Proof of Theorem 3.1

In the first part of the proof it will be shown that the lower and the upper extremes of the design space are points of the three–point equally weighted design maximizing the determinant. Then the explicit expression of the intermediate point will be found.

The transformation $x = \phi_C - \phi$ will be used to prove these results. Thus, the design space is $[x_L, x_U]$. A design of type

$$\xi = \left\{ \begin{array}{ccc} x_1 & x_2 & x_3 \\ 1/3 & 1/3 & 1/3 \end{array} \right\}, \tag{1}$$

is considered. We need to prove that the determinant of the FIM for this designs increases in x_3 and decreases in x_1. Let define $h = x_3 - x_1$, then the determinant is proportional to

$$\frac{\left(-x_1 \left(h + x_1 - x_2\right) \log\left(x_1\right) + \left(h + x_1\right)\left(x_1 - x_2\right) \log\left(h + x_1\right) + h x_2 \log\left(x_2\right)\right)^2}{x_1^2 \left(h + x_1\right)^2 x_2^2}.$$

It is enough to prove that the squared root is either an increasing or a decreasing function of h for fixed values of x_1, x_2,

$$\frac{-x_1(h + x_1 - x_2) \log\left(x_1\right) + (h + x_1)(x_1 - x_2) \log\left(h + x_1\right) + h x_2 \log\left(x_2\right)}{x_1(h + x_1)x_2}.$$

In particular, we have to prove that the derivative of the square root of the determinant of the information matrix for the design ξ with respect to h is less than or equal to 0 for any value of h, while $0 < x_1 < x_2$ are assumed constant,

$$\frac{\partial \sqrt{|M(\xi, \theta)|}}{\partial h} = \frac{(h + x_1)(x_1 - x_2) + x_1 x_2(-\log\left(x_1\right) + \log\left(x_2\right))}{x_1(h + x_1)^2 x_2}.$$

Thus, the denominator of this expression is always positive. We will see that the numerator is negative for all h. Consequently, the square root of the determinant will be a decreasing function on h and the determinant will be an increasing function on h.

After some algebra it may be seen that the numerator will be negative if the following inequality is satisfied,

$$\frac{x_1}{x_2} + \frac{x_1 \log\left(\frac{x_2}{x_1}\right)}{(h + x_1)} \quad < \quad 1.$$

If $h = 0$ (this would mean $x_1 = x_2 = x_3$), the preceding expression is equal to 1. And while increasing the value of h, with the constraint $x_1 < x_2$, it decreases. Thus, it is satisfied.

This proves that the greatest possible value of x_3 has to be in the design, that is $x_3 = x_U$. Using a symmetrical argument the lowest possible value has to be in the design as well. This is true for any middle point x_2, which will be optimized in what follows.

The optimal middle point, $x^\star$, has to satisfy the equation

$$\frac{\partial |M(\xi,\theta)|}{\partial x_2} \;=\; 0.$$

The following solutions of this equation are obtained,

$$\begin{aligned} x_1^\star &= \frac{x_L x_U(\log(x_L) - \log(x_U))}{x_L - x_U}\,, \\ x_2^\star &= \frac{x_L x_U(-\log(x_L) + \log(x_U))}{(x_L - x_U)W(g(x_L, x_U))}, \end{aligned}$$

where

$$g(x_L, x_U) = \frac{-x_L x_U \left(\frac{x_U^{x_U/x_L}}{x_L}\right)^{\frac{x_L}{x_L - x_U}} (\log(x_L) - \log(x_U))}{x_L - x_U}$$

and $W(g(x_L, x_U))$ is the Lambert W function that provides the main solution of the equation $z = W(z)e^{W(z)}$ for any complex number. It is well known that the Lambert function is double-valued on $(-1/e, 0)$. It will be proved that $g(x_L, x_U)$ remains if $z \geq -1/e$ for $0 < x_L < x_U$.

Let $\Delta = x_U - x_L$ the range of the interval, then

$$g(x_L, x_L + \Delta) = \frac{x_L(\Delta + x_L)\left(\frac{(\Delta+x_L)^{\frac{\Delta+x_L}{x_L}}}{x_L}\right)^{-\frac{x_L}{\Delta}} (\log(x_L) - \log(\Delta + x_L))}{\Delta}.$$

It is satisfied that

$$\begin{aligned} \lim_{\Delta\to 0} g(x_L, x_L + \Delta) &= \frac{-1}{e}, \\ \lim_{\Delta\to\infty} g(x_L, x_L + \Delta) &= 0. \end{aligned}$$

On the other hand the function $g(x_L, x_L + \Delta)$ will increase as Δ increases if

$$\frac{\partial g(x_L, x_L + \Delta)}{\partial \Delta} \;>\; 0,\ \Delta > 0,$$

which is true if

$$\left(\Delta - x_L \log\left(\frac{x_L + \Delta}{x_L}\right)\right)\left(\Delta \log\left(\frac{x_L + \Delta}{x_L}\right) - \Delta + x_L \log\left(\frac{x_L + \Delta}{x_L}\right)\right) > 0.$$

It is straightforward to see that both factors are positive. Thus, $g(x_L, x_U) \in (-1/e, 0)$, $0 < x_L < x_U$ and then $x_2^\star$ has actually two values, say $x_{21}^\star$ and $x_{22}^\star$, and the derivative of the determinant of the information matrix vanishes at three different points.

Let us prove that $x_L < x_1^* < x_U$. Applying the Mean Value Theorem to the logarithm it is easily checked that

$$x_L < \frac{x_L x_U (\log(x_L) - \log(x_U))}{x_L - x_U} < x_U.$$

This means there are either two maxima and one minimum or one maximum and two minima. But the determinant is always non-negative and it vanishes for both $x_{21}^\star = x_L$ and $x_{22}^\star = x_U$. Thus, they should be minima and $x_1^\star$ needs to be the maximum. This proves the theorem.