\lambda_j) - E[I_v(\lambda_j)]\} \frac{\partial f(\lambda_j; \theta_0)^{-1}}{\partial \theta} + o_p(1). \quad (8)$$

Dunsmuir (1979, Theorem 2.1 and Corollary 2.2) has shown that the first term in the above expression is asymptotically normal with mean zero and covariance matrix $2\Omega + \Pi$. Finally, since $\tilde{\theta}_T \xrightarrow{p} \theta_0$, Lemma 3 and (6) imply the statement of the theorem. ■

Proof of Corollary 1: Assumption ARCH-4 and 1(b) imply Assumption 1(a) since Assumption

1(b) implies that $f(\lambda; \theta)$ has an integrable logarithm (see the proof of Theorem 2.1 in Giraitis and Robinson, 2001) so that the result follows from Theorem 1. ■

We may still use three out of the four lemmas used to prove Theorem 2 to prove the asymptotic normality of $\hat{\theta}_T$ for the ARCH model (Lemmas 2-4). However, Lemma 5 must be proved under an entirely different set of assumptions, manifest in the following lemma.

Lemma 6. *Let $\eta(\cdot) : [-\pi, \pi] \rightarrow \mathbb{R}$ be any mapping that satisfies $|\eta(\lambda)| \leq K$ for all $\lambda \in [-\pi, \pi]$. Under Assumption ARCH-8: $\sum_{j=-l}^l \{I_v(\lambda_j) - E[I_v(\lambda_j)]\} \eta(\lambda_j) = O_p(l^{1/2})$.*

Proof: Note $\sum_{j=-l}^l \{I_v(\lambda_j) - E[I_v(\lambda_j)]\} = \sum_{t=1}^T \sum_{s=1}^T d_T(t-s)(X_t X_s - E[X_t X_s])$ where $d_T(t-s) \equiv (1/2\pi T) \sum_{j=-l}^l \exp(-i\lambda_j(t-s))$. Then, using the representation of Giraitis and Robinson (2001, pp. 619-620):

$$\text{Var} \left(\sum_{j=-l}^l \{I_v(\lambda_j) - E[I_v(\lambda_j)]\} \right) = \psi_0^4 \sum_{k_1=0}^{\infty} \dots \sum_{k_4=0}^{\infty} \text{Cov}(Q_T^{(k_1, k_2)}, Q_T^{(k_3, k_4)}), \quad (9)$$

where

$$Q_T^{(l, k)} \equiv \sum_{t=1}^T \sum_{s=1}^T d_T(t-s)(m_l(t) - E[m_l(t)])(m_k(t) - E[m_k(t)])$$

$$m_l(t) \equiv \sum_{j_1=-\infty}^{t-1} \sum_{j_2=-\infty}^{j_1-1} \dots \sum_{j_l=-\infty}^{j_{l-1}-1} \psi_{t-j_1} \psi_{j_1-j_2} \dots \psi_{j_{l-1}-j_l} \varepsilon_t^2 \varepsilon_{j_1}^2 \dots \varepsilon_{j_l}^2,$$

for $l \geq 1$, and $m_0(t) \equiv \varepsilon_t^2$. Now, write $c(t_1, \dots, t_4) = \text{Cov}(Y_{t_1}^{(k_1)} Y_{t_2}^{(k_2)}, Y_{t_3}^{(k_3)} Y_{t_4}^{(k_4)})$, where for $k_1, \dots, k_4 \geq 1$,

$$Y_t^{(k_i)} \equiv \sum_{j_1=-\infty}^{t-1} \sum_{j_2=-\infty}^{j_1-1} \dots \sum_{j_l=-\infty}^{j_{l-1}-1} g_{t-j_1, j_1-j_2, \dots, j_{k_i-1}-j_{k_i}}^{(k_i)} (\varepsilon_t^2 \varepsilon_{j_1}^2 \dots \varepsilon_{j_{k_i}}^2 - E[\varepsilon_t^2 \varepsilon_{j_1}^2 \dots \varepsilon_{j_{k_i}}^2])$$

with $g_{j_1, \dots, j_l}^{(l)} \equiv \psi_{j_1} \dots \psi_{j_l} I(j_1 \geq 1, \dots, j_l \geq 1)$ and $Y_t^{(0)} \equiv \varepsilon_t^2$. Then,

$$\begin{aligned} \frac{1}{T} \left| \text{Cov}(Q_T^{(k_1, k_2)}, Q_T^{(k_3, k_4)}) \right| &= \frac{1}{T} \left| \sum_{t_1=1}^T \dots \sum_{t_4=1}^T d_T(t_1-t_2) d_T(t_3-t_4) c(t_1, \dots, t_4) \right| \\ &\leq \frac{2}{T} \sum_{t_1=1}^T \dots \sum_{t_4=1}^T (d_T(t_1-t_2)^2 + d_T(t_3-t_4)^2) |c(t_1, \dots, t_4)| \\ &= \frac{2}{T} \sum_{t_1=1}^T \sum_{t_2=1}^T \sum_{u=1-T}^{T-1} \sum_{s=1}^{T-|u|} d_T(u)^2 |c(t_1, t_2, s, s+|u|)| \\ &\quad + \frac{2}{T} \sum_{t_3=1}^T \sum_{t_4=1}^T \sum_{u=1-T}^{T-1} \sum_{s=1}^{T-|u|} d_T(u)^2 |c(s, s+|u|, t_3, t_4)| \end{aligned}$$

$$\begin{aligned}
&\leq K \sum_{u=1-T}^{T-1} \sum_{t_1=1-n+|u|}^{T-1} \sum_{t_2=1-n+|u|}^{T-1} d_T(u)^2 |c(t_1, t_2, 0, |u|)| \\
&\quad + K \sum_{u=1-T}^{T-1} \sum_{t_3=1-n+|u|}^{T-1} \sum_{t_4=1-n+|u|}^{T-1} d_T(u)^2 |c(0, |u|, t_3, t_4)| \\
&\leq K \sum_{u=1-T}^{T-1} d_T(u)^2 \left(\sup_{t_3, t_4} \sum_{t_1=-\infty}^{\infty} \sum_{t_2=-\infty}^{\infty} |c(t_1, t_2, t_3, t_4)| \right) \\
&\quad + K \sum_{u=1-T}^{T-1} d_T(u)^2 \left(\sup_{t_1, t_2} \sum_{t_3=-\infty}^{\infty} \sum_{t_4=-\infty}^{\infty} |c(t_1, t_2, t_3, t_4)| \right) \\
&\leq K \sum_{u=1-T}^{T-1} d_T(u)^2 \prod_{i=1}^4 \{ \|g^{(k_i)}\|_1 (k_i + 1)^2 (E[\varepsilon_0^8])^{k_i/4} \} \\
&\leq K \sum_{u=-T}^{T-1} d_T(u)^2 [(k_1 + 1) \dots (k_4 + 1)]^2 D^{k_1 + \dots + k_4}, \tag{10}
\end{aligned}$$

where $D \equiv E[\varepsilon_0^8]^{1/4} \sum_{j=1}^{\infty} \psi_j < 1$. The second inequality follows from the stationarity assumption and the fourth inequality comes from (3.23) of Giraitis and Robinson (2001). Focusing on the sum in (10), for T large enough that $2l < T$, we have

$$\begin{aligned}
\sum_{u=-T}^{T-1} d_T(u)^2 &= \frac{1}{(2\pi)^2 T^2} \sum_{u=-T}^{T-1} \sum_{j=-l}^l \sum_{k=-l}^l e^{-i(\lambda_j - \lambda_k)u} = \frac{1}{(2\pi)^2 T^2} \sum_{j=-l}^l \sum_{k=-l}^l \sum_{u=-T}^{T-1} e^{-i\lambda_u(j-k)} \tag{11} \\
&= \frac{1}{(2\pi)^2 T^2} \sum_{j=-l}^l 2T = \frac{2l+1}{2\pi^2 T}
\end{aligned}$$

since the final sum in (11) is nonzero only when $j = k$ or $|j - k|$ is a multiple of T . Hence, by (9):

$$\frac{1}{T} \text{Var} \left(\sum_{j=-l}^l \{I_v(\lambda_j) - E[I_v(\lambda_j)]\} \right) \leq K \frac{1}{T} \left(\sum_{k=0}^{\infty} (k+1)^2 D^k \right)^4 = O(l/T),$$

which implies $\sum_{j=-l}^l \{I_v(\lambda_j) - E[I_v(\lambda_j)]\} = O_p(l^{1/2})$ as claimed. ■

Proof of Theorem 3: Lemma 2 continues to hold since Assumption ARCH-8 implies $\sum_{k=0}^{\infty} |\gamma_k| < \infty$ (see Giraitis and Robinson, 2001 and Giraitis et al., 2000 for details). Similarly, Lemmas 3 and 4 still hold as they rely upon assumptions that are still directly enforced. However, Lemma 5 is no longer applicable but Lemma 6 can be used in its place for this ARCH context. Hence, the expressions leading

up to equations (7) and (8) remain valid so that, given Lemma 3, all that is left to show is

$$\frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_1} \{I_v(\lambda_j) - E[I_v(\lambda_j)]\} \frac{\partial f(\lambda_j; \theta_0)^{-1}}{\partial \theta} \xrightarrow{d} N(0, 2\Omega + \Pi).$$

But the Hölder continuity condition on $\partial f(\lambda; \theta_0)^{-1} / \partial \theta_k$ ensures that

$$\frac{1}{T^{1/2}} \sum_{j \in \mathcal{F}_1} \{I_v(\lambda_j) - E[I_v(\lambda_j)]\} \frac{\partial f(\lambda_j; \theta_0)^{-1}}{\partial \theta_k} = T^{1/2} \int_{-\pi}^{\pi} \{I_v(\lambda) - E[I_v(\lambda)]\} \frac{\partial f(\lambda; \theta_0)^{-1}}{\partial \theta_k} d\lambda + o_p(1)$$

for $k = 1, \dots, s$ (see, e.g., Hannan, 1973), and the proof of Theorem 2.2 of Giraitis and Robinson (2001) shows $T^{1/2} \int_{-\pi}^{\pi} \{I_v(\lambda) - E[I_v(\lambda)]\} (\partial / \partial \theta) f(\lambda; \theta_0)^{-1} d\lambda \xrightarrow{d} N(0, 2\Omega + \Pi)$ since $x' f(\lambda; \theta_0)^{-1} / \partial \theta$ is square integrable (see (3.2) of Giraitis and Robinson, 2001 and Parseval's Theorem). ■

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