Supplementary Material for "Geometry of Multiprimary Display Colors II: Metameric Control Sets and Gamut Tilings"

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This document provides supplementary material, for the paper [1]. In Section S.I, we present the full collection of facet spans and all the complete sets of compatible facet spans for the $K = 5$ primary systems $\mathbf{P}_e^{(5)}$ and $\mathbf{P}_w^{(5)}$. In Section S.I-B we show all the tilings of for the five primary system $\mathbf{P}_w^{(5)}$ and highlight the fact that the corresponding progressive tilings presented in the companion Part I paper [2, Appendix E] form a strict subset. In Section S.II we provide the computation time requirements for the tiling enumeration results presented in Table 3 of the main paper.

S.I. Examples of Facet Spans, Complete Sets, and Tilings

In this section, we present examples of facet spans, complete sets of compatible facet spans, and tilings for primary systems with $K = 5$, specifically, for the five-primary systems $\mathbf{P}_e^{(5)}$ and $\mathbf{P}_w^{(5)}$, specified in Appendix C.

A. Facet spans and Complete Sets for $\mathbf{P}_e^{(5)}$

Figure S.1 shows the full collection of facets spans for the $K = 5$ primary system $\mathbf{P}_e^{(5)}$. The facet spans in Fig. S.1 are organized such that each row shows the full set of $(K-2)! = 6$ facet spans for some $\mathbb{J} \in \mathfrak{C}^2(\langle K \rangle)$. Additionally, the facet spans for each $\mathbb{J}$ are organized in two halves to illustrate the symmetry of facet spans. From left-to-right, the first $(K-2)!/2 = 3$ facet spans in the row comprise one half, and the remaining $(K-2)!/2 = 3$ facet spans in the row comprise the second half, and each half contains the mirror symmetric versions of the other. For instance, the facet spans $\mathcal{F}^{([1\,2],[3\,4\,5])}$ and $\mathcal{F}^{([12],[5\,4\,3])}$ for $\mathbb{J} = [1\,2]$ located in the first and second half of the first row are mirror symmetric versions of each other.

All the ten complete sets of compatible facet spans for $\mathbf{P}_e^{(5)}$ are shown in Fig. S.2, where each of the subfigures (a)-(e) shows a pair of symmetric complete sets, with boxes delimiting the ten facet spans comprising each complete set, with the mirror symmetric facet spans located in corresponding positions within each box.

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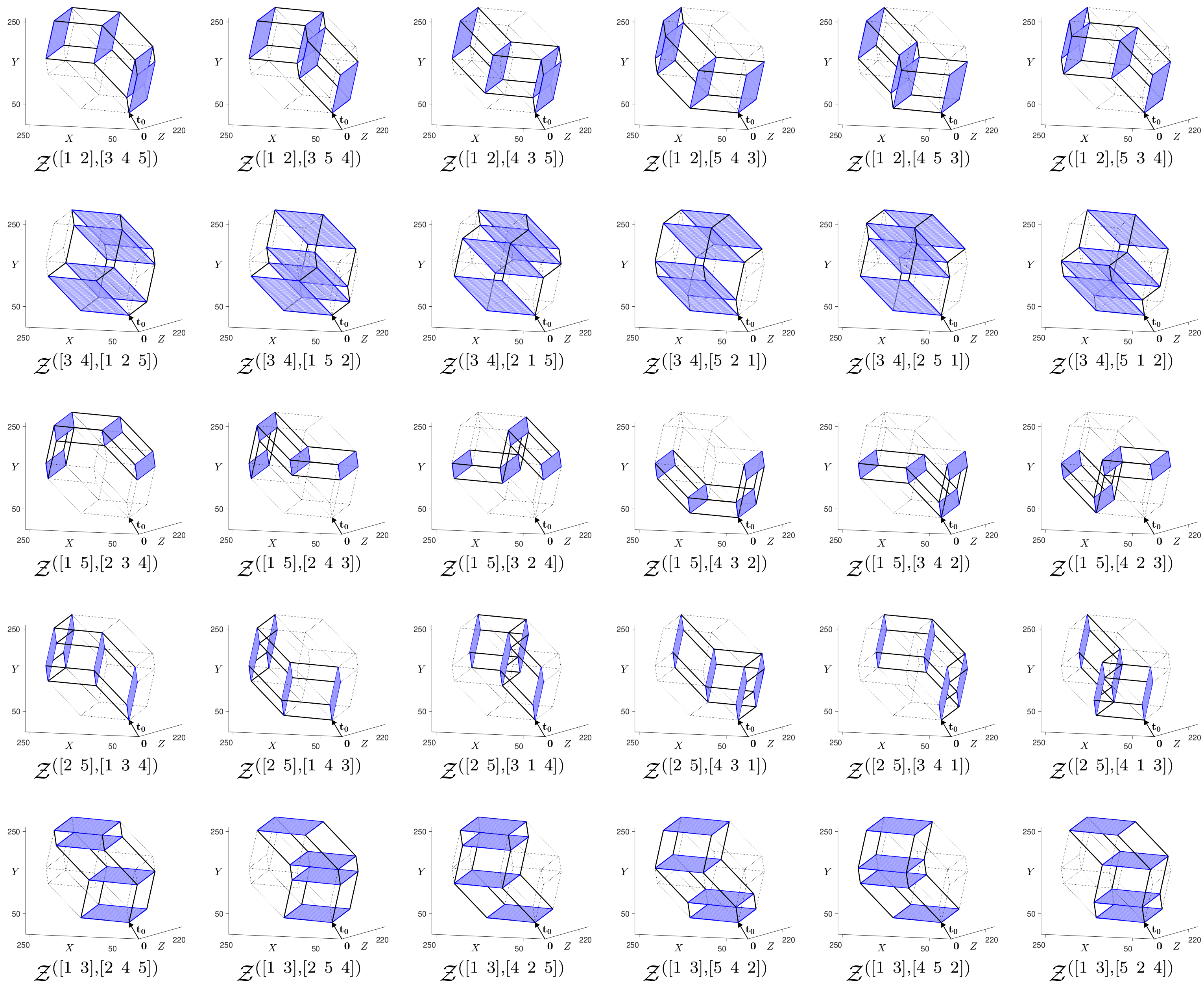
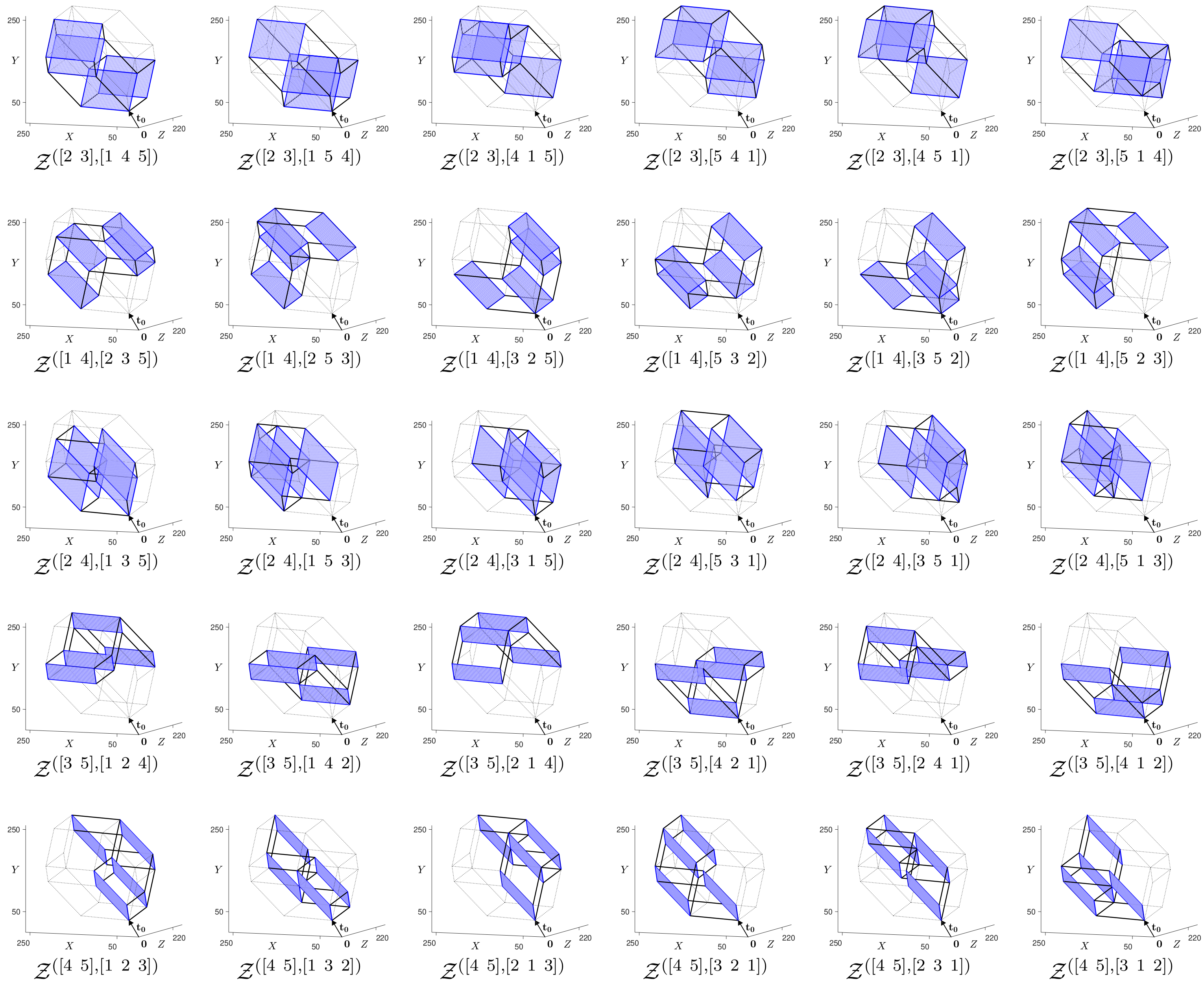
We also note all the tilings for $\mathbf{P}_e^{(5)}$ are produced by the progressive methodology introduced in the companion Part I Paper [2].

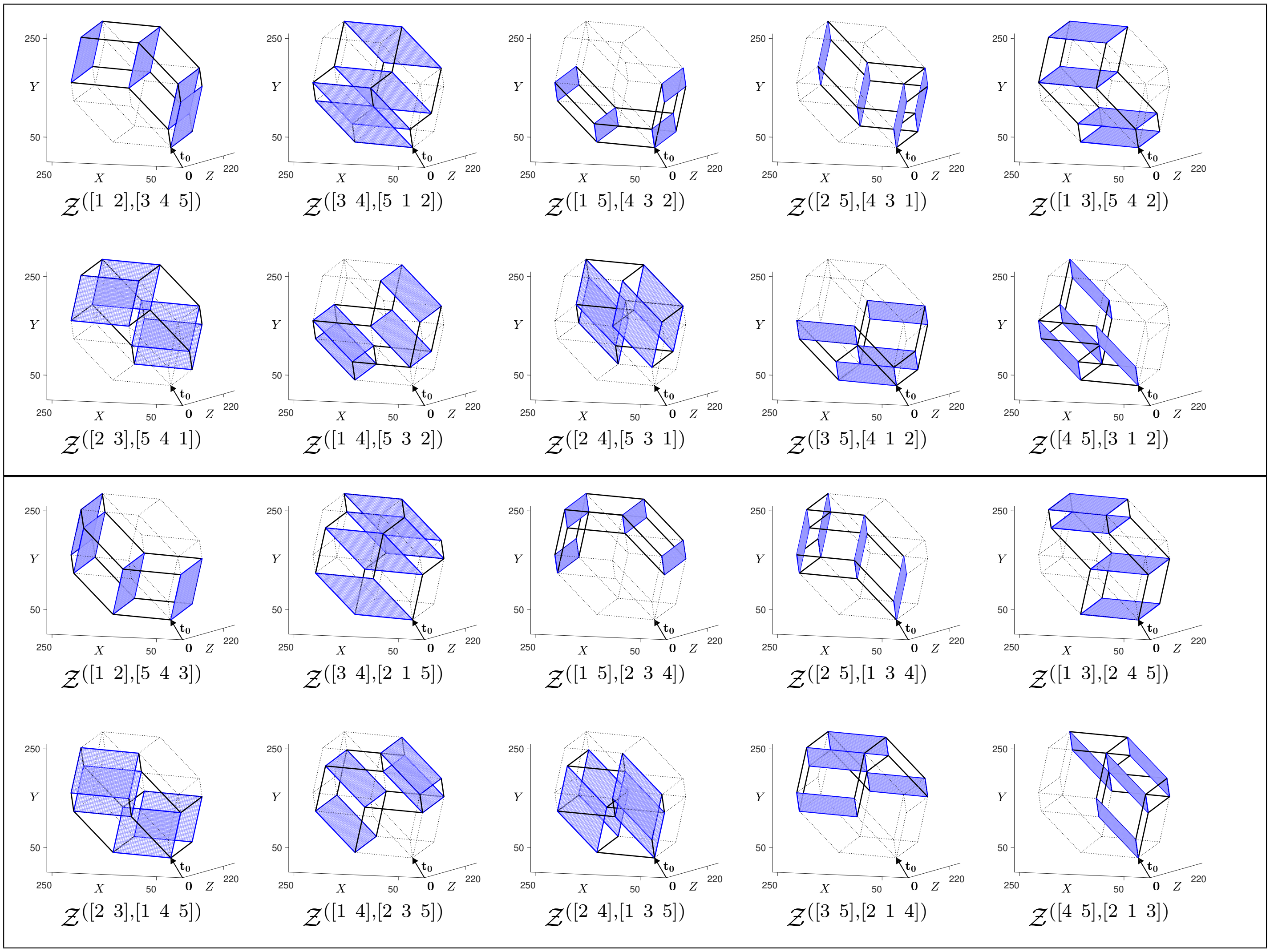


(a) *(See page S.4 for caption)*

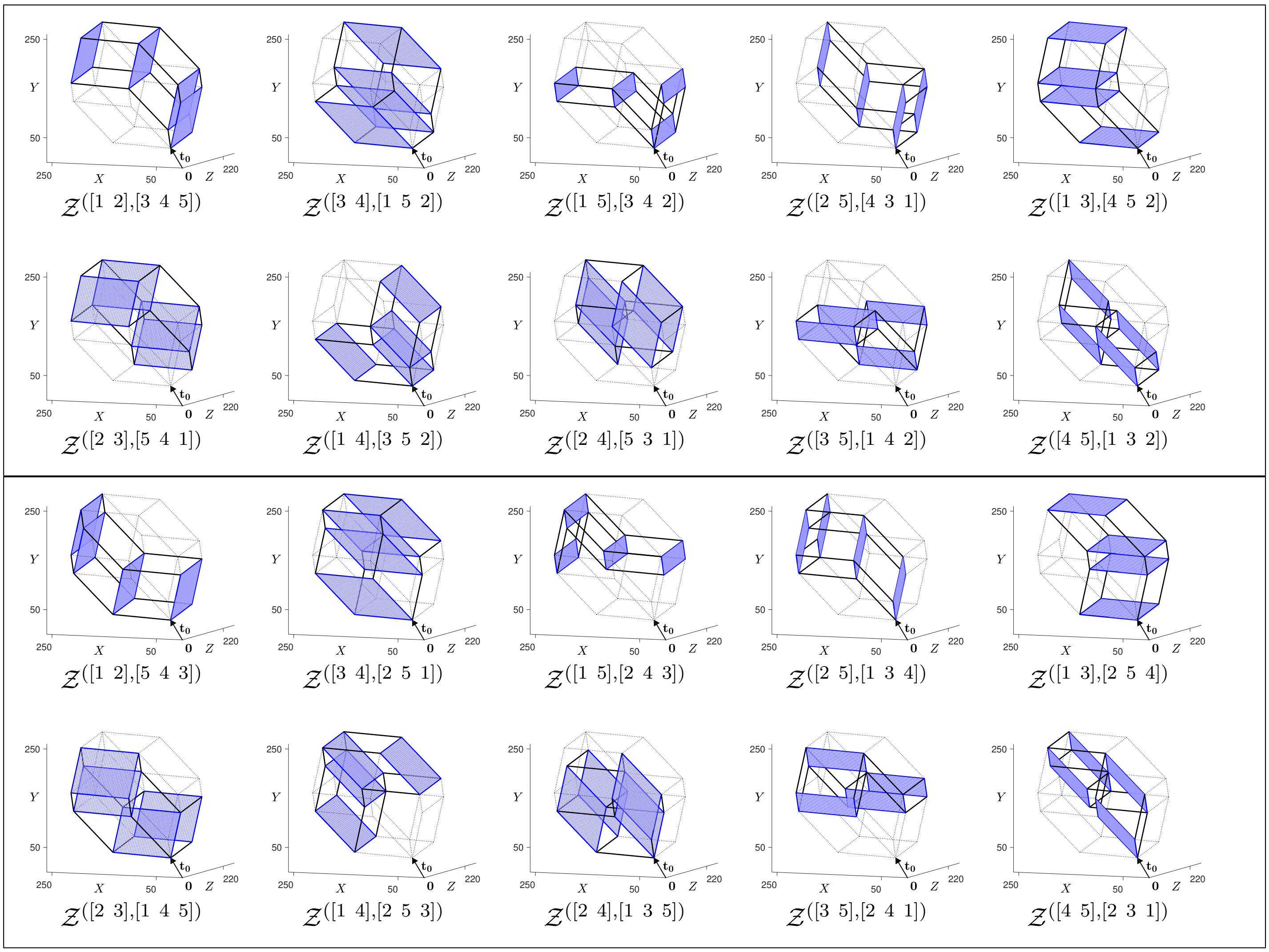


(b)

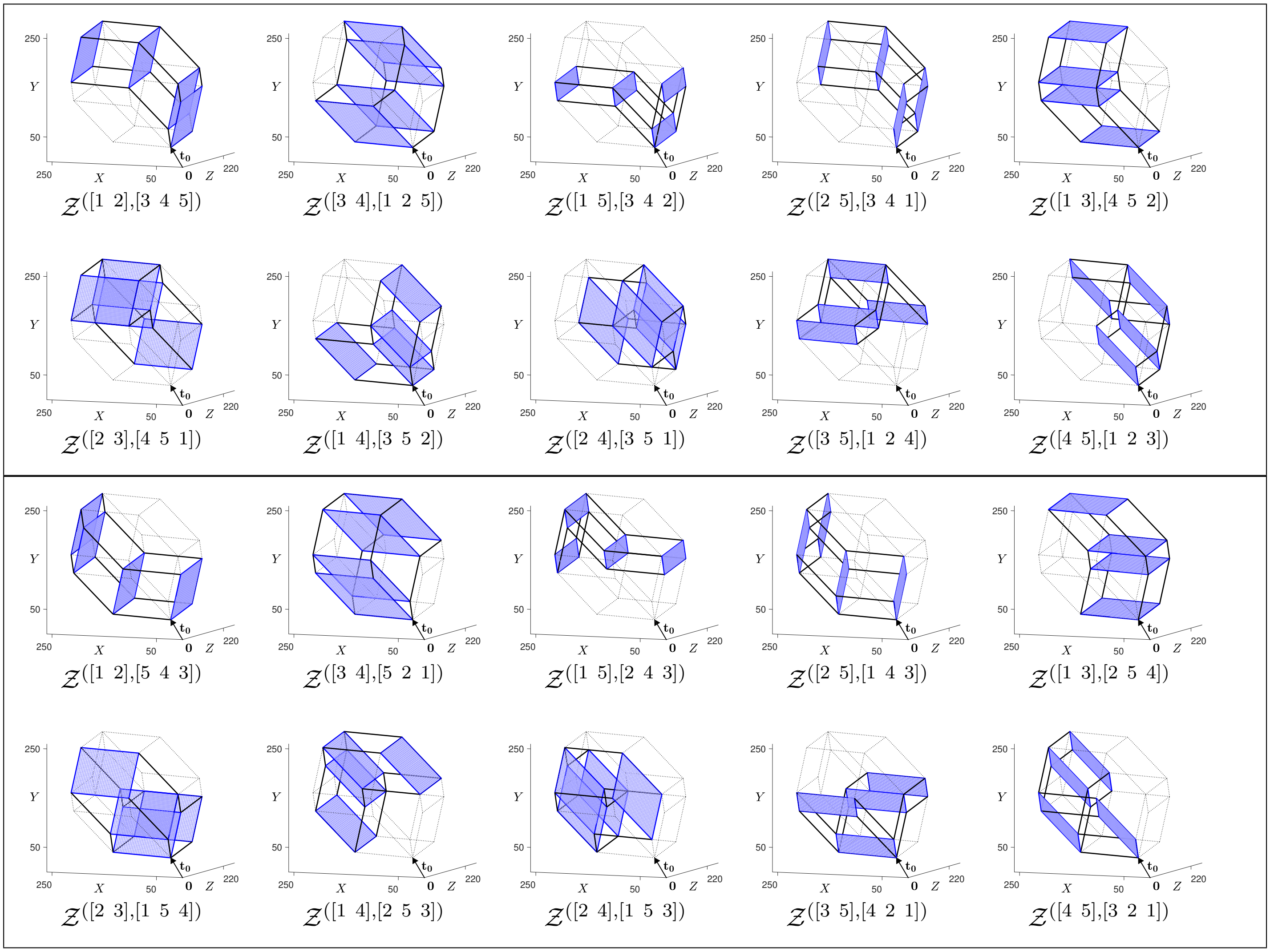
Fig. S.1: The full collection of facet spans for the $K = 5$ primary system $\mathbf{P}_e^{(5)}$. All the $(K-2)! = 6$ facet spans $\mathcal{Z}^{(\mathbb{J},\mathbb{I})}$ are shown in each row in (a) for $\mathbb{J} = [1\,2],\ [3\,4],\ [1\,5],\ [2\,5],\ [1\,3]$ and in (b) for $\mathbb{J} = [2\,3],\ [1\,4],\ [2\,4],\ [3\,5],\ [4\,5]$.



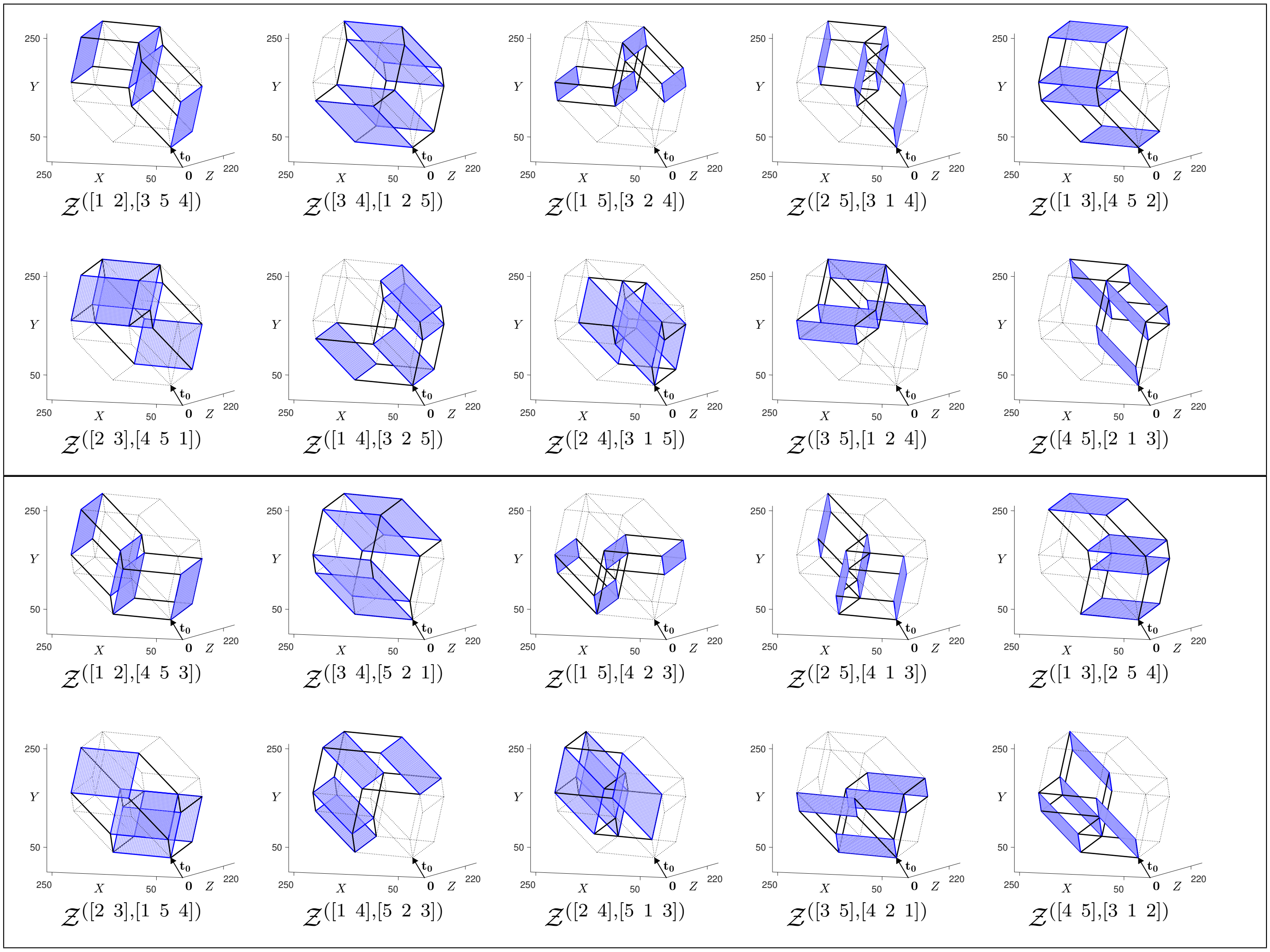
(a) *(See page S.9 for caption)*



(b) *(See page S.9 for caption)*



(c) *(See page S.9 for caption)*



(d) *(See page S.9 for caption)*

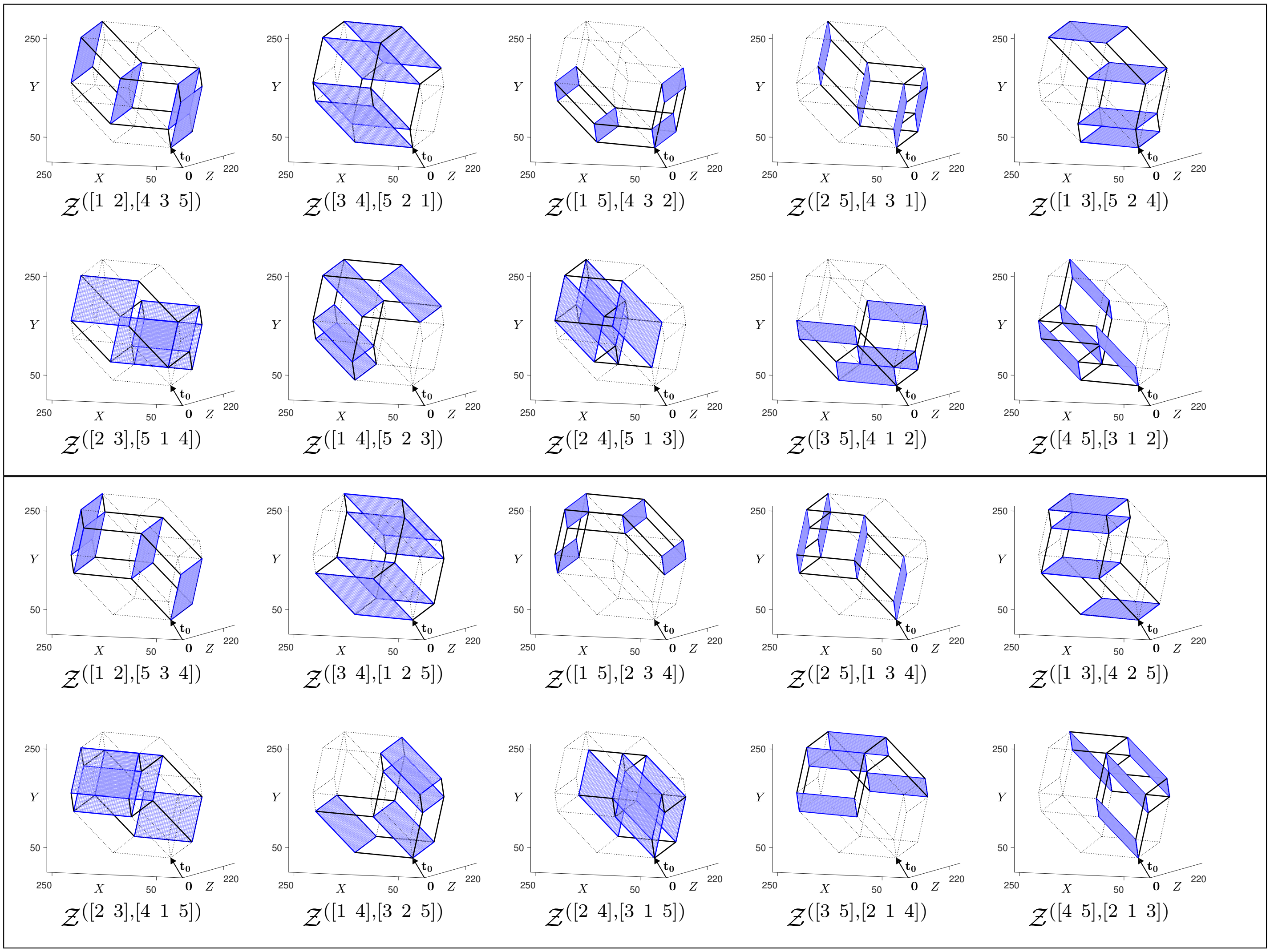


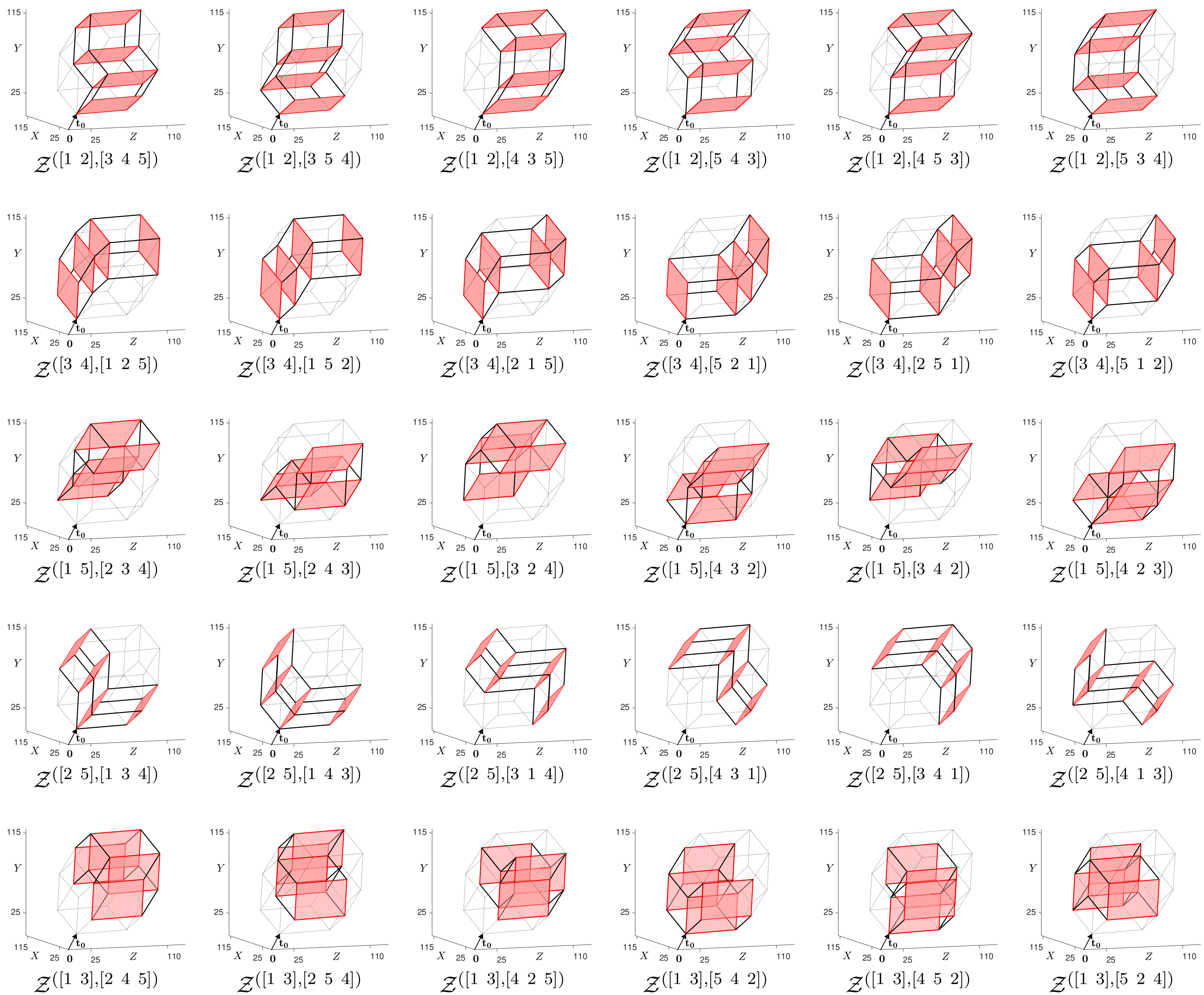
Fig. S.2: The ten complete sets for primary system $\mathbf{P}_e^{(5)}$, shown in pairs in subfigures (a) to (e). In each subfigure, the top box delimits the facet spans of a complete set $\mathcal{K}$, and the bottom box delimits the corresponding mirror symmetric complete set $\tilde{\mathcal{K}}$.

B. *Facet spans, Complete Sets, and Tilings for* $\mathbf{P}_w^{(5)}$

Figure S.3 shows[1] the full collection of facets spans for the primary system $\mathbf{P}_w^{(5)}$, and Fig. S.4 shows the ten complete sets of compatible facet spans for $\mathbf{P}_w^{(5)}$. The facet spans in Fig. S.3 and Fig. S.4 are organized to illustrate the symmetry of the facet spans, as described in Section S.I-A.

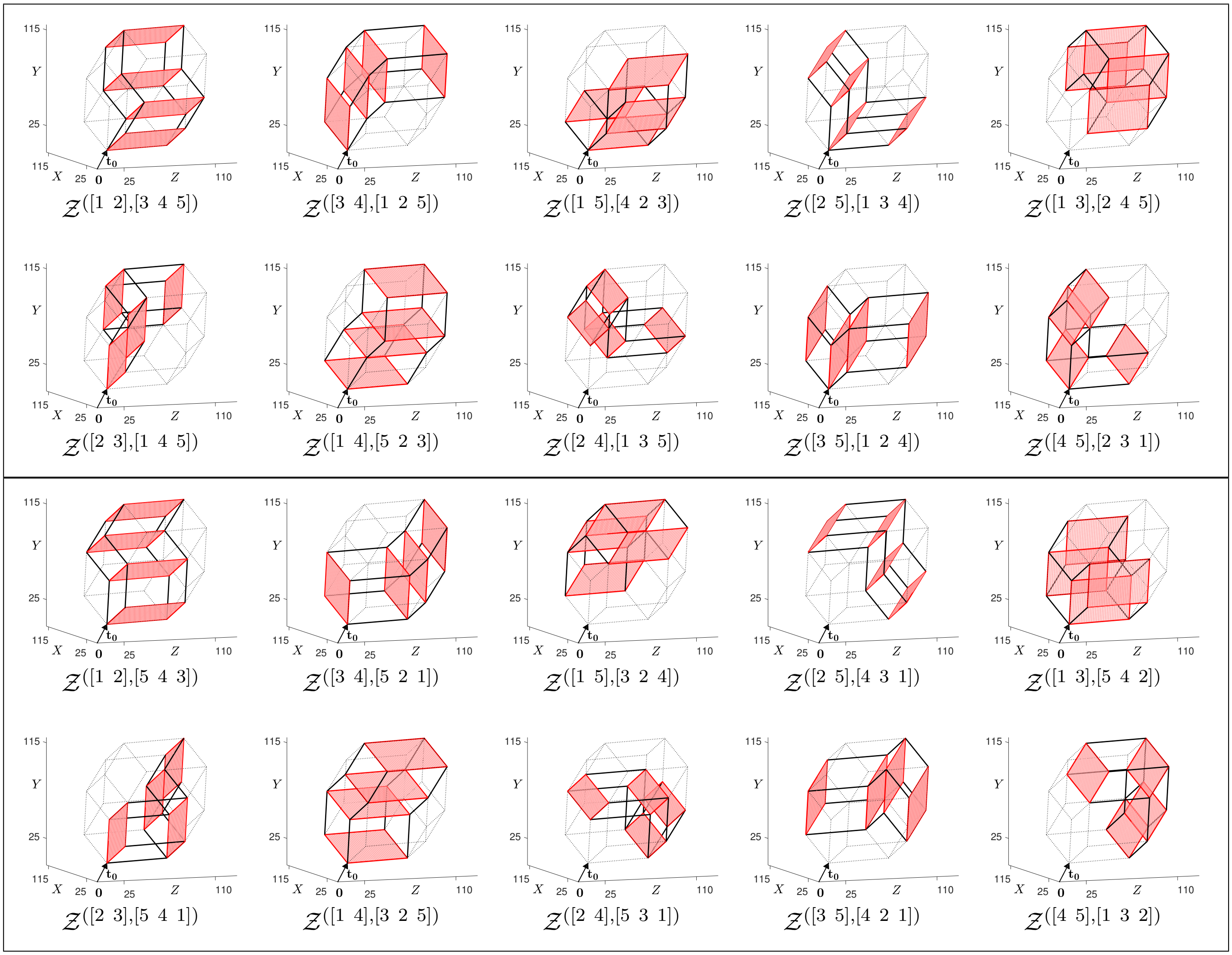
The tilings of the gamut for the system $\mathbf{P}_w^{(5)}$ are shown in Fig. S.5, where the subcaptions indicate the corresponding complete set in Fig. S.2. To allow the distinct parallelepipeds in the tilings to be seen, the centers of the parallelepipeds have been displaced radially outward with respect to the center of the gamut (by scaling the original displacements by a factor of 1.8) the axes in Fig. S.5 are therefore labeled as X', Y', Z'. Note that the eight tilings in (a)–(h) are progressive tilings and correspond to the eight tilings shown, respectively, in Fig. 13 (a)–(h) of the companion Part I paper [2]. The two additional tilings, shown in (i) and (j), corresponding to complete sets $\mathcal{K}_4$ and $\tilde{\mathcal{K}}_5$, are not progressive tilings. In particular, the facet spans $\mathcal{Z}^{([2\ 4],[3\ 5\ 1])} \in \tilde{\mathcal{K}}_4$ and $\mathcal{Z}^{([3\ 5],[4\ 1\ 2])} \in \mathcal{K}_5$ do not belong to any progressive tiling. We note that the clear vertex in Fig. 2(a) in the main paper [1], which did not have an associated progressive tiling control vector, correspond to the tiling control vector associated with the tiling $\mathcal{T}_{\tilde{\mathcal{K}}_4}$ shown in Fig. 5(i).

[1]All the gamut and facet span visualizations in this section use a value of $\mathbf{t_0} = [15, 15, 15]^T$, which is chosen to improve the visualizations by distancing the axes from the gamut facets and edges.

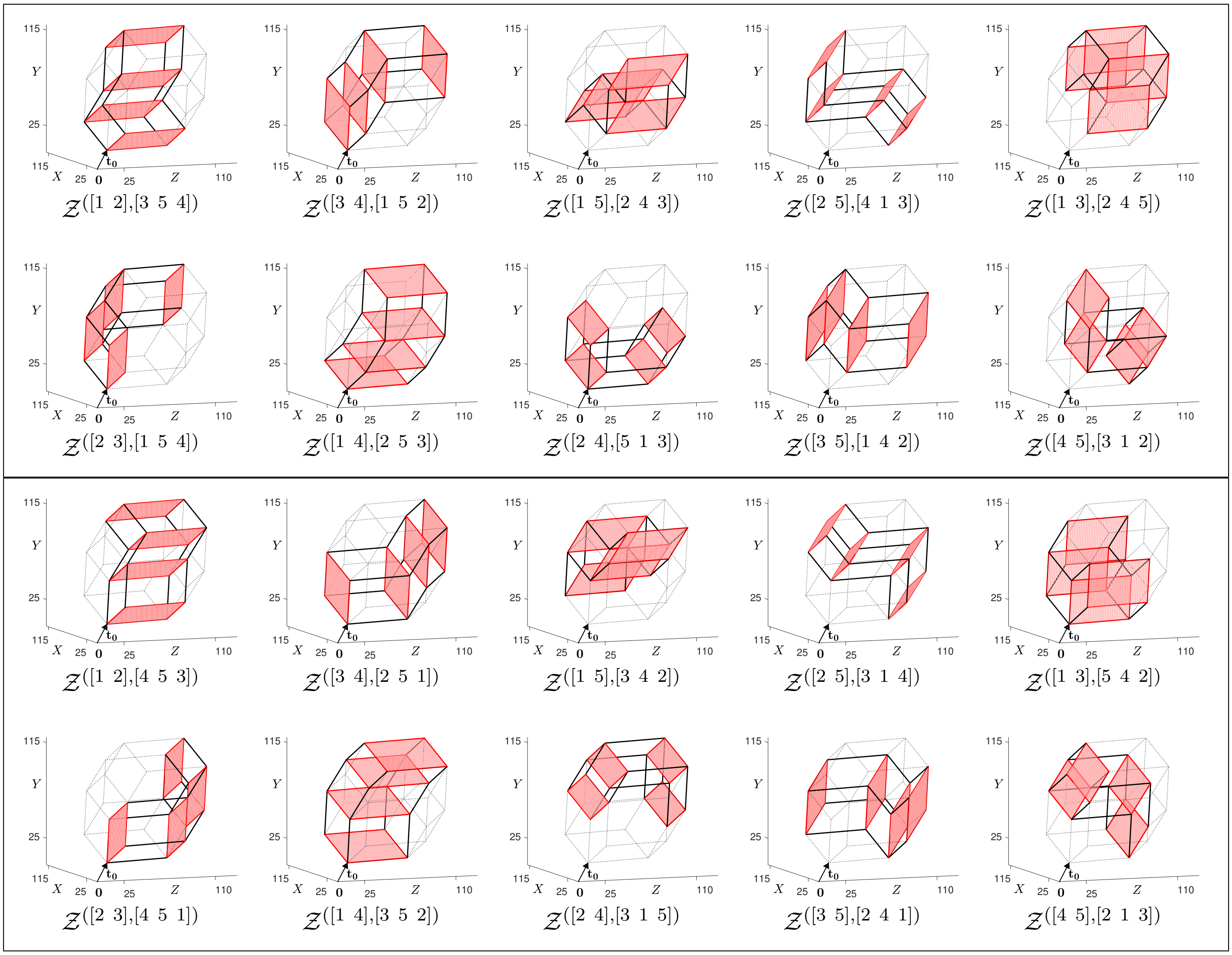


(a) *(See page S.12 for caption)*

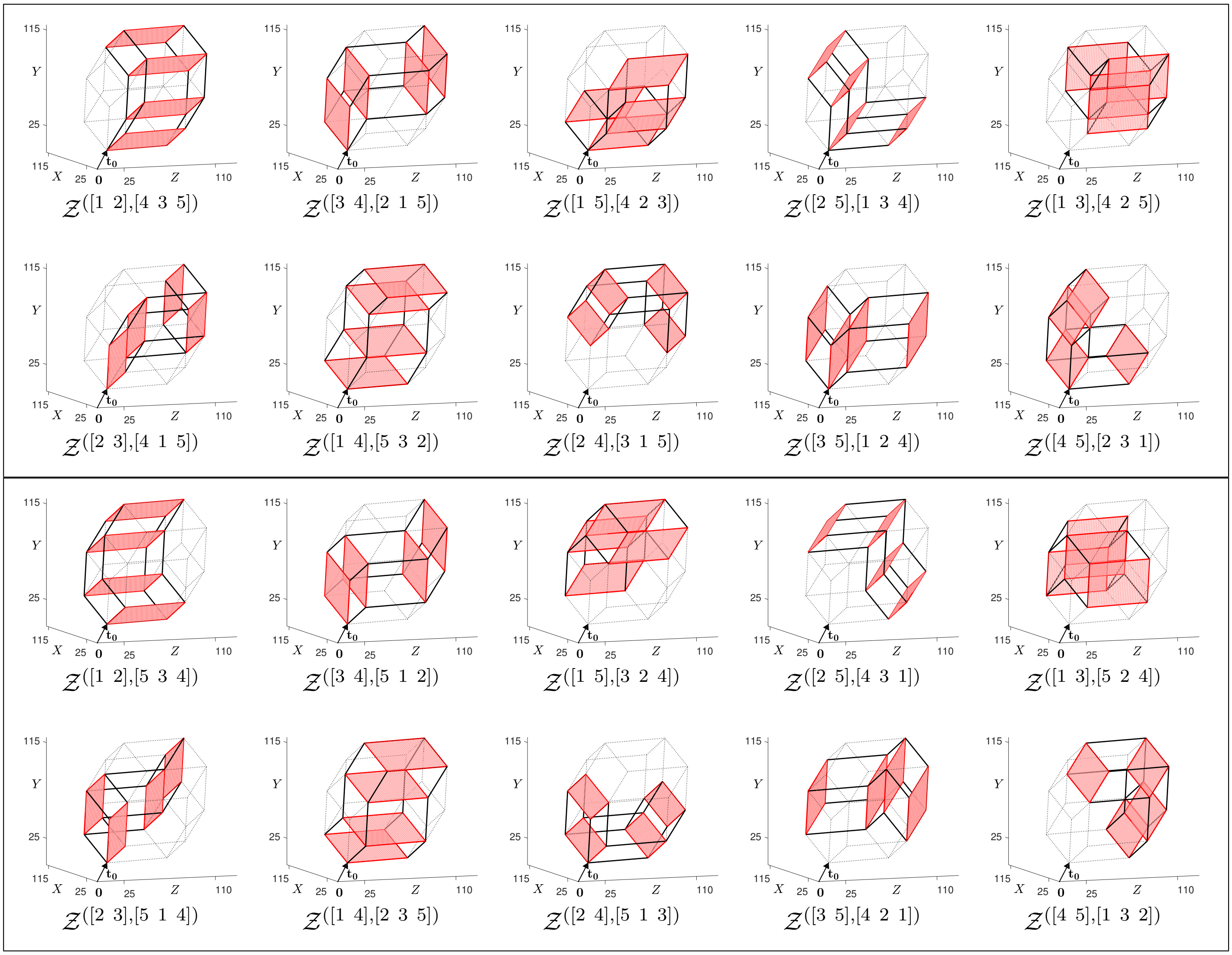
Fig. S.3: The full collection of facet spans for the $K = 5$ primary system $\mathbf{P}_w^{(5)}$. All the $(K-2)! = 6$ facet spans $\mathcal{Z}^{(\mathbb{J},\mathbb{I})}$ are shown in each row in (a) for $\mathbb{J} = [1\,2],\ [3\,4],\ [1\,5],\ [2\,5],\ [1\,3]$ and in (b) for $\mathbb{J} = [2\,3],\ [1\,4],\ [2\,4],\ [3\,5],\ [4\,5]$.



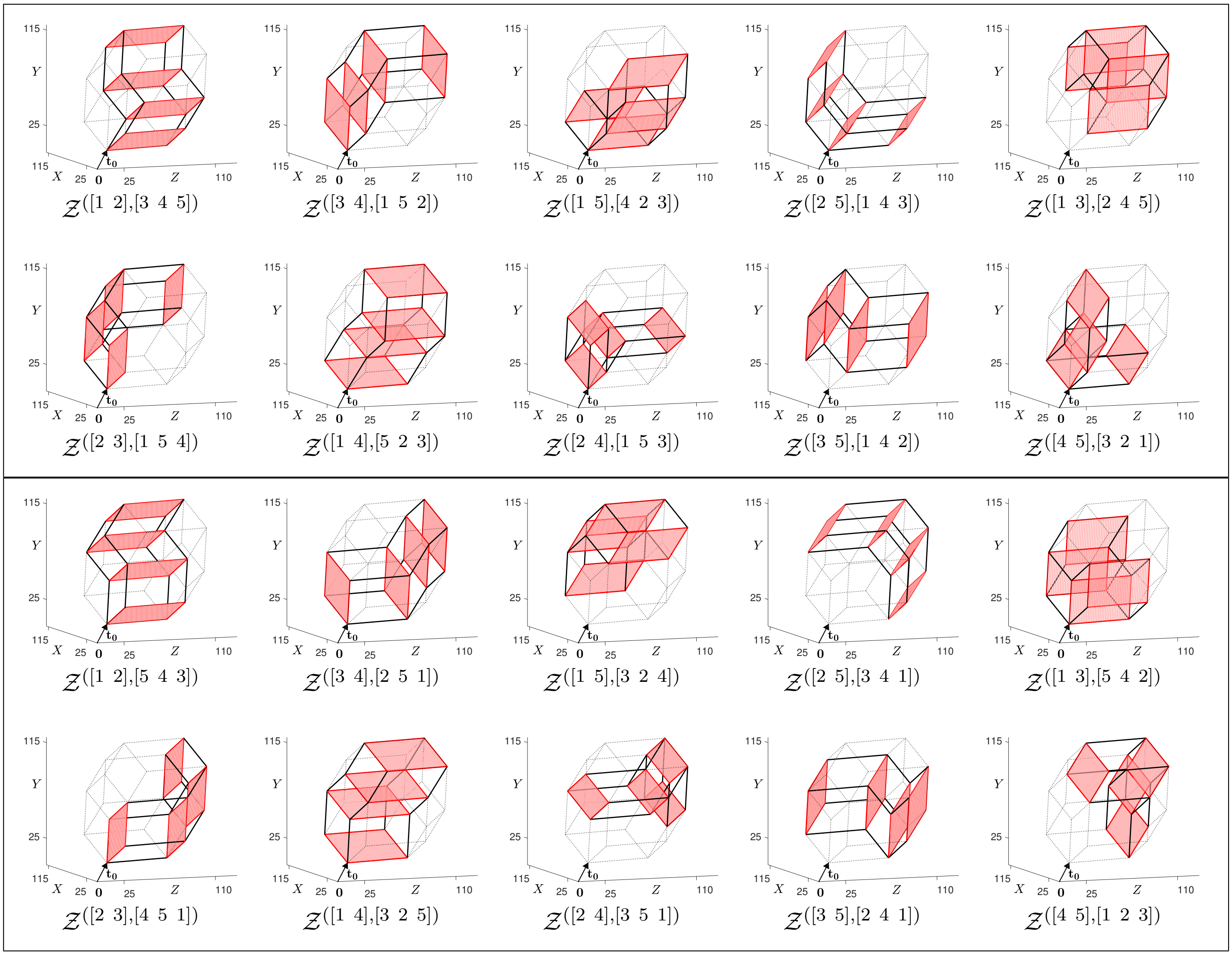
(a) $\mathcal{K}_1$ (top) and $\tilde{\mathcal{K}}_1$ (bottom) *(See page S.17 for caption)*



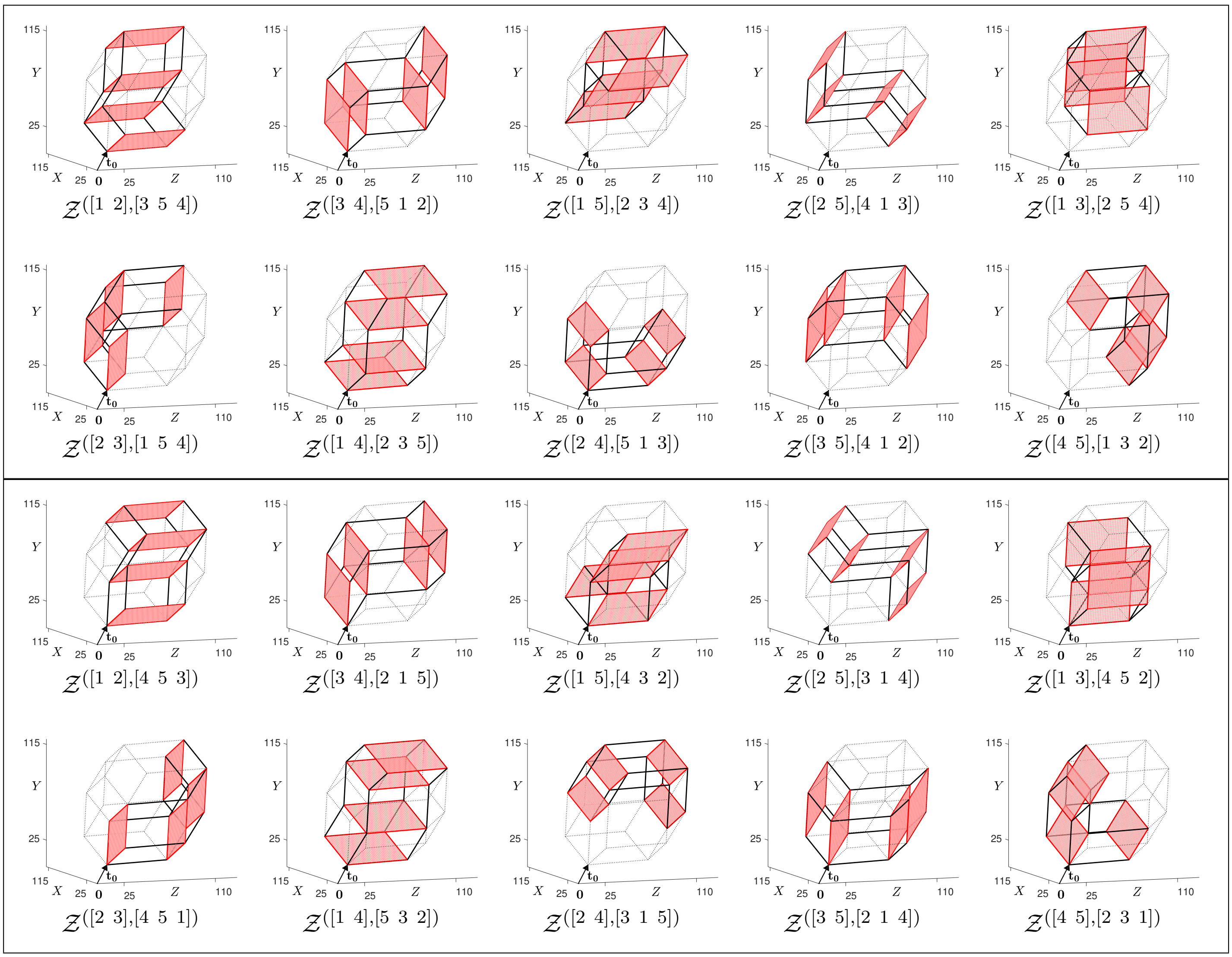
(b) $\mathcal{K}_2$ (top) and $\tilde{\mathcal{K}}_3$ (bottom) *(See page S.17 for caption)*



(c) $\mathcal{K}_3$ (top) and $\tilde{\mathcal{K}}_3$ (bottom) *(See page S.17 for caption)*

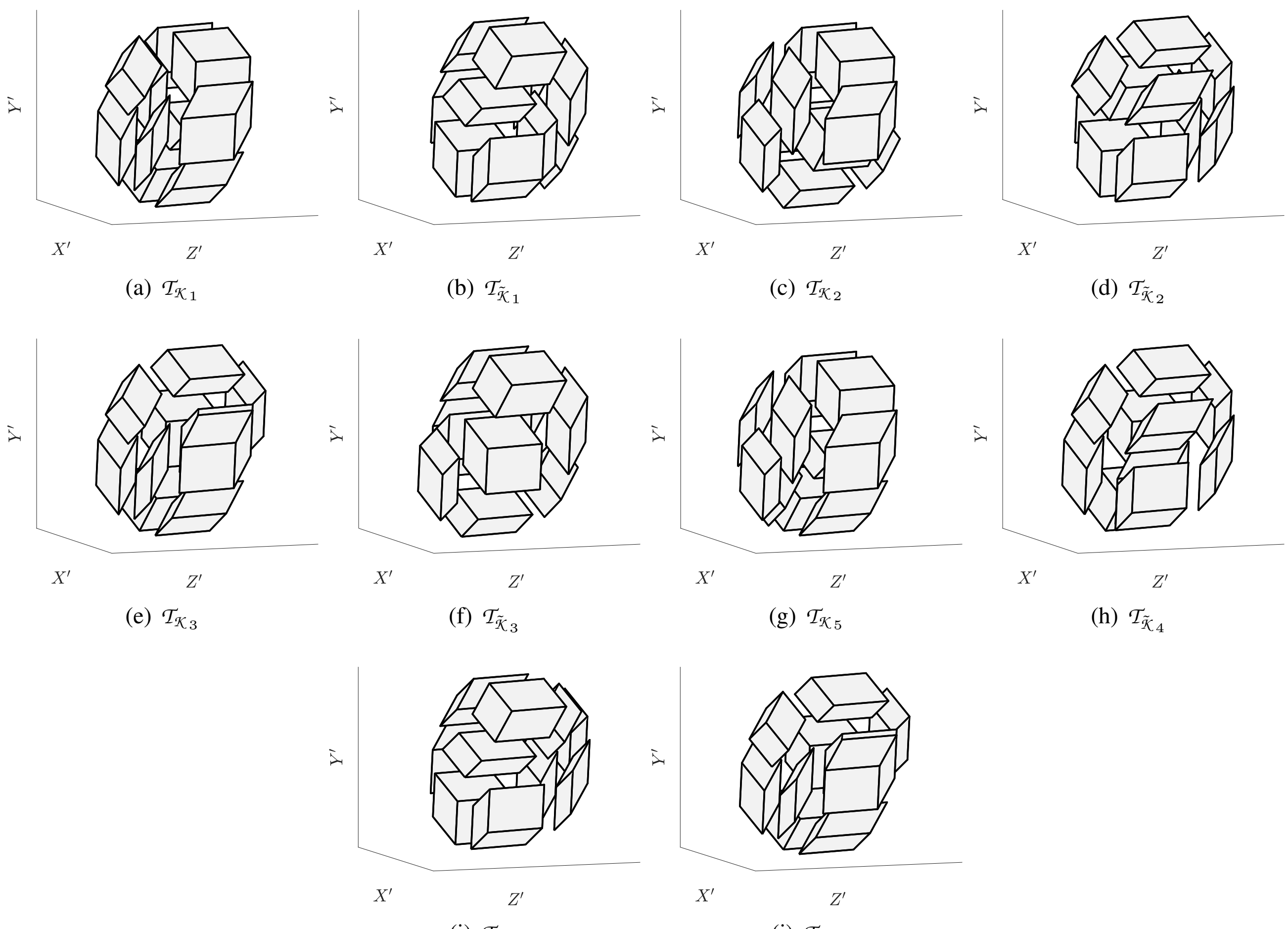


(d) $\mathcal{K}_4$ (top) and $\tilde{\mathcal{K}}_4$ (bottom) *(See page S.17 for caption)*



(e) $\mathcal{K}_5$ (top) and $\tilde{\mathcal{K}}_5$ (bottom)

Fig. S.4: The ten complete sets for primary system $\mathbf{P}_w^{(5)}$, shown in pairs in subfigures (a) to (e). In each subfigure, the top box delimits the facet spans of a complete set $\mathcal{K}$, and the bottom box delimits the corresponding mirror symmetric complete set $\tilde{\mathcal{K}}$.



(a) $\mathcal{T}_{\mathcal{K}_1}$ (b) $\mathcal{T}_{\tilde{\mathcal{K}}_1}$ (c) $\mathcal{T}_{\mathcal{K}_2}$ (d) $\mathcal{T}_{\tilde{\mathcal{K}}_2}$

(e) $\mathcal{T}_{\mathcal{K}_3}$ (f) $\mathcal{T}_{\tilde{\mathcal{K}}_3}$ (g) $\mathcal{T}_{\mathcal{K}_5}$ (h) $\mathcal{T}_{\tilde{\mathcal{K}}_4}$

(i) $\mathcal{T}_{\tilde{\mathcal{K}}_4}$ (j) $\mathcal{T}_{\mathcal{K}_5}$

Fig. S.5: The full set of ten tilings of the gamut for the $K = 5$ primary system $\mathbf{P}_w^{(5)}$. Each of the subfigures (a) to (j) illustrates a tiling that is labeled based on its corresponding complete set in Fig. S.2.

S.II. Enumeration of Tilings: Computational Time Requirements

Despite the efficiencies introduced in the proposed approach, the enumeration of tilings becomes particularly computationally demanding with increasing number of primaries K. In this section, we provide the computation time requirements for obtaining the results presented in Table 3 of the main paper, where we enumerated the tilings for primary systems where each primary expands the chromaticity gamut. All computations were performed in a Matlab environment running on a linux cluster [3] using computational nodes comprising of 8 CPU Intel Xeon E5-2695 processors operating at a 2.40GHz clock speed and having 40GB of memory.

We implemented the enumeration strategy in two stages, seeking to avoid the execution of many repetitive steps. The first stage uses Algorithm 2 to evaluates the s-compatibility between all the facet spans for the index-pairs in the sequence $\mathbb{J}_1, ..., \mathbb{J}_M$. The computation of the compatibility function was executed on a single computational node of the cluster using parallelization over the 8 processor cores for the node. The second stage, which enumerated complete sets using Alg. 1, was parallelized by enumerating each set $\mathfrak{A}_M^{(\mathbb{J}_1,\mathbb{I}_i)}$, $i \in \langle (K-2)!/2 \rangle$ independently on a computational node of the cluster. Figure S.6 plots the computation time requirements and the number of tilings as a function of K, showing both the total time required, as well as the time required for the first s-compatibility check stage (Algorithm 2) and second enumeration stage (Algorithm 1). Numerical values of the runtimes are tabulated in Table S.I. To highlight the speed-up that the proposed approach for efficient enumeration of tilings offers, we also include in Table S.II the computation time requirements for a "brute-force approach" that tests all sets of facet spans including one facet span for each $\mathbb{J} \in \mathfrak{C}^2(\langle K \rangle)$ to determine which of these are compatible and therefore constitute complete sets. By comparing the corresponding entries in Tables S.I and S.II, we see that the proposed efficient enumeration offers a very significant speed-up for $K = 7$. In fact, for $K = 8$, the brute force approach did not complete in the time allocated for the job.

Stage	K					
	4	5	6	7	8	9
Total Computational Time (s)	2.3	3.6	13.6	45.7	789.5	1359433.5
Strong Compatibility Function Computation (Alg. 2) (s)	2.1	2.2	8.3	31.3	531.7	39501.5
Enumeration of Complete Sets (Alg. 1) (s)	0.2	1.4	5.3	14.5	257.8	1319932.0

TABLE S.I: Runtime in seconds for enumerating the tilings ($\mathscr{N}(\mathfrak{K})$) in Table 3. The time requirements are listed for the two stages and overall. See text for the description of the stages and the hardware specifications.

	K			
	4	5	6	7
Total Computational Time (s)	1.41	3.88	42.83	1586.60

TABLE S.II: Runtime in seconds for enumerating the tilings ($\mathscr{N}(\mathfrak{K})$) in Table 3, using a "brute force" approach. See text for the hardware specifications.

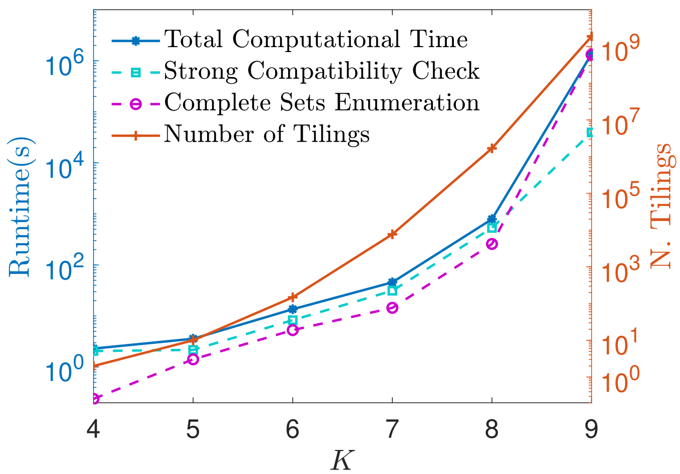


Fig. S.6: Runtime for enumerating the number of tilings ($\mathscr{N}(\mathfrak{K})$) in Table 3 for different number of primaries K: The solid blue line represents the total computational time, whereas the dashed cyan and magenta lines represent, respectively, the time required for the first stage evaluation of strong compatibility for all pairs of facet spans for the index-pairs in the sequence $\mathbb{J}_1, ..., \mathbb{J}_M$ using Algorithm 2 and the second stage enumeration of complete sets using Algorithm 1. For reference, the number of tilings are also plotted as the solid orange line (with the ordinate labels on the right hand side of the plot).

References

[1] C. E. Rodríguez-Pardo and G. Sharma, "Geometry of multiprimary display colors II: Metameric control sets and gamut tilings," submitted for review.

[2] G. Sharma and C. E. Rodríguez-Pardo, "Geometry of multiprimary display colors I: Gamut and color control," submitted for review.

[3] University of Rochester, Center for Integrated Research Computing, "BlueHive Cluster." [Online]. Available: http://www.circ.rochester.edu/wiki/index.php/BlueHive_Cluster