Proof of Theorem 1

First notice that because the blocks are constructed from fold-over pairs and center runs, it follows that the sum of the each of the columns of $\mathbf{L}$ within each block must be zero, so that $\mathbf{L}'\mathbf{Z} = \mathbf{0}$. We conclude that blocks are orthogonal to linear main effects.

We show that the design matrix ($\mathbf{L}$, $\mathbf{Q}$, $\mathbf{Z}$) constructed using the fixed effects blocking scheme has rank $2m + B$. Note first that since $\mathbf{Q}$ and $\mathbf{Z}$ are in the null space of $\mathbf{L}$ and since $\mathbf{L}$ has rank m, we only need to show that rank($\mathbf{Q}$, $\mathbf{Z}$) $= m + B$. Note further that each row of ($\mathbf{Q}$, $\mathbf{Z}$) is duplicated by the row corresponding to its mirror image in the design. For simplicity, we will delete one row from each duplicate pair giving the smaller matrix ($\mathbf{Q}^*$, $\mathbf{Z}^*$), which has the same rank as ($\mathbf{Q}$, $\mathbf{Z}$). The form of ($\mathbf{Q}^*$, $\mathbf{Z}^*$) is shown in Table 1. The partitioning of the matrix shown in Table 1 will be identified as:

$$(\mathbf{Q}^*, \mathbf{Z}^*) = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \\ \mathbf{E} & \mathbf{F} \end{pmatrix}$$

Consider the first $m + k$ rows.

$$\begin{aligned} \text{rank} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} &\geq \text{rank} \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \\ &= \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{D}) \\ &= m + \min(k, B) \end{aligned}$$

Suppose $\min(k, B) = B$. Then the first $m + k$ rows have rank $m + B$, and there is no need for any added center values. If $\min(k, B) = k < B$, we add $B - k$ center-value runs sequentially, each to a block that has not yet appeared in rows $m + k$ through the most recently added center-value run. When all required center-value runs have been added, every block is represented in the last $m + k$ through n rows. Write

$$\mathbf{C}^* = \begin{pmatrix} \mathbf{C} \\ \mathbf{E} \end{pmatrix} \quad \text{and} \quad \mathbf{D}^* = \begin{pmatrix} \mathbf{D} \\ \mathbf{F} \end{pmatrix}$$

Then

$$\text{rank} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \\ \mathbf{E} & \mathbf{F} \end{pmatrix} = \text{rank} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}^* & \mathbf{D}^* \end{pmatrix} \geq \text{rank} \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^* \end{pmatrix} = \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{D}^*) = m + B$$

Thus $\text{rank}(\mathbf{Q}, \mathbf{Z}) \geq m + b$. But $\text{rank}(\mathbf{Q}, \mathbf{Z}) \leq m + B$ since $(\mathbf{Q}, \mathbf{Z})$ has $m + B$ columns. We conclude $\text{rank}(\mathbf{Q}, \mathbf{Z}) = m + B$.

Table 1: Design matrix structure for the quadratic and block columns ($\mathbf{Q}$, $\mathbf{Z}$) of the design matrix.

Foldover Pair	$\mathbf{X}_1^2$	$\mathbf{X}_2^2$	$\mathbf{X}_3^2$	$\mathbf{X}_4^2$	$\mathbf{X}_5^2$	Starting Block Assignment $\mathbf{z}_1$	$\mathbf{z}_2$	$\mathbf{z}_3$
1	0	1	1	1	1	1	0	0
2	1	0	1	1	1	0	1	0
3	1	1	0	1	1	0	0	1
4	1	1	1	0	1	1	0	0
$m = 5$	1	1	1	1	0	0	1	0
$m + k = 6$	1	1	1	1	1	0	0	1
Center	0	0	0	0	0	1	0	0
runs	0	0	0	0	0	0	1	0