Supplementary Appendix

Simulation for the cross-sectional dependence test

We conduct simulation studies to examine the finite sample properties of our cross-sectional dependence test. DGP1 and DGP2 in Section 5.1 are employed to generate the data. Since this test is based on the spatial HAC estimator $\tilde{\Gamma}_T$ and does not employ AV-SHAC, we consider the MSE criterion and parametric plug-in method proposed by Kim and Sun (2011) to select the bandwidth. We first approximate the MSE of $\alpha' V^{-1} \tilde{\Gamma}_T V^{-1} \alpha$ with

$$AMSE = \frac{1}{d_n^{2q}} K_q^2 \left(\alpha' V^{-1} \Gamma_T^{(q)} V^{-1} \alpha \right)^2 + \frac{\ell_n}{n} 2 \bar{\mathcal{K}}_T \left(\alpha' V^{-1} \Gamma_T^H V^{-1} \alpha \right)^2, \tag{S.1}$$

where $\Gamma_T^{(q)} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n E\left[\eta_{iT}^H \left(\eta_{jT}^H \right)' \right] d_{ij}^q$. To avoid the effect of an unduly small value of d_n, our bandwidth selection is based on the following modified MSE criterion,

$$d_n^* = \max \left\{ \arg\min_{d_n} AMSE, \underline{d} \right\}, \tag{S.2}$$

where $\underline{d}$ is the prespecified minimum value of the bandwidth.

For the plug-in method, we employ the SAR(1) model

$$\eta_{a,iT}^H = \phi_a \mathcal{W} \eta_{a,iT}^H + u_{iT}, \tag{S.3}$$

where $\eta_{a,iT}^H$ is the ath component of η_{iT}^H and $u_{iT} \sim^{iid} (0,1)$, and we estimate ϕ_a by the QML method, which is given by

$$\hat{\phi}_a = \arg\max_{\phi} \log L \left(\tilde{\eta}_{a,iT} | \phi_a \right) \tag{S.4}$$

with

$$\log L \left(\tilde{\eta}_{a,iT} | \phi_a \right) = -\frac{n}{2} \log \left(\tilde{\eta}_{a,iT} - \phi_a \mathcal{W} \tilde{\eta}_{a,iT} \right)' \left(\tilde{\eta}_{a,iT} - \phi_a \mathcal{W} \tilde{\eta}_{a,iT} \right) - \log \left| \mathbb{I}_n - \phi_a \mathcal{W} \right| + const.$$

$\mathcal{W}$ is a contiguity matrix in which we treat units i and j as neighbors if $d_{ij} \leq 1$. For this matrix, row standardization is applied and all the diagonal elements are zero. See Kim and Sun (2011) for details of this plug-in method.

Table S1 below reports the empirical rejection probabilities (ERPs) of our cross-sectional dependence test. The table shows that the test works very well. The empirical sizes of $\mathcal{T}_T$ are always close to the nominal level $\alpha = 0.05$. In the presence of cross-sectional dependence, the ERP becomes larger than α and grows as the strength of the dependence and/or n, T increase. We set $\underline{d} = 20$ for DGP1, and $\underline{d} = 4$ for DGP2. Additional simulations (not reported here to save space) show that the test tends to lose power when $\underline{d}$ becomes larger.

Table S1: Empirical rejection probabilities of the cross-sectional dependence test ($\alpha = 0.05$)

DGP1				DGP2			
n	T	γ	ERP	n	T	ϱ	ERP
		0.0	0.045			0.0	0.065
100	50	0.3	0.303	100	50	0.1	0.308
		0.5	0.656			0.3	0.690
		0.7	0.925			0.5	0.790
		0.0	0.050			0.0	0.058
100	100	0.3	0.328	100	100	0.1	0.323
		0.5	0.686			0.3	0.710
		0.7	0.939			0.5	0.800
		0.0	0.060			0.0	0.060
100	200	0.3	0.331	100	200	0.1	0.333
		0.5	0.682			0.3	0.718
		0.7	0.944			0.5	0.817
		0.0	0.044			0.0	0.060
150	50	0.3	0.406	144	50	0.1	0.396
		0.5	0.821			0.3	0.783
		0.7	0.986			0.5	0.869
		0.0	0.050			0.0	0.057
150	100	0.3	0.413	144	100	0.1	0.421
		0.5	0.827			0.3	0.798
		0.7	0.990			0.5	0.890
		0.0	0.050			0.0	0.049
150	200	0.3	0.427	144	200	0.1	0.445
		0.5	0.845			0.3	0.813
		0.7	0.993			0.5	0.894

Proof of Proposition A1

Lemma S1 *Under Assumption T5,*

$$\frac{1}{n}\sum_{i_1=1}^{L_n}\sum_{i_2=1}^{M_n}\psi_b\left(\frac{i_1}{L_n},\frac{i_2}{M_n}\right)\lambda^H_{(i_1,i_2)}\left(\lambda^H_{(i_1,i_2)}\right)' \to^p \frac{1}{n}\sum_{i_1=1}^{L_n}\sum_{i_2=1}^{M_n}\psi_b\left(\frac{i}{L_n},\frac{i_2}{M_n}\right)\Sigma_{\Lambda^H}.$$

The proof of this lemma is included in the proof of Lemma 1 in Kim and Sun (2013).

Proof of Proposition A1 Let $\Gamma_t^0 = \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n K\left(\frac{d_{ij}}{d_n}\right)\lambda_i^H\left(\lambda_j^H\right)' e_{it}e_{jt}$ denote the infeasible spatial HAC estimator in time period t. Then

$$\begin{aligned}
\tilde{\Gamma}_t - \Gamma_t^0 &= \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n K\left(\frac{d_{ij}}{d_n}\right)\left[\tilde{\lambda}_i\tilde{\lambda}_j\tilde{e}_{it}\tilde{e}_{jt} - \lambda_i^H\lambda_j^H e_{it}e_{jt}\right] \\
&= \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n K\left(\frac{d_{ij}}{d_n}\right)\left[\left(\tilde{\lambda}_i\tilde{\lambda}_j - \lambda_i^H\lambda_j^H\right)\tilde{e}_{it}\tilde{e}_{jt} + \lambda_i^H\lambda_j^H\left(\tilde{e}_{it}\tilde{e}_{jt} - e_{it}e_{jt}\right)\right] \\
&:= C_1 + C_2,
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n K\left(\frac{d_{ij}}{d_n}\right)\left(\tilde{\lambda}_i\tilde{\lambda}_j - \lambda_i^H\lambda_j^H\right)\tilde{e}_{it}\tilde{e}_{jt}, \\
C_2 &= \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n K\left(\frac{d_{ij}}{d_n}\right)\lambda_i^H\lambda_j^H\left(\tilde{e}_{it}\tilde{e}_{jt} - e_{it}e_{jt}\right).
\end{aligned}$$

For C_1, we have

$$\begin{aligned}
C_1 = &\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n K\left(\frac{d_{ij}}{d_n}\right)\left[\left(\tilde{\lambda}_i - \lambda_i^H\right)\left(\tilde{\lambda}_j - \lambda_j^H\right)' + 2\left(\tilde{\lambda}_i - \lambda_i^H\right)\left(\lambda_j^H\right)'\right] e_{it}e_{jt} \\
&- \frac{2}{n}\sum_{i=1}^n\sum_{j=1}^n K\left(\frac{d_{ij}}{d_n}\right)\left[\left(\tilde{\lambda}_i - \lambda_i^H\right)\left(\tilde{\lambda}_j - \lambda_j^H\right) + 2\left(\tilde{\lambda}_i - \lambda_i^H\right)\lambda_j^H\right] e_{it} \\
&\times\left\{\tilde{F}_t\left(\tilde{\lambda}_j - \lambda_j^H\right) + \left(\tilde{F}_t - F_t^H\right)\lambda_j^H\right\} \\
&+ \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n K\left(\frac{d_{ij}}{d_n}\right)\left[\left(\tilde{\lambda}_i - \lambda_i^H\right)\left(\tilde{\lambda}_j - \lambda_j^H\right) + 2\left(\tilde{\lambda}_i - \lambda_i^H\right)\lambda_j^H\right] \\
&\times\left\{\tilde{F}_t\left(\tilde{\lambda}_i - \lambda_i^H\right) + \left(\tilde{F}_t - F_t^H\right)\lambda_i^H\right\}\left\{\tilde{F}_t\left(\tilde{\lambda}_j - \lambda_j^H\right) + \left(\tilde{F}_t - F_t^H\right)\lambda_j^H\right\} \\
&:= C_{11} + C_{12} + C_{13}.
\end{aligned}$$

For C_{11},

$$\begin{aligned}
C_{11} = &\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n K\left(\frac{d_{ij}}{d_n}\right)\left(\tilde{\lambda}_i - \lambda_i^H\right)\left(\tilde{\lambda}_j - \lambda_j^H\right) e_{it}e_{jt} \\
&+ \frac{2}{n}\sum_{i=1}^n\sum_{j=1}^n K\left(\frac{d_{ij}}{d_n}\right)\left(\tilde{\lambda}_i - \lambda_i^H\right)\lambda_j^H e_{it}e_{jt} \\
&:= C_{11}^{(1)} + C_{11}^{(2)}.
\end{aligned}$$

For $C_{11}^{(1)}$

$$\left|C_{11}^{(1)}\right| \leq \frac{\ell_n}{T}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s^H e_{is}\right)^4\right)^{1/4}\left(\frac{1}{n}\sum_{i=1}^{n}e_{it}^4\right)^{1/4}$$
$$\times\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\ell_n}\sum_{j\in\{d_{ij}\leq d_n\}}\left(\frac{1}{\sqrt{T}}\sum_{\tau=1}^{T}F_\tau^H e_{j\tau}\right)^2\right)\left(\frac{1}{\ell_n}\sum_{j\in\{d_{ij}\leq d_n\}}e_{jt}^2\right)\right)^{1/2}+o_P(1)$$
$$= O_P\left(\frac{\ell_n}{T}\right). \tag{S.5}$$

For $C_{11}^{(2)}$,

$$C_{11}^{(2)} = \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\left(\tilde{\lambda}_i-\lambda_i^H\right)\lambda_j^H e_{it}e_{jt}$$
$$= \frac{2}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)F_s^H e_{is}e_{it}\eta_{jt}^H + o_P(1)$$

and

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)F_s^H e_{is}e_{it}\eta_{jt}^H$$
$$= \frac{\sqrt{\ell_n}}{nT}\sum_{i=1}^{n}\sum_{s=1}^{T}\left(F_s^H e_{is}e_{it}-E\left(F_s^H e_{is}e_{it}\right)\right)\frac{1}{\sqrt{\ell_n}}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\eta_{jt}^H$$
$$+ \frac{\sqrt{\ell_n}}{nT}\sum_{i=1}^{n}\sum_{s=1}^{T}E\left(F_s^H e_{is}e_{it}\right)\frac{1}{\sqrt{\ell_n}}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\eta_{jt}^H$$
$$= c_1 + c_2.$$

For c_1,

$$|c_1| \leq \sqrt{\frac{\ell_n}{nT}}\left|\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{s=1}^{T}\left(F_s^H e_{is}e_{it}-E\left(F_s^H e_{is}e_{it}\right)\right)\right|\sup_a\left|\frac{1}{\sqrt{\ell_n}}\sum_{j=1}^{n}K\left(\frac{d_{aj}}{d_n}\right)\eta_{jt}^H\right|$$
$$= O_P\left(\sqrt{\frac{\ell_n}{nT}}\right)$$

under Assumption T6.

For c_2,

$$
\begin{aligned}
|c_2| &\le \frac{\sqrt{\ell_n}}{nT}\sum_{i=1}^{n}\sum_{s=1}^{T}\left|E\left(F_s^H\right)\right|\left|E\left(e_{is}e_{it}\right)\right|\left|\frac{1}{\sqrt{\ell_n}}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\eta_{jt}^H\right| \\
&\le \frac{\sqrt{\ell_n}}{T}\frac{M^2}{n}\sum_{i=1}^{n}\left|\frac{1}{\sqrt{\ell_n}}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\eta_{jt}^H\right| \\
&= O_P\left(\frac{\sqrt{\ell_n}}{T}\right).
\end{aligned}
$$

Thus we have

$$
C_{11} = o_P\left(1\right), \tag{S.6}
$$

as $\ell_n/T \to 0$ and $d_n, \ell_n, n, T \to \infty$.

C_{12} can be rewritten as

$$
\begin{aligned}
C_{12} = &-\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\left[\left(\tilde{\lambda}_i-\lambda_i^H\right)\left(\tilde{\lambda}_j-\lambda_j^H\right)^2 e_{it}\tilde{F}_t\right. \\
&+\left(\tilde{\lambda}_i-\lambda_i^H\right)\left(\tilde{\lambda}_j-\lambda_j^H\right)e_{it}\lambda_j^H\left(\tilde{F}_t-F_t^H\right)+2\left(\tilde{\lambda}_i-\lambda_i^H\right)\lambda_j^H e_{iT}\tilde{F}_t\left(\tilde{\lambda}_j-\lambda_j^H\right) \\
&\left.+2\left(\tilde{\lambda}_i-\lambda_i^H\right)\left(\lambda_j^H\right)^2 e_{it}\left(\tilde{F}_t-F_t^H\right)\right] \\
&:= C_{12}^{(1)}+C_{12}^{(2)}+C_{12}^{(3)}+C_{12}^{(4)}.
\end{aligned}
$$

For $C_{12}^{(1)}$,

$$
\begin{aligned}
\left|C_{12}^{(1)}\right| \le\ & 2\left|\tilde{F}_t\right|\frac{\ell_n}{T\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s e_{is}\right)^4\right)^{1/4}\left(\frac{1}{n}\sum_{i=1}^{n}e_{it}^4\right)^{1/4} \\
&\times\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\ell_n}\sum_{j\in\{d_{ij}\le d_n\}}\left(\frac{1}{\sqrt{T}}\sum_{\tau=1}^{T}F_\tau e_{j\tau}\right)^2\right)^2\right)^{1/2}+o_P\left(1\right) \\
&= o_P\left(1\right).
\end{aligned} \tag{S.7}
$$

For $C_{12}^{(2)}$,

$$
\begin{aligned}
\left|C_{12}^{(2)}\right| \le\ & 2\frac{\ell_n}{T\sqrt{n}}\left|\sqrt{n}\left(\tilde{F}_t-F_t^H\right)\right|\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s^H e_{is}\right)^4\right)^{1/4}\left(\frac{1}{n}\sum_{i=1}^{n}e_{it}^4\right)^{1/4} \\
&\times\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\ell_n}\sum_{j\in\{d_{ij}\le d_n\}}\left(\frac{1}{\sqrt{T}}\sum_{\tau=1}^{T}F_\tau^H e_{j\tau}\right)^2\right)\left(\frac{1}{\ell_n}\sum_{j\in\{d_{ij}\le d_n\}}\left(\lambda_j^H\right)^2\right)\right)^{1/2} \\
&= o_P\left(1\right)
\end{aligned} \tag{S.8}
$$

For $C_{12}^{(3)}$,

$$\begin{aligned}
&\left|C_{12}^{(3)}\right| \\
&\leq 4\left|\tilde{F}_t\right|\frac{\ell_n}{T}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s^H e_{is}\right)^4\right)^{1/4}\left(\frac{1}{n}\sum_{i=1}^{n}e_{it}^4\right)^{1/4} \\
&\times\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\ell_n}\sum_{j\in\{d_{ij}\leq d_n\}}\left(\frac{1}{\sqrt{T}}\sum_{\tau=1}^{T}F_\tau^H e_{j\tau}\right)^2\right)\left(\frac{1}{\ell_n}\sum_{j\in\{d_{ij}\leq d_n\}}\left(\lambda_j^H\right)^2\right)\right)^{1/2}+o_P(1) \\
&=o_P(1)\,. \qquad \text{(S.9)}
\end{aligned}$$

For $C_{12}^{(4)}$,

$$\begin{aligned}
\left|C_{12}^{(4)}\right| &\leq 4\frac{\ell_n}{\sqrt{nT}}\left|\sqrt{n}\left(\tilde{F}_t-F_t^H\right)\right|\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s^H e_{is}\right)^4\right)^{1/4}\left(\frac{1}{n}\sum_{i=1}^{n}e_{it}^4\right)^{1/4} \\
&\times\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\ell_n}\sum_{j\in\{d_{ij}\leq d_n\}}\left(\lambda_j^H\right)^2\right)^2\right)^{1/2}+o_P(1) \\
&=o_P(1)\,. \qquad \text{(S.10)}
\end{aligned}$$

From (S.7), (S.8), (S.9), and (S.10), we have

$$C_{12}=o_P(1)\,,$$

as $\ell_n/T, \ell_n/\sqrt{nT}\to 0$ and $d_n,\ell_n,n,T\to\infty$.

For C_{13},

$$\begin{aligned}
C_{13}&=\frac{\tilde{F}_t^2}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\left[\left(\tilde{\lambda}_i-\lambda_i^H\right)^2\left(\tilde{\lambda}_j-\lambda_j^H\right)^2+2\left(\tilde{\lambda}_i-\lambda_i^H\right)^2\left(\tilde{\lambda}_j-\lambda_j^H\right)\lambda_j^H\right] \\
&+\frac{2\tilde{F}_t}{n}\left(\tilde{F}_t-F_t^H\right)\sum_{i=1}^{n}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\left[\left(\tilde{\lambda}_i-\lambda_i^H\right)^2\left(\tilde{\lambda}_j-\lambda_j^H\right)\lambda_j^H+2\left(\tilde{\lambda}_i-\lambda_i^H\right)^2\left(\lambda_j^H\right)^2\right] \\
&+\frac{\left(\tilde{F}_t-F_t^H\right)^2}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\left[\left(\tilde{\lambda}_i-\lambda_i^H\right)\left(\tilde{\lambda}_j-\lambda_j^H\right)\lambda_i^H\lambda_j^H+2\left(\tilde{\lambda}_i-\lambda_i^H\right)\lambda_i^H\left(\lambda_j^H\right)^2\right] \\
&=A_{13}^{(1)}+A_{13}^{(2)}+A_{13}^{(3)}
\end{aligned}$$

For $C_{13}^{(1)}$,

$$\begin{aligned}
\left|C_{13}^{(1)}\right| &\leq \frac{\ell_n}{T^2}\tilde{F}_t^2\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s^He_{is}\right)^4\right)^{1/2} \\
&\times\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\ell_n}\sum_{j\in\{d_{ij}\leq d_n\}}\left(\frac{1}{\sqrt{T}}\sum_{\tau=1}^{T}F_\tau^He_{j\tau}\right)^2\right)^2\right)^{1/2} \\
&+2\frac{\ell_n}{T\sqrt{T}}\tilde{F}_t^2\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s^He_{is}\right)^4\right)^{1/2} \\
&\times\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\ell_n}\sum_{j\in\{d_{ij}\leq d_n\}}\left(\frac{1}{\sqrt{T}}\sum_{\tau=1}^{T}F_\tau^He_{j\tau}\right)^2\right)\left(\frac{1}{\ell_n}\sum_{j\in\{d_{ij}\leq d_n\}}\left(\lambda_j^H\right)^2\right)\right)^{1/2} \\
&= o_P(1)
\end{aligned}$$

Based on similar procedures, it is easy to show that

$$C_{13}^{(2)} = C_{13}^{(3)} = o_P(1).$$

Thus we have

$$C_1 = o_P(1). \tag{S.11}$$

For C_2,

$$\begin{aligned}
C_2 &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\lambda_i^H\lambda_j^H\left[\left(X_{it}-\tilde{\lambda}_i\tilde{F}_t\right)\left(X_{jt}-\tilde{\lambda}_j\tilde{F}_t\right)-e_{it}e_{jt}\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\lambda_i^H\lambda_j^H\left[\left(\tilde{\lambda}_i\tilde{F}_t-\lambda_i^HF_t^H\right)\left(\tilde{\lambda}_j\tilde{F}_t-\lambda_j^HF_t^H\right)-2e_{it}\left(\tilde{\lambda}_j\tilde{F}_t-\lambda_j^HF_t^H\right)\right] \\
&:= C_{21}+C_{22},
\end{aligned}$$

where

$$\begin{aligned}
C_{21} &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\lambda_i^H\lambda_j^H\left(\tilde{\lambda}_i\tilde{F}_t-\lambda_i^HF_t^H\right)\left(\tilde{\lambda}_j\tilde{F}_t-\lambda_j^HF_t^H\right), \\
C_{22} &= -\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\lambda_i^H\lambda_j^He_{it}\left(\tilde{\lambda}_j\tilde{F}_t-\lambda_j^HF_t^H\right).
\end{aligned}$$

C_{21} can be rewritten as

$$
\begin{aligned}
C_{21} &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} K\left(\frac{d_{ij}}{d_n}\right)\lambda_i^H\lambda_j^H\left[\left(\tilde{\lambda}_i-\lambda_i^H\right)\tilde{F}_t+\lambda_i^H\left(\tilde{F}_t-F_t^H\right)\right] \\
&\times\left[\left(\tilde{\lambda}_j-\lambda_j^H\right)\tilde{F}_t+\lambda_j^H\left(\tilde{F}_t-F_t^H\right)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} K\left(\frac{d_{ij}}{d_n}\right)\lambda_i^H\lambda_j^H\left[\left(\tilde{\lambda}_i-\lambda_i^H\right)\left(\tilde{\lambda}_j-\lambda_j^H\right)\tilde{F}_t^2\right. \\
&\left.+2\left(\tilde{\lambda}_i-\lambda_i^H\right)\lambda_j^H\tilde{F}_t\left(\tilde{F}_t-F_t^H\right)+\lambda_i^H\lambda_j^H\left(\tilde{F}_t-F_t^H\right)^2\right] \\
&= C_{21}^{(1)}+C_{21}^{(2)}+C_{21}^{(3)}.
\end{aligned}
$$

For $C_{21}^{(1)}$,

$$
\begin{aligned}
&\left|C_{21}^{(1)}\right| \\
&\leq \tilde{F}_t^2\frac{\ell_n}{T}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s^H e_{is}\right)^4\right)^{1/4}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\lambda_i^H\right)^4\right)^{1/4} \\
&\times\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\ell_n}\sum_{j\in\{d_{ij}\leq d_n\}}\left(\frac{1}{\sqrt{T}}\sum_{\tau=1}^{T}F_\tau^H e_{j\tau}\right)^2\right)\left(\frac{1}{\ell_n}\sum_{j\in\{d_{ij}\leq d_n\}}\left(\lambda_j^H\right)^2\right)\right)^{1/2}+o_P(1) \\
&= o_P(1).
\end{aligned}
$$

For $C_{21}^{(2)}$,

$$
\begin{aligned}
\left|C_{21}^{(2)}\right| &\leq \frac{2\ell_n}{\sqrt{nT}}\left|\tilde{F}_t\right|\left|\sqrt{n}\left(\tilde{F}_t-F_t^H\right)\right|\left(\frac{1}{n}\sum_{i=1}^{n}\left(\lambda_i^H\right)^4\right)^{1/4}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}F_s^H e_{is}\right)^2\right)^{1/4} \\
&\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\ell_n}\sum_{j\in\{d_{ij}\leq d_n\}}\left(\lambda_j^H\right)^2\right)^2\right)^{1/2}+o_P(1) \\
&= o_P(1),
\end{aligned}
$$

as $\ell_n/\sqrt{nT}\to 0$.

For $C_{21}^{(3)}$,

$$
\begin{aligned}
&\left|C_{21}^{(3)}\right| \\
&\leq \frac{\ell_n}{n}\left(\sqrt{n}\left(\tilde{F}_t-F_t^H\right)\right)^2\left(\frac{1}{n}\sum_{i=1}^{n}\left(\lambda_i^H\right)^4\right)^{1/2}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\ell_n}\sum_{j\in\{d_{ij}\leq d_n\}}\left(\lambda_j^H\right)^2\right)^2\right)^{1/2} \\
&= O_P\left(\frac{\ell_n}{n}\right).
\end{aligned}
$$

For C_{22},

$$
\begin{aligned}
C_{22} &= -\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\lambda_i^H e_{it}\lambda_j^H\left[\left(\tilde{\lambda}_j-\lambda_j^H\right)\tilde{F}_t+\lambda_j^H\left(\tilde{F}_t-F_t^H\right)\right] \\
&= C_{22}^{(1)}+C_{22}^{(2)}.
\end{aligned}
$$

For $C_{22}^{(1)}$,

$$
\begin{aligned}
& C_{22}^{(1)} \\
&\leq \frac{2\ell_n}{\sqrt{nT}}\left|\tilde{F}_t\right|\left(\frac{1}{n}\sum_{i=1}^{n}(\lambda_j^H)^2\left(\frac{1}{\sqrt{T}}\sum_{\tau=1}^{T}F_\tau^H e_{j\tau}\right)^2\right)^{1/2} \\
&\times\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{\ell_n}}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\eta_{it}^H\right)^2\right)^{1/2}+o_P(1) \\
&\leq \frac{2\ell_n}{\sqrt{nT}}\left|\tilde{F}_t\right|\left(\frac{1}{n}\sum_{i=1}^{n}(\lambda_j^H)^4\right)^{1/4}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{\tau=1}^{T}F_\tau^H e_{j\tau}\right)^4\right)^{1/4} \\
&\times\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{\ell_n}}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)\eta_{it}^H\right)^2\right)^{1/2}+o_P(1) \\
&= o_P(1),
\end{aligned}
$$

as $\ell_n/\sqrt{nT}\to 0$.

For $C_{22}^{(2)}$,

$$
\begin{aligned}
& \left|C_{22}^{(2)}\right| \\
&\leq \frac{2\ell_n}{n}\left|\sqrt{n}\left(\tilde{F}_t-F_t^H\right)\right|\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\lambda_i^H e_{it}\frac{1}{\ell_n}\sum_{j=1}^{n}K\left(\frac{d_{ij}}{d_n}\right)(\lambda_j^H)^2\right| \\
&\leq \frac{2\ell_n}{n}\left|\sqrt{n}\left(\tilde{F}_t-F_t^H\right)\right|\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\lambda_i^H e_{it}\right|\left|\sup_c\frac{1}{\ell_n}\sum_{j=1}^{n}K\left(\frac{d_{cj}}{d_n}\right)(\lambda_j^H)^2\right| \\
&= O_P\left(\frac{\ell_n}{n}\right).
\end{aligned}
$$

Under the rate condition $\ell_n/n\to b^o$ with $\ell_n/T=o(1)$, we have

$$
\tilde{\Gamma}_t=\tilde{\Gamma}_t^0+C_{21}^{(3)}+C_{22}^{(2)}+o_P(1).
$$

Using matrix notation, we have

$$\tilde{\Gamma}_t^0 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} K\left(\frac{d_{ij}}{d_n}\right)\eta_{it}^H \left(\eta_{jt}^H\right)',$$

$$C_1 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} K\left(\frac{d_{ij}}{d_n}\right)\lambda_i^H \left(\lambda_i^H\right)'\left(\tilde{F}_t - F_t^H\right)\left(\tilde{F}_t - F_t^H\right)'\lambda_j^H \left(\lambda_j^H\right)',$$

$$C_2 = -\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} K\left(\frac{d_{ij}}{d_n}\right)\left[\eta_{it}^H\left(\tilde{F}_t - F_t^H\right)'\lambda_j^H \left(\lambda_j^H\right)' + \lambda_i^H \left(\lambda_i^H\right)'\left(\tilde{F}_t - F_t^H\right)\left(\eta_{it}^H\right)'\right].$$

By the Fourier series representation, $\tilde{\Gamma}_t$ can be written as

$$\begin{aligned}
\tilde{\Gamma}_t &= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} K\left(\frac{d_{ij}}{d_n}\right)\left[\eta_{it}^H - \lambda_i^H \left(\lambda_i^H\right)'\left(\tilde{F}_t - F_t^H\right)\right]\left[\eta_{jt}^H - \lambda_j^H \left(\lambda_j^H\right)'\left(\tilde{F}_t - F_t^H\right)\right]' + o_P(1) \\
&= \sum_{\iota=1}^{\infty}\kappa_\iota\left(\frac{1}{\sqrt{n}}\sum_{i_1=1}^{L_n}\sum_{i_2=1}^{M_n}\psi_b\left(\frac{i_1}{L_n},\frac{i_2}{M_n}\right)\left[\eta_{(i_1,i_2),t}^H - \lambda_{(i_1,i_2)}^H\left(\lambda_{(i_1,i_2)}^H\right)'\left(\tilde{F}_t - F_t^H\right)\right]\right) \\
&\times \left(\frac{1}{\sqrt{n}}\sum_{j_1=1}^{L_n}\sum_{j_2=1}^{M_n}\psi_b\left(\frac{j}{L_n},\frac{j_2}{M_n}\right)\left[\eta_{(j_1,j_2),t}^H - \lambda_{(j_1,j_2)}^H\left(\lambda_{(j_1,j_2)}^H\right)'\left(\tilde{F}_t - F_t^H\right)\right]\right)' + o_P(1) \\
&= \sum_{\iota=1}^{\infty}\kappa_\iota \frac{1}{\sqrt{n}}\sum_{i_1=1}^{L_n}\sum_{i_2=1}^{M_n}\psi_b\left(\frac{i_1}{L_n},\frac{i_2}{M_n}\right)\left(\eta_{(i_1,i_2),t}^H - \lambda_{(i_1,i_2)}^H\left(\lambda_{(i_1,i_2)}^H\right)' V^{-1}\left(\frac{\tilde{F}'F^H}{T}\right)\frac{1}{n}\sum_{a_1=1}^{L_n}\sum_{a_2=1}^{M_n}\eta_{(a_1,a_2),t}^H\right) \\
&\times \frac{1}{\sqrt{n}}\sum_{j_1=1}^{L_n}\sum_{j_2=1}^{M_n}\psi_b\left(\frac{j_1}{L_n},\frac{j_2}{M_n}\right)\left(\eta_{(j_1,j_2),t}^H - \lambda_{(j_1,j_2)}^H\left(\lambda_{(j_1,j_2)}^H\right)' V^{-1}\left(\frac{\tilde{F}'F^H}{T}\right)\frac{1}{n}\sum_{a_1=1}^{L_n}\sum_{a_2=1}^{M_n}\eta_{(a_1,a_2),t}^H\right)' \\
&+ o_P(1) \qquad \text{(S.12)}
\end{aligned}$$

According to Lemma 1, we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i_1=1}^{L_n}\sum_{i_2=1}^{M_n}\psi_b\left(\frac{i_1}{L_n},\frac{i_2}{M_n}\right)\lambda_{(i_1,i_2)}^H\left(\lambda_{(i_1,i_2)}^H\right)' V^{-1}\left(\frac{\tilde{F}'F^H}{T}\right)\frac{1}{n}\sum_{a_1=1}^{L_n}\sum_{a_2=1}^{M_n}\eta_{(a_1,a_2),t}^H \\
&= \frac{1}{n}\sum_{i_1=1}^{L_n}\sum_{i_2=1}^{M_n}\psi_b\left(\frac{i_1}{L_n},\frac{i_2}{M_n}\right)\left(\Sigma_{\Lambda^H} + o_P(1)\right) V^{-1}\left(\frac{\tilde{F}'F^H}{T}\right)\frac{1}{\sqrt{n}}\sum_{a_1=1}^{L_n}\sum_{a_2=1}^{M_n}\eta_{(a_1,a_2),t}^H \\
&= \frac{1}{\sqrt{n}}\sum_{i_1=1}^{L_n}\sum_{i_2=1}^{M_n}\psi_b\left(\frac{i_1}{L_n},\frac{i_2}{M_n}\right)\frac{1}{n}\sum_{i_1=1}^{L_n}\sum_{i_2=1}^{M_n}\eta_{(i_{1t},i_{2t}),t}^H + o_P(1), \qquad \text{(S.13)}
\end{aligned}$$

where we use $\Sigma_{\Lambda^H} V^{-1} \left(\tilde{F}' F^H / T \right) \to^p \mathbb{I}_p$. Substituting this result in (S.12), we have

$$
\begin{aligned}
\tilde{\Gamma}_t &= \sum_{\iota=1}^{\infty} \kappa_\iota \frac{1}{\sqrt{n}} \sum_{i_1=1}^{L_n} \sum_{i_2=1}^{M_n} \psi_b \left(\frac{i_1}{L_n}, \frac{i_2}{M_n} \right) \left(\eta^H_{(i_1,i_2),t} - \frac{1}{n} \sum_{a_1=1}^{L_n} \sum_{a_2=1}^{M_n} \eta^H_{(a_1,a_2),t} \right) \\
&\quad \times \frac{1}{\sqrt{n}} \sum_{j_1=1}^{L_n} \sum_{j_2=1}^{M_n} \psi_b \left(\frac{j_1}{L_n}, \frac{j_2}{M_n} \right) \left(\eta^H_{(j_1,j_2),t} - \frac{1}{n} \sum_{a_1=1}^{L_n} \sum_{a_2=1}^{M_n} \eta^H_{(a_1,a_2),t} \right)' + o_P(1) \\
&= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} K \left(\frac{d_{ij}}{d_n} \right) \left(\eta^H_{it} - \frac{1}{n} \sum_{a=1}^{n} \eta^H_{at} \right) \left(\eta^H_{jt} - \frac{1}{n} \sum_{a=1}^{n} \eta^H_{at} \right)' + o_P(1) \\
&\overset{a}{\sim} J_t \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} K \left(\frac{d_{ij}}{d_n} \right) \left(\xi_i - \bar{\xi} \right) \left(\xi_j - \bar{\xi} \right)' J_t' + o_P(1), \qquad \text{(S.14)}
\end{aligned}
$$

where the asymptotic equivalence is a direct application of Proposition 2 in Kim and Sun (2013). This completes the proof.

References

[1] Kim, M. S., and Sun, Y. (2011) Spatial heteroskedasticity and autocorrelation consistent estimation of covariance matrix, *Journal of Econometrics*, 160, 349–371.

[2] Kim, M. S., and Sun, Y. (2013) Heteroskedasticity and spatiotemporal dependence robust inference for linear panel models with fixed effects, *Journal of Econometrics*, 177, 85–108.