Appendix

Let $F_{r,s,\delta}(\cdot)$ denote the noncentral F-distribution with degrees of freedom r and s and non-centrality parameter δ, and let $F_{r,s}(\cdot) = F_{r,s,0}(\cdot)$. The mean and variance of $F_{r,s,\delta}(\cdot)$ are

$$\frac{s(r+\delta)}{r(s-2)} \text{ and } 2\,\frac{(r+\delta)^2 + (r+2\delta)(s-2)}{(s-2)^2(s-4)}, \tag{11}$$

assuming that $s > 2$ and $s > 4$, respectively.

We use the following representation of these distributions (Johnson et al. (1995), eq. (30.10)),

$$\begin{aligned} F_{r,s,\delta}(u) &= \sum_{l=0}^{\infty} \frac{e^{-\frac{\delta}{2}}(\frac{\delta}{2})^l}{l!} F_{r+2l,s}\left(\frac{ru}{r+2l}\right) && (12)\\ F_{r,s}(u) &= I_{\frac{ru}{ru+s}}\left(\frac{r}{2}, \frac{s}{2}\right), && (13) \end{aligned}$$

where $I_u(a,b)$ is the regularized incomplete beta function (i.e., beta distribution function) given by

$$I_u(a,b) = \frac{1}{B(a,b)} \int_0^u t^{a-1}(1-t)^{b-1}dt, \tag{14}$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the usual beta function.

Proof of Lemma 1. The conditional distribution of the projected data matrix $\mathbf{X}\mathbb{R}$ and $\mathbf{Y}\mathbb{R}$, given $\mathbb{R}$, are independent $N_k(\mathbb{R}'\mu_1, \mathbb{R}'\Sigma\mathbb{R})$ and $N_k(\mathbb{R}'\mu_2, \mathbb{R}'\Sigma\mathbb{R})$, respectively. Note that $S_{\mathbb{R}} = \mathbb{R}'S\mathbb{R}$, given $\mathbb{R}$, is distributed as Wishart $W_k\left(\frac{1}{n_1+n_2-1}\mathbb{R}'\Sigma\mathbb{R}, n_1+n_2-2\right)$. According to Theorem 3.4.8 of Mardia et al. (1979),

$$|S_{\mathbb{R}}| = |\mathbb{R}'\Sigma\mathbb{R}| \prod_{j=1}^{k} \chi^2_{n_1+n_2-j-1}, \tag{15}$$

where $\chi^2_{n_1+n_2-j-1}$ for $j = 1, \dots, k$ are independent χ^2 random variables. From expression (15), the proof is completed by showing that $\lambda_{\min}(\mathbb{R}'\Sigma\mathbb{R}) > 0$ with probability 1, where $\lambda_{\min}(A)$ is the minimum eigenvalue of the matrix A. Now, observe that

$$\begin{aligned} \lambda_{\min}(\mathbb{R}'\Sigma\mathbb{R}) &= \inf_{||u||_2=1} u'\mathbb{R}'\Sigma\mathbb{R}u \\ &\geq \inf_{||v||_2=1} v'\Sigma v \inf_{||u||_2=1} ||\mathbb{R}u||^2 = \lambda_{\min}(\Sigma) > 0. \end{aligned}$$

□

Proof of Theorem 1

Part (a). Note that

$$E[\phi(T^2_{\mathbb{R}})] = E_{\mathbb{R}}\left\{E_{\mathbf{X},\mathbf{Y}}\left[\phi(T^2_{\mathbb{R}})\middle|\mathbb{R}\right]\right\} = E_{\mathbb{R}}\left\{P_{\mathbf{X},\mathbf{Y}}\left[\frac{n-k+1}{k}\cdot\frac{T^2_{\mathbb{R}}}{n} > c_\alpha\middle|\mathbb{R}\right]\right\}. \qquad (16)$$

Under $\mathbf{H_0}$, the conditional distribution of $\frac{n-k+1}{k}\ \frac{T^2_{\mathbb{R}}}{n}$ is $F_{k,n-k+1}$, independent of $\mathbb{R}$. By (5), we have $E[\phi(T^2_{\mathbb{R}})|\mathbf{H_0}] = \mathbf{E}_{\mathbb{R}}\left\{\alpha\right\} = \alpha$.

Part (b). Under $\mathbf{H_1^*}$ and for fixed $\mathbb{R}$, the conditional distribution of $\frac{n-k+1}{k}\ \frac{T^2_{\mathbb{R}}}{n}$ is $F_{k,n-k+1,(n_1^{-1}+n_2^{-1})^{-1}\mathbf{\Delta}_{\mathbb{R}}}$. (Recall that $\mathbf{\Delta}_{\mathbb{R}} = (\mu_1-\mu_2)'\mathbb{R}(\mathbb{R}'\Sigma\mathbb{R})^{-1}\mathbb{R}'(\mu_1-\mu_2)$.) By (11) with $r = k$, $s = n-k+1$, and $\delta = 0$ we have that $c_\alpha \to 1$. By (11) with $r = k$, $s = n-k+1$, and $\delta = (n_1^{-1}+n_2^{-1})^{-1}\Delta_{\mathbb{R}}$ we have under $\mathbf{H_1^*}$, and for fixed $\mathbb{R}$, that the mean and variance of $\frac{n-k+1}{k}\ \frac{T^2_{\mathbb{R}}}{n}$ behave asymptotically as $c_\alpha + (n_1^{-1}+n_2^{-1})^{-1}\Delta_{\mathbb{R}}/k$ and $2/n$, respectively. (We say that a behaves asymptotically as b if $a/b \to 1$.)

It then follows from (5), (6), (16), and Chebychev's inequality that

$$E[\phi(T^2_{\mathbb{R}})|\mathbf{H_1^*}] = \mathbf{E}_{\mathbb{R}}\left\{\mathbf{E}_{\mathbf{X},\mathbf{Y}}\left[\phi(\mathbf{T^2_{\mathbb{R}}})\middle|\mathbb{R},\mathbf{H_1^*}\right]\right\} \to \mathbf{1}. \qquad (17)$$

Part (c). Using the property that $I_u(a+1,b) \leq I_u(a,b)$, and (13), we have

$$\begin{aligned} I_{\frac{kc_\alpha}{kc_\alpha+n-k+1}}\left(\frac{k}{2}+l, \frac{n-k+1}{2}\right) &\leq I_{\frac{kc_\alpha}{kc_\alpha+n-k+1}}\left(\frac{k}{2}, \frac{n-k+1}{2}\right) \\ &= F_{k,n-k+1}(c_\alpha) = 1-\alpha. \end{aligned} \qquad (18)$$

Thus, using (16) and (18), we have $E[\phi(T^2_{\mathbb{R}})|\mathbf{H_1}] \geq \alpha$. □

Proof of Theorem 2 By evaluating the conditional probability that $\bar{\theta}^* < u$ given the data, and then taking an expectation over the data, we have

$$P\left[\bar{\theta}^* < u\right] = E_{\mathbf{X},\mathbf{Y}}\left\{P_{\mathbb{R}}\left[\bar{\theta}^* < u\middle|\mathbf{X},\mathbf{Y}\right]\right\}. \qquad (19)$$

Note that

$$P_{\mathbb{R}}\left[\bar{\theta}^* < u\middle|\mathbf{X},\mathbf{Y}\right] = P_{\mathbb{R}}\left[\frac{\bar{\theta}^* - E_{\mathbb{R}}\left(\theta_1^*|\mathbf{X},\mathbf{Y}\right)}{\sqrt{V_{\mathbb{R}}\left(\theta_1^*|\mathbf{X},\mathbf{Y}\right)/m}} < \frac{u - E_{\mathbb{R}}\left(\theta_1^*|\mathbf{X},\mathbf{Y}\right)}{\sqrt{V_{\mathbb{R}}\left(\theta_1^*|\mathbf{X},\mathbf{Y}\right)/m}}\ \middle|\mathbf{X},\mathbf{Y}\right], \qquad (20)$$

where $E_{\mathbb{R}}\left(\theta_1^*|\mathbf{X},\mathbf{Y}\right)$ and $V_{\mathbb{R}}\left(\theta_1^*|\mathbf{X},\mathbf{Y}\right)$ are the conditional mean and variance of θ_1^* given the data, $\mathbf{X},\mathbf{Y}$. Further, given $\mathbf{X},\mathbf{Y}$, the random variables $\{\theta_i^*,\ i = 1,2\ldots,m\}$ are independent and identically distributed with finite variance. Now by using the Central Limit Theorem, we have

$$\lim_{m\to\infty}\left\{P_{\mathbb{R}}\left[\bar{\theta}^{*}<u\middle|\mathbf{X},\mathbf{Y}\right]-\Phi\left(\frac{u-E_{\mathbb{R}}\left(\theta_1^{*}\middle|\mathbf{X},\mathbf{Y}\right)}{\sqrt{V_{\mathbb{R}}\left(\theta_1^{*}\middle|\mathbf{X},\mathbf{Y}\right)/m}}\right)\right\}=0, \tag{21}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function. From (7),

$$\begin{aligned}
&E_{\mathbb{R}}(\theta_1^{*}|\mathbf{X},\mathbf{Y})\\
=\;& E_{\mathbb{R}}\left[1-F_{k,n-k+1}\left(\frac{n-k+1}{k}\cdot\frac{T^2_{\mathbb{R}_1}}{n}\right)\middle|\mathbf{X},\mathbf{Y}\right]\\
=\;& \int\left\{1-F_{k,n-k+1}\left(\frac{n-k+1}{k}\cdot\frac{\frac{n_1n_2}{n_1+n_2}(\overline{X}-\overline{Y})'R(R'SR)^{-1}R'(\overline{X}-\overline{Y})}{n_1+n_2-2}\right)\right\}d\mathbf{P}_R,
\end{aligned}$$

where $\mathbf{P}_R$ is the probability measure corresponding to random matrix $\mathbb{R}$. We claim that distribution of $E_{\mathbb{R}}(\theta_1^{*}|\mathbf{X},\mathbf{Y})$ does not depend upon the parameters μ_1, μ_2 and Σ. To verify this claim, it suffices to show that

$$\begin{aligned}
&E_{\mathbf{X},\mathbf{Y}}\left[E_{\mathbb{R}}(\theta_1^{*}|\mathbf{X},\mathbf{Y})\right]^r\\
=&\int\left[\int\left\{1-F_{k,n-k+1}\left(\frac{n-k+1}{k}\cdot\frac{\frac{n_1n_2}{n_1+n_2}(\overline{X}-\overline{Y})'R(R'SR)^{-1}R'(\overline{X}-\overline{Y})}{n_1+n_2-2}\right)\right\}d\mathbf{P}_R\right]^r d\mathbf{P}_{\mathbf{X},\mathbf{Y}}
\end{aligned} \tag{22}$$

does not depend upon (μ_1,μ_2,Σ) for $r=1,2,\ldots$, where $\mathbf{P}_{\mathbf{X},\mathbf{Y}}$ is the probability measure corresponding to the data $\mathbf{X},\mathbf{Y}$.

Note that $0\le E_{\mathbb{R}}(\theta_1^{*}|\mathbf{X},\mathbf{Y})\le 1$. Observe that

$$\begin{aligned}
&\int\!\!\int\left\{1-F_{k,n-k+1}\left(\frac{n-k+1}{k}\cdot\frac{\frac{n_1n_2}{n_1+n_2}(\overline{X}-\overline{Y})'R(R'SR)^{-1}R'(\overline{X}-\overline{Y})}{n_1+n_2-2}\right)\right\}^r d\mathbf{P}_R d\mathbf{P}_{\mathbf{X},\mathbf{Y}} \qquad (23)\\
=&\int\left[\int\left\{1-F_{k,n-k+1}\left(\frac{n-k+1}{k}\cdot\frac{\frac{n_1n_2}{n_1+n_2}(\overline{X}-\overline{Y})'R(R'SR)^{-1}R'(\overline{X}-\overline{Y})}{n_1+n_2-2}\right)\right\}^r d\mathbf{P}_{\mathbf{X},\mathbf{Y}}\right]d\mathbf{P}_R,
\end{aligned}$$

where the interchange of integral are permitted by Fubini's theorem. Now, observe that under $\mathbf{H_0}$, the distribution of $F_{k,n-k+1}\left(\frac{n-k+1}{k}\cdot\frac{\frac{n_1n_2}{n_1+n_2}(\overline{X}-\overline{Y})'R(R'SR)^{-1}R'(\overline{X}-\overline{Y})}{n_1+n_2-2}\right)$ is $U(0,1)$ for any given Projection matrix R. Therefore, the inner integral

$$\int\left\{1-F_{k,n-k+1}\left(\frac{n-k+1}{k}\cdot\frac{\frac{n_1n_2}{n_1+n_2}(\overline{X}-\overline{Y})'R(R'SR)^{-1}R'(\overline{X}-\overline{Y})}{n_1+n_2-2}\right)\right\}^r d\mathbf{P}_{\mathbf{X},\mathbf{Y}} \tag{24}$$

does not depend upon the parameter (μ_1, μ_2, Σ). This imply that (23) does not depend upon the parameter for any positive integer r.

Now note that, from (22) and by using Fubini theorem, we have

$$E_{\mathbf{X},\mathbf{Y}}\left[E_{\mathbb{R}}(\theta_1^*|\mathbf{X},\mathbf{Y})\right]^r$$
$$= \int \cdots \int \left[\int \prod_{i=1}^{r} \left\{ 1 - F_{k,n-k+1}\left(\frac{n-k+1}{k} \cdot \frac{\frac{n_1 n_2}{n_1+n_2}(\overline{X}-\overline{Y})' R_i (R_i' S R_i)^{-1} R_i' (\overline{X}-\overline{Y})}{n_1+n_2-2} \right) \right\} d\mathbf{P}_{\mathbf{X},\mathbf{Y}} \right] \prod_{i=1}^{r} d\mathbf{P}_{R_i} \tag{25}$$

In (25), observe that R_i , $i = 1, \ldots, r$, are iid with probability measure P_R. By using this and (24), it follows that

$$\int \prod_{i=1}^{r} \left\{ 1 - F_{k,n-k+1}\left(\frac{n-k+1}{k} \cdot \frac{\frac{n_1 n_2}{n_1+n_2}(\overline{X}-\overline{Y})' R_i (R_i' S R_i)^{-1} R_i' (\overline{X}-\overline{Y})}{n_1+n_2-2} \right) \right\} d\mathbf{P}_{\mathbf{X},\mathbf{Y}},$$

does not depend upon the parameter (μ_1, μ_2, Σ) which in turn implies that (22) holds for any positive integer r. Similarly, under $\mathbf{H_0}$, the distribution of $V_{\mathbb{R}}\left(\theta_1^* \middle| \mathbf{X}, \mathbf{Y}\right)$ also does not depend on the parameters. Now note that

$$\left| P_{\mathbb{R}}\left[\bar{\theta}^* < u \middle| \mathbf{X}, \mathbf{Y}\right] - \Phi\left(\frac{u - E_{\mathbb{R}}\left(\theta_1^* \middle| \mathbf{X}, \mathbf{Y}\right)}{\sqrt{V_{\mathbb{R}}\left(\theta_1^* \middle| \mathbf{X}, \mathbf{Y}\right)/m}} \right) \right| < 2. \tag{26}$$

From (19), (21), (26) and the dominated convergence theorem, we have

$$\lim_{m\to\infty} \left\{ P\left[\bar{\theta}^* < u\right] - E_{\mathbf{X},\mathbf{Y}}\left[\Phi\left(\frac{u - E_{\mathbb{R}}\left(\theta_1^* \middle| \mathbf{X}, \mathbf{Y}\right)}{\sqrt{V_{\mathbb{R}}\left(\theta_1^* \middle| \mathbf{X}, \mathbf{Y}\right)/m}} \right) \right] \right\} = 0$$

Thus, for any n_1, n_2, as $m \to \infty$, the asymptotic distribution of $\frac{1}{m}\sum_{i=1}^{m} \theta_i^*$ does not depend on the parameters μ_1, μ_2, and Σ. This completes the proof. □

Proof of Theorem 3 The power of the test (8) is

$$E[\phi^*|\mathbf{H_1^*}] \;=\; P\left[\bar{\theta}^* < u_{\{\alpha,n_1,n_2\}} \middle| \mathbf{H_1^*}\right],$$

where $u_{\{\alpha,n_1,n_2\}}$ is such that

$$P\left[\bar{\theta}^* < u_{\{\alpha,n_1,n_2\}} \middle| \mathbf{H_0}\right] = \alpha.$$

For a given α, n_1, and n_2, we have $0 < u_{\{\alpha,n_1,n_2\}} < 1$. Thus, there exists a convergent subsequence of $u_{\{\alpha,n_1,n_2\}}$. With an abuse of the notation, let this subsequence be $u_{\{\alpha,n_1,n_2\}}$, converging to u_α.

We claim that $u_\alpha > 0$. To see this, note first that for all (n_1, n_2), $P(\bar{\theta}^* \leq \epsilon|\mathbf{H_0}) \leq P(m^{-1}\theta_1 \leq \epsilon|\mathbf{H_0}) = \epsilon m$, since θ_i is uniform(0,1) distributed under $\mathbf{H_0}$. Thus, there exists a positive ϵ such that $P(\bar{\theta}^* \leq \epsilon|\mathbf{H_0}) < \alpha$ for all (n_1, n_2). It follows that $u_{\alpha,n_1,n_2} \geq \epsilon$ for all (n_1, n_2) and therefore $u_\alpha \geq \epsilon > 0$.

Let ν be positive. Since θ_i is the p-value of the test $\phi(T^2_{\mathbb{R}})$, it follows from Theorem 1 (b) with $\alpha = \nu$ that $P(\theta_i < \nu|\mathbf{H_1^*}) = P(\phi(T^2_{\mathbb{R}}) = 1|\mathbf{H_1^*}) \to 1$. Therefore, since m is fixed and finite, $P(\theta_i < \nu,\ i = 1, \ldots, m|\mathbf{H_1^*}) \to 1$ and consequently, $P(\bar{\theta}^* < \nu|\mathbf{H_1^*}) \to 1$. This result holds for all $\nu > 0$. Since $u_{\{\alpha,n_1,n_2\}} \to u_\alpha > 0$, it follows that $P(\bar{\theta}^* < u_{\{\alpha,n_1,n_2\}}|\mathbf{H_1^*}) \to 1$, that is, $\lim_{n_1,n_2\to\infty} E[\phi^*|\mathbf{H_1^*}] = 1$.

$\square$