1 Supplementary proofs

1.1 Derivation of the Estimator

For the special case of $\alpha = 2$ and $q = 2$, $f_n(\boldsymbol{\gamma};\lambda)$ in (3.1) can be rewritten in the matrix form as follows:

$$f_n(\boldsymbol{\gamma};\lambda) = (\mathbf{y} - \boldsymbol{X}\boldsymbol{\gamma})^{\mathrm{T}}(\mathbf{y} - \boldsymbol{X}\boldsymbol{\gamma}) + \lambda(\boldsymbol{A}\boldsymbol{\gamma})^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{\gamma}).$$

The first derivative of $f_n(\boldsymbol{\gamma};\lambda)$ with respect to $\gamma(k)$ are

$$\begin{aligned}
\frac{\partial f_n(\boldsymbol{\gamma};\lambda)}{\partial \gamma(0)} =& 2n\gamma(0) - 2\boldsymbol{z}_0^{\mathrm{T}}\mathbf{1}_n, \\
\frac{\partial f_n(\boldsymbol{\gamma};\lambda)}{\partial \gamma(k)} =& 2(n-k)\gamma(k) - 2\boldsymbol{z}_k^{\mathrm{T}}\mathbf{1}_{n-k} + 2\lambda k^2\gamma(k).
\end{aligned}$$

Let $\frac{\partial f_n(\boldsymbol{\gamma};\lambda)}{\partial \gamma(k)} = 0$ for $k = 0, 1, \ldots, n-1$. Solving these equations gives

$$\hat{\gamma}(0) = \frac{1}{n}\mathbf{1}_n^{\mathrm{T}}\boldsymbol{z}_0, \quad \hat{\gamma}(k) = \frac{1}{(n-k) + \lambda k^2}\mathbf{1}_{n-k}^{\mathrm{T}}\boldsymbol{z}_k, \quad \text{for } k = 1, \ldots, n-1.$$

We note that if $k \leq \lambda k^2$,

$$|\hat{\gamma}(k)| \leq |\bar{\gamma}(k)| \leq \bar{\gamma}(0) = \hat{\gamma}(0).$$

Hence, the proposed estimator satisfies the restriction $\mathcal{C} = \{\gamma : \gamma(0) \geq |\gamma(k)|\,, k = 1, \ldots, n-1\}$. Here, we use the fact that $\bar{\gamma}(k)$, the sample autocovariance function defined in (1.2), is nonnegative definite, and thus $|\bar{\gamma}(k)| \leq \bar{\gamma}(0)$.

1.2 Proofs of Theorems 1 and 2

We present the proofs of Theorems 1 and 2 by following the setting and notation in Wu (2009). Let $\kappa_p = \|Y_i\|_p$.

For Theorem 1, we use the coupling argument and central limit theorem for stationary processes (Hannan, 1973). Similar calculations to the proof of Theorem 1 in Wu (2009) give

$$\sum_{i=0}^{\infty} \|\mathcal{P}_0(Y_i Y_{i-k})\| \leq 2\kappa_4 \sum_{i=0}^{\infty} \delta_4(i) < \infty.$$

Also, for $0 \leq j \leq k$ and appropriate λ, $\frac{n}{(n-j)+\lambda j^2} \leq 1$, which results in $\sum_{i=0}^{\infty} \|\mathcal{P}_0(Y_i \mathbf{A}_k \boldsymbol{\eta}_i)\| < \infty$. Hence, by the Cramer-Wold device, (3.4) follows from Theorem 1 in (Hannan, 1973).

For Theorem 2, let $T_{nj} = \sum_{i=1}^{n-j}(Y_i Y_{i+j} - \gamma(j))$. The calculations of the proof are similar to the proof of Theorem 7 (i) in Wu (2011). In particular following Wu (2011) we obtain:

$$\|\mathcal{P}_0(Y_i Y_{i+j})\|_{p/2} \leq \|Y_i\|_p \delta_p(i+j) + \delta_p(i)\|Y_{i+j}^{'}\|_p,$$

for $j \geq 0$. By the triangle inequality,

$$\|T_{nj}\|_{p/2} = \|\sum_{i=1}^{n-j}\sum_{l\in\mathbb{Z}} \mathcal{P}_{i-l} Y_i Y_{i+j}\|_{p/2} \leq \sum_{l\in\mathbb{Z}} \|\sum_{i=1}^{n-j} \mathcal{P}_{i-l} Y_i Y_{i+j}\|_{p/2}.$$

Note that $\mathcal{P}_{i-l} Y_i Y_{i+j}$, $i = 1, \ldots, n$, form stationary martingale differences. Using Burkholder's moment inequality for martingale differences (Burkholder, 1988), the fact $p/4 \leq 1$ and simple algebra we obtain

$$\|\sum_{i=1}^{n-j} \mathcal{P}_{i-l} Y_i Y_{i+j}\|_{p/2}^{p/2} \leq \frac{\mathbb{E}\{[\sum_{i=1}^{n-j}(\mathcal{P}_{i-l} Y_i Y_{i+j})^2]^{p/4}\}}{(p/2-1)^{p/2}} \leq \frac{(n-j)\|\mathcal{P}_0 Y_l Y_{l+j}\|_{p/2}^{p/2}}{(p/2-1)^{p/2}}.$$

From these three facts

$$\begin{aligned}
\|T_{nj}\|_{p/2} \leq& \sum_{l\in\mathbb{Z}} \|\sum_{i=1}^{n-j} \mathcal{P}_{i-l} Y_i Y_{i+j}\|_{p/2} \\
\leq& \sum_{l\in\mathbb{Z}} \frac{(n-j)^{2/p}\|\mathcal{P}_0 Y_l Y_{l+j}\|_{p/2}}{p/2-1} \\
=& \frac{(n-j)^{2/p}}{p/2-1} \sum_{l\in\mathbb{Z}} \Big(\|Y_l\|_p \delta_p(l+j) + \delta_p(l)\|Y_{l+j}^{'}\|_p \Big) \\
\leq& \frac{4(n-j)^{2/p}\kappa_p \Delta_p}{p-2}.
\end{aligned}$$

Since $\delta_p(i) = 0$ if $i < 0$,

$$\begin{aligned}
\hat{\gamma}(k) - \frac{n-k}{(n-k)+\lambda k^2}\gamma(k) &= \frac{1}{(n-k)+\lambda k^2}\sum_{i=1}^{n-k}(Y_i Y_{i+k} - \gamma(k)) \\
&\leq \frac{1}{n-k}\sum_{i=1}^{n-k}(Y_i Y_{i-k} - \gamma(k)) \\
&= \frac{1}{n-k} T_{nk}
\end{aligned}$$

for appropriate λ. Hence,

$$\begin{aligned}
\left\|\hat{\gamma}(k) - \left(1 - \frac{\lambda k^2}{(n-k)+\lambda k^2}\right)\gamma(k)\right\|_{p/2} &\leq \frac{4(n-k)^{2/p}\kappa_p \Delta_p}{(p-2)n} \\
&= \frac{4\|Y_1\|_p \Delta_p}{(p-2)n^{1-2/p}}.
\end{aligned}$$

References

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