Supplementary Material for

Statistical Inference for Power Law Process with Competing Risks

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Proof of Theorem 1 in Section 2.2.1

Proof. From the expressions in Equation (4) in the main text, we see that $\hat{\mu} = \sum_{j=1}^{J} \hat{\mu}_j = n/t_n^{\hat{\beta}}$, which can be used to arrive at the identity (after rearranging some terms),

$$\sqrt{n}(\log n)^{-1}(\log \hat{\mu} - \log \mu) = \sqrt{n}(\log n)^{-1}(\log n - \log \mu - \beta \log t_n) - \sqrt{n}\left(\hat{\beta} - \beta\right)\frac{\log t_n}{\log n}. \quad (1)$$

From Equation (5) in the article, we note that $2\mu t_n^{\beta} \sim \chi^2_{2n}$, which readily yields the large sample result

$$\sqrt{n}\left(\frac{\mu t_n^{\beta}}{n} - 1\right) \xrightarrow{d} N(0,1). \quad (2)$$

By an application of delta method on (2),

$$\sqrt{n}(\log \mu + \beta \log t_n - \log n) \xrightarrow{d} N(0,1) \quad (3)$$

and hence the first term in (1) is $o_p(1)$. This yields $\log t_n / \log n \xrightarrow{p} 1/\beta$ and thus (1) can be reexpressed as

$$\sqrt{n}(\log n)^{-1}(\log \hat{\mu} - \log \mu) = -\beta^{-1} W_{3n} + o_p(1),$$

which, upon applying delta method, simplifies to the relation $W_{2n} = -\mu\beta^{-1}W_{3n} + o_p(1)$. Thus the limiting distributions (if they exist) of W_{2n}, W_{3n} are constant multiples of each other. Since $2n\beta/\hat{\beta} \sim \chi^2_{2(n-1)}$, the limiting distribution of W_{3n} is easily seen to be $N(0, \beta^2)$. Combining these, we get

$$(W_{2n}, W_{3n})' \xrightarrow{d} N_2\left(\mathbf{0}, \begin{bmatrix} \mu^2 & -\mu\beta \\ -\mu\beta & \beta^2 \end{bmatrix}\right).$$

We arrive at our result upon using the normal approximation to multinomial distribution. $\square$

Proof of Fact 1 in Section 2.2.1

Proof. We shall drop the argument $\boldsymbol{\phi}^0$ in the expressions of the likelihood and its derivatives for brevity. The log-likelihood function is written as

$$\log L = \sum_{j=1}^{J} n_j \log \mu_j - t_n^{\beta} \sum_{j=1}^{J} \mu_j + (\beta - 1)\sum_{i=1}^{n} \log t_i + n \log \beta,$$

which, upon differentiating w.r.t. the parameters yields the score vector $(l_{1n}, \ldots, l_{J+1,n})'$ as

$$\begin{aligned} l_{jn} &= \frac{\partial \log L}{\partial \mu_j} = \frac{n_j}{\mu_j} - t_n^{\beta}, \;\; j = 1, \ldots, J, \\ l_{J+1,n} &= \frac{\partial \log l}{\partial \beta} = -t_n^{\beta} \log t_n \sum_{j=1}^{J} \mu_j + \sum_{i=1}^{n} \log t_i + \frac{n}{\beta}. \end{aligned}$$

Denote by Z_n the random variable

$$Z_n \equiv n^{1/2} \left[\frac{t_n^{\beta} \sum_{j=1}^{J} \mu_j}{n} - 1 \right].$$

Recalling that $\rho_j = \mu_j / \sum_{j=1}^{J} \mu_j$, and $\hat{\rho}_j = n_j/n$, we have

$$\begin{aligned} n^{-1/2} l_{jn} &= n^{1/2} \left(\frac{\hat{\rho}_j}{\mu_j} - \frac{t_n^{\beta}}{n} \right) \\ &= \frac{n^{1/2}}{\sum_{j=1}^{J} \mu_j} \left(\frac{\hat{\rho}_j}{\rho_j} - \frac{t_n^{\beta} \sum_{j=1}^{J} \mu_j}{n} \right) \\ &= \frac{1}{\sum_{j=1}^{J} \mu_j} \left[\rho_j^{-1} n^{1/2} (\hat{\rho}_j - \rho_j) - Z_n \right], \end{aligned} \tag{4}$$

and

$$\begin{aligned} n^{-1/2} (\log n)^{-1} l_{J+1,n} &= -n^{1/2} \left[\frac{\log t_n}{\log n} \left\{ \frac{t_n^{\beta} \sum_{j=1}^{J} \mu_j}{n} - 1 \right\} \right] - n^{1/2} (\log n)^{-1} \left[\hat{\beta}^{-1} - \beta^{-1} \right] \\ &= -\beta^{-1} Z_n + \beta^{-2} (\log n)^{-1} W_{3n} + o_p(1), \end{aligned} \tag{5}$$

where W_{3n} is as in Theorem 1. The second equality uses the fact that $(\log t_n / \log n) \xrightarrow{p} 1/\beta$. The second term in (5) is $o_p(1)$ since W_{3n} has an asymptotic normal distribution. Further, for each j, $\hat{\rho}_j$ and Z_n are independent, asymptotically normal in view of Theorem 1 and Equation (4) in the main article. **Fact 1** is then established from (4) and (5). $\square$

Proof of Fact 2 in Section 2.2.1

Proof. It suffices to prove the convergence of the (scaled) second derivative matrix of the log-likelihood at the true parameter $\boldsymbol{\phi}^{\mathbf{0}}$. The uniform convergence in a shrinking neighborhood around $\boldsymbol{\phi}^{\mathbf{0}}$ follows by continuity. The components of the $(J+1) \times (J+1)$ second-derivative matrix of the log-likelihood are given for $j = 1, \ldots, J$, by:

$$
\begin{aligned}
a_{jj} &= \frac{\partial^2 \log L}{\partial \mu_j^2} = -\frac{n_j}{\mu_j^2}, \quad a_{jj'} = \frac{\partial^2 \log L}{\partial \mu_j \mu_{j'}} = 0, \quad j' = 1, \ldots, J; j' \neq j, \\
a_{J+1,J+1} &= \frac{\partial^2 \log L}{\partial \beta^2} = -t_n^{\beta} (\log t_n)^2 \sum_{j=1}^{J} \mu_j - \frac{n}{\beta^2} \\
a_{J+1,j} &= \frac{\partial^2 \log L}{\partial \beta \partial \mu_j} = -t_n^{\beta} \log t_n.
\end{aligned}
$$

Let us define

$$
\boldsymbol{C_n} = n^{-1} \begin{bmatrix}
a_{11} & \cdot\ \cdot\ \cdot & a_{1J} & a_{1,J+1}/(\log n) \\
\cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\
\cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\
a_{1J} & \cdot\ \cdot\ \cdot & a_{JJ} & a_{J,J+1}/(\log n) \\
a_{1,J+1}/(\log n) & \cdot\ \cdot\ \cdot & a_{J,J+1}/(\log n) & a_{J+1,J+1}/(\log n)^2.
\end{bmatrix}
$$

The Taylor series expansion in Equation (9) of the main article follows. That $\boldsymbol{C_n}$ converges in probability to a non-stochastic matrix follows by noting for $j = 1, \ldots, J$

$$
\begin{aligned}
n^{-1} a_{jj} &\xrightarrow{p} -\rho_j \mu_j^{-2}, \\
n^{-1} (\log n)^{-2} a_{J+1,J+1} &= -\left(\frac{\log t_n}{\log n}\right)^2 \frac{t_n^{\beta} \sum_{j=1}^{J} \mu_j}{n} \xrightarrow{p} -\beta^{-2}, \\
(n \log n)^{-1} a_{J+1,j} &\xrightarrow{p} -(\beta \mu)^{-1}.
\end{aligned}
$$

□

We require a lemma to prepare the groundwork for proving Theorem 2.

Lemma 1. *Let us assume without loss of generality,* $\beta_1 = \max_j \beta_j$. *Then* $n \to \infty$,
(a) $n_1/n \overset{a.s.}{\to} 1$, (b) $n_j / \left[\mu_j (n/\mu_1)^{\beta_j/\beta_1}\right] \overset{a.s.}{\to} 1$ *for* $j = 2, 3, \ldots, J$,
(c) $\log n_1 / \log n \overset{a.s.}{\to} 1$, (d) $\log n_j / \log(\mu_j (n/\mu_1)^{\beta_j/\beta_1}) \overset{a.s.}{\to} 1$ *for* $j = 2, 3, \ldots, J$.

Proof. Since overall system failures follow NHPP with cumulative intensity function $\Lambda(t) = \sum_{j=1}^{J} \mu_j t^{\beta_j}$ with $\lim_{t \to \infty} \Lambda(t) = \infty$, we have, from properties of a NHPP,

$$
n \Big/ \left(\sum_{j=1}^{J} \mu_j t_n^{\beta_j} \right) \overset{a.s.}{\to} 1 \text{ and } t_n \overset{a.s.}{\to} \infty, \text{ as } n \to \infty. \tag{6}
$$

With the second result in (6), we also get

$$n_j/\mu_j t_n^{\beta_j} \overset{a.s.}{\to} 1, \text{ as } n \to \infty. \tag{7}$$

Since $\beta_1 = \max_j \beta_j$, $\lim_{t\to\infty} \mu_1 t^{\beta_1}/(\sum_{j=1}^J \mu_j t^{\beta_j}) = 1$. Hence,

$$\frac{n_1}{n} = \frac{n_1}{\mu_1 t_n^{\beta_1}} \times \frac{\mu_1 t_n^{\beta_1}}{\sum_{j=1}^J \mu_j t^{\beta_j}} \times \frac{\sum_{j=1}^J \mu_j t^{\beta_j}}{n} \overset{a.s.}{\to} 1 \text{ and } \frac{\mu_1 t_n^{\beta_1}}{n} \overset{a.s.}{\to} 1, \text{ as } n \to \infty. \tag{8}$$

By applying (7) and (8), it follows that

$$\frac{n_j}{\mu_j (n/\mu_1)^{\beta_j/\beta_1}} = \frac{n_j}{\mu_j t_n^{\beta_j}} \times \left(\frac{\mu_1 t_n^{\beta_1}}{n}\right)^{\beta_j/\beta_1} \overset{a.s.}{\to} 1, \text{ as } n \to \infty. \tag{9}$$

Parts (c) and (d) of Lemma 1 follow in a similar manner. □

Proof of Theorem 2

Proof. Let

$$S_{jn} = \beta_j \sum_{i=1}^n \log(t_n/t_i) I(\delta_i = j) = n_j \beta_j/\hat{\beta}_j, \text{ and } \quad \upsilon_{jn} = (S_{jn} - n_j)/\sqrt{n_j}.$$

We need to show that $(\upsilon_{1n}, \upsilon_{2n}, \ldots, \upsilon_{Jn})' \overset{d}{\to} N_J(\mathbf{0}, \boldsymbol{I_J})$, as $n \to \infty$ with $\boldsymbol{I_J}$ denotes $J \times J$ identity matrix. This can be shown by argument on moment generating functions of $\upsilon_{1n}, \upsilon_{2n}, \ldots, \upsilon_{Jn}$. First, we assume $\delta_n = 1$. Using Equation (13) from the article, we have $S_{1n}|n_1 \sim Gamma(n_1, 1)$. Hence,

$$\mathbb{E}[e^{\upsilon_{1n} t}] = \mathbb{E}_{n_1}\left[\mathbb{E}\left(e^{\frac{1}{\sqrt{n_1}}(S_{1n} - n_1)} \mid n_1\right)\right] = \mathbb{E}_{n_1}\left[e^{\sqrt{n_1} t}\left(\frac{1}{1 - \frac{t}{\sqrt{n_1}}}\right)^{n_1}\right].$$

By applying Taylor series expansion, we get

$$\log\left[e^{\sqrt{n_1} t}\left(\frac{1}{1 - \frac{t}{\sqrt{n_1}}}\right)^{n_1}\right] = t^2/2 + o(n_1^{-1/2}).$$

Using Lemma 1(a), it follows that $\mathbb{E}[e^{\upsilon_{1n} t}] = \exp\left[t^2/2 + o_p\left(n^{-1/2}\right)\right]$. The result remains the same if $\delta_n \neq 1$. Thus, $\upsilon_{1n} \to^d N(0, 1)$. Apply the same strategy in conjunction with Lemma 1(c) to conclude $\upsilon_{jn} \to^d N(0, 1)$ for $j = 2, 3, \ldots, J$. Furthermore, with given t_n, $S_{1n}, S_{2n}, \ldots,$ and S_{Jn} are mutually independent because they are functions of failure times corresponding to J independent failure modes. Additionally, by Lemma 1(a), $t_n/(n/\mu_1)^{1/\beta_1} \overset{a.s.}{\to} 1$ as $n \to \infty$, implying that $S_{1n}, S_{2n}, \dots, S_{Jn}$ are asymptotically mutually independent unconditional on t_n. Hence, $(\upsilon_{1n}, \upsilon_{2n}, \dots, \upsilon_{Jn})' \xrightarrow{d} N_J(\mathbf{0}, \boldsymbol{I_J})$.

For $j = 2, \dots, J$, let $U^*_{jn} = \sqrt{g_j(n)}(\log g_j(n))^{-1}(\hat{\mu}_j - \mu_j)$ and $V^*_{jn} = \sqrt{g_j(n)}(\hat{\beta}_j - \beta_j)$ with $g_j(n) = \mu_j(n/\mu_1)^{\beta_j/\beta_1}$. Note that $\hat{\beta}_j = n_j\beta_j/S_{jn}$ for $j = 1, 2, \dots, J$. By Lemma 1, it follows that

$$V_{1n} = -\beta_1\upsilon_{1n} + o_p(1) \text{ and } V^*_{jn} = -\beta_j\upsilon_{jn} + o_p(1). \tag{10}$$

Note that

$$\log\hat{\mu}_1 = (\log n_1)\left(1 - \frac{\hat{\beta}_1}{\beta_1}\right) - \frac{\hat{\beta}_1}{\beta_1}\log\left(\frac{\mu_1 t_n^{\beta_1}}{n_1}\right) + \frac{\hat{\beta}_1}{\beta_1}\log\mu_1. \tag{11}$$

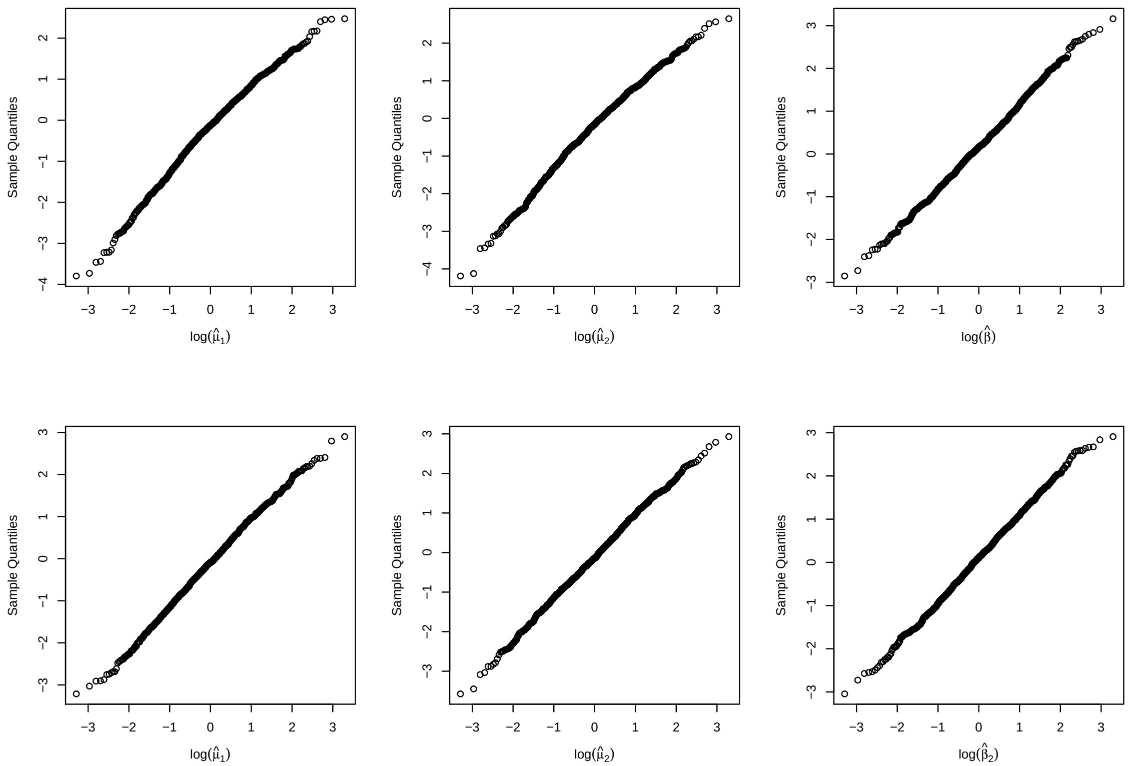
Algebraic manipulation of the terms in (11) and a subsequent application of the results from Lemma 1 yield

$$\sqrt{n}(\log n)^{-1}(\log\hat{\mu}_1 - \log\mu_1) = \upsilon_{1n} + o_p(1).$$

By Taylor series expansion, we have $U_{1n} = \mu_1\upsilon_{1n} + o_p(1)$. Similarly, $U^*_{jn} = \mu_j\upsilon_{jn} + o_p(1)$, and one finally arrives at

$$U_{jn} = \sqrt{\frac{\mu_1^{\beta_j/\beta_1}}{\mu_j}} \cdot \frac{\beta_j}{\beta_1}U^*_{jn} + o_p(1) \text{ and } V_{jn} = \sqrt{\frac{\mu_1^{\beta_j/\beta_1}}{\mu_j}}V^*_{jn}. \tag{12}$$

The result follows in view of (10) and (12). $\square$



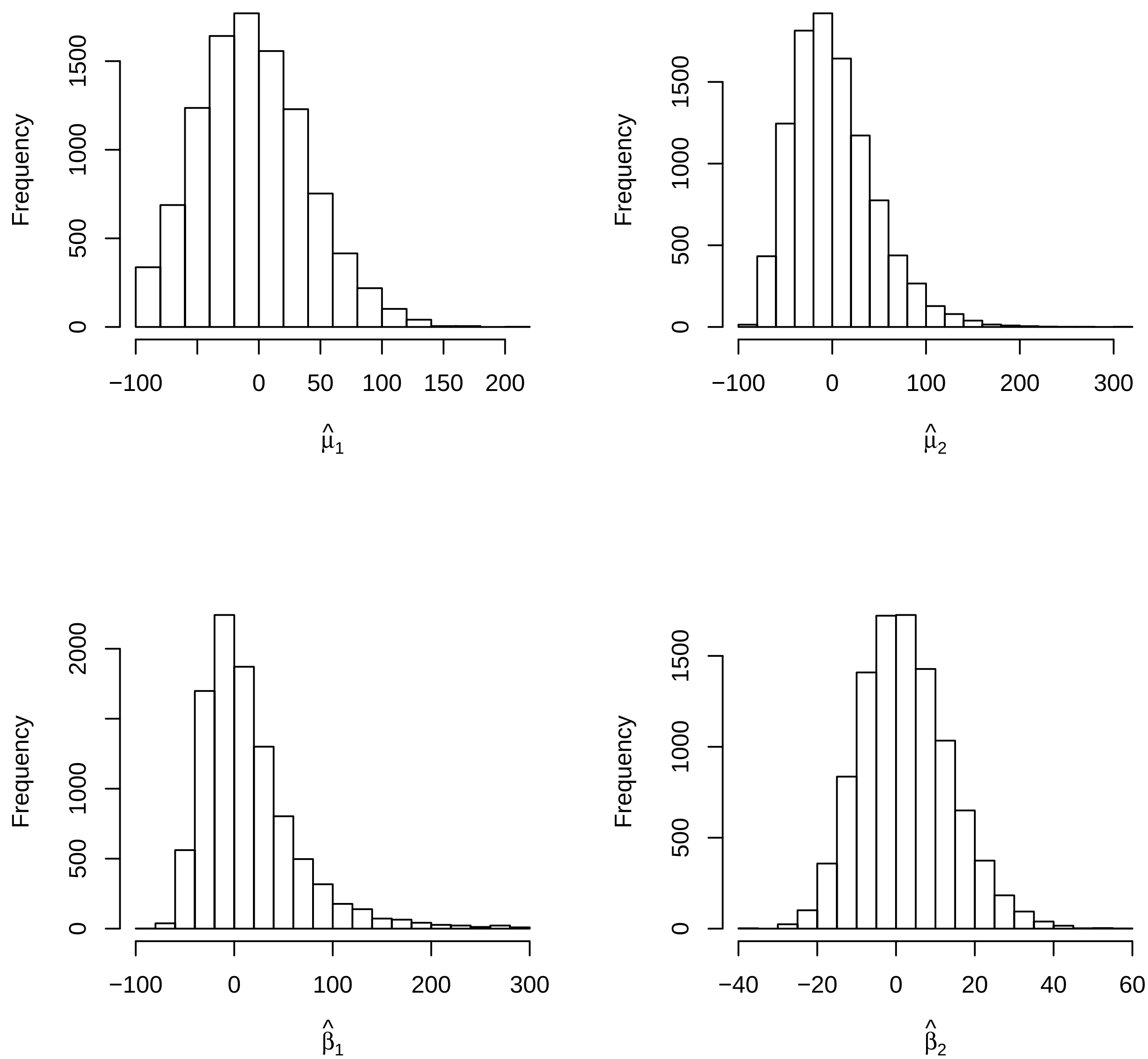
Supplementary Figure 1: Q-Q plots of the scaled and centered log-MLEs for true $\mu_1 = .65, \mu_2 = .35, \beta = .5$ with sample sizes of 100 (top row) and 500 (bottom row)

Supplementary Table 1: Automotive Warranty Data

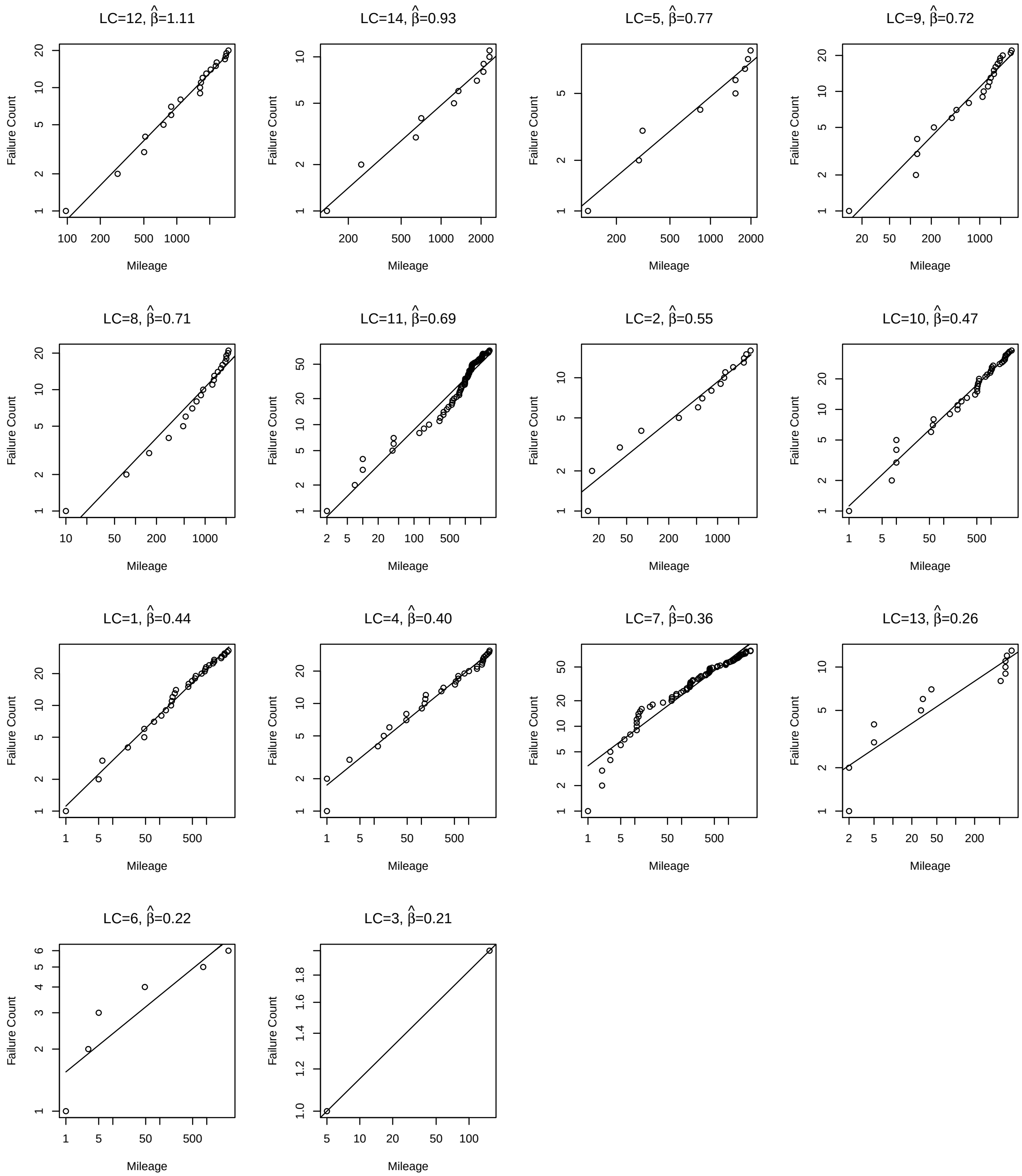
Mileage	Mode	Mileage	Mode	Mileage	Mode	Mileage	Mode	Mileage	Mode	Mileage	Mode	Mileage	Mode	Mileage	Mode
1.21	1	30.00	1	174.00	1	514.00	2	879.00	3	1347.00	3	1938.00	3	2574.00	2
1.48	2	38.00	3	175.00	2	515.00	3	887.00	1	1349.00	1	1951.00	3	2606.00	3
1.53	2	40.05	2	182.00	2	519.00	2	888.26	3	1350.00	3	1953.00	2	2719.00	2
1.66	2	40.35	3	189.00	2	520.00	1	888.91	3	1355.00	3	1957.00	3	2720.00	2
1.76	1	40.72	1	196.07	2	525.62	3	914.00	2	1370.00	2	1961.03	2	2750.00	3
1.81	2	40.92	3	196.21	2	526.00	2	920.00	1	1375.00	3	1961.84	2	2755.00	2
2.23	1	41.00	1	196.81	3	532.00	2	938.00	3	1380.00	3	1979.00	3	2767.00	2
2.30	3	47.10	2	210.00	2	535.00	2	944.00	2	1393.00	1	1987.00	3	2800.00	3
2.41	1	47.66	2	218.00	3	543.00	3	960.00	2	1440.00	3	2016.00	2	2801.35	3
2.49	1	48.20	1	223.85	1	558.10	2	975.00	3	1447.00	3	2017.05	3	2801.68	3
2.75	1	48.41	2	223.87	2	558.79	2	979.00	2	1457.10	2	2017.17	3	2840.00	3
3.23	1	48.93	2	233.00	1	559.00	3	984.00	3	1457.54	2	2017.96	2	2850.00	3
3.61	2	54.00	2	237.00	2	562.00	3	985.71	3	1488.00	1	2026.01	2	2868.00	3
3.64	1	59.00	2	248.00	1	567.00	2	985.84	3	1499.00	2	2026.72	3	2883.00	3
3.76	1	61.59	2	250.00	3	569.00	1	996.11	2	1507.00	2	2028.00	2	2898.00	1
5.09	2	61.65	1	259.00	1	590.55	1	996.29	3	1512.00	3	2040.00	3	2921.00	3
5.27	1	62.37	1	265.00	2	590.98	2	996.98	3	1521.30	2	2055.00	2	2939.00	2
5.34	1	62.71	1	279.00	2	604.09	3	1015.00	2	1521.72	3	2083.00	3	2959.63	1
5.37	1	74.00	3	288.00	3	604.39	2	1037.00	1	1535.55	3	2090.00	1	2959.85	2
5.47	1	76.00	2	293.00	2	604.54	2	1041.22	2	1535.83	3	2091.00	3	2964.00	3
5.51	1	77.20	1	294.00	3	604.82	2	1041.48	2	1599.00	1	2097.00	3	2966.00	3
6.23	1	77.22	1	300.00	3	620.45	1	1081.00	3	1603.22	3	2104.00	3	2972.00	1
6.39	2	81.00	2	309.00	2	620.80	1	1086.00	3	1603.64	3	2105.00	2		
7.00	3	97.00	3	313.45	3	620.82	1	1097.00	3	1622.00	1	2126.00	2		
8.28	2	98.00	1	313.65	3	645.00	3	1104.00	2	1631.07	3	2127.00	3		
8.62	1	102.00	2	325.00	3	654.26	3	1107.00	2	1631.61	3	2144.07	3		
10.21	3	109.27	1	329.00	1	654.27	1	1116.00	3	1633.00	3	2144.28	3		
10.26	2	109.55	2	331.00	1	665.00	1	1135.00	1	1662.00	3	2144.30	3		
10.33	2	117.00	2	367.00	1	677.00	3	1137.00	2	1668.00	2	2164.00	3		
10.74	3	120.00	3	374.00	3	696.00	3	1140.69	3	1692.00	3	2168.00	3		
10.78	2	121.00	2	378.00	3	709.00	3	1140.78	3	1693.00	3	2176.00	3		
10.82	3	123.00	3	386.00	1	753.00	3	1140.91	3	1696.09	2	2198.00	2		
11.19	1	124.00	2	395.00	3	756.00	3	1181.00	3	1696.24	1	2205.00	3		
11.46	1	125.06	3	396.44	1	760.03	3	1200.04	3	1696.37	3	2270.00	3		
11.55	1	125.31	3	396.76	1	760.65	3	1200.57	3	1722.00	3	2296.42	1		
11.64	1	127.00	3	397.59	1	760.70	2	1200.69	3	1731.00	1	2296.73	1		
12.01	2	129.43	1	397.99	1	776.00	3	1240.70	1	1733.00	1	2313.12	3		
12.08	1	129.99	1	400.00	1	777.00	1	1240.77	2	1779.50	3	2314.00	3		
12.76	1	135.00	2	408.00	2	787.00	2	1250.00	3	1779.62	3	2316.00	3		
13.15	3	136.00	2	416.00	2	801.00	3	1270.00	3	1805.00	3	2324.00	2		
13.74	1	138.00	3	436.00	3	816.61	2	1287.00	2	1812.00	3	2329.00	2		
14.18	1	149.07	1	447.00	1	816.65	2	1301.00	3	1838.39	2	2338.00	1		
14.59	2	149.07	1	461.68	2	827.49	3	1303.00	1	1838.53	3	2368.05	1		
16.42	2	153.08	1	461.97	3	827.55	3	1315.00	3	1842.00	1	2368.88	2		
16.43	2	153.26	1	470.00	3	827.82	2	1322.08	3	1854.00	3	2387.00	2		
21.19	2	154.00	1	484.00	3	840.17	3	1322.35	3	1861.54	3	2411.00	1		
21.31	1	155.00	1	490.00	2	840.47	1	1330.00	3	1861.95	3	2420.00	2		
21.56	2	156.00	3	503.00	3	870.00	3	1333.76	3	1902.00	3	2424.00	2		
24.00	1	157.00	3	509.30	2	878.20	1	1334.00	3	1911.00	1	2446.00	2		
28.00	1	170.00	1	509.49	2	878.56	1	1343.00	3	1912.00	2	2524.00	2		

Supplementary Table 2: MLEs from ten generations of data with random tie breaking

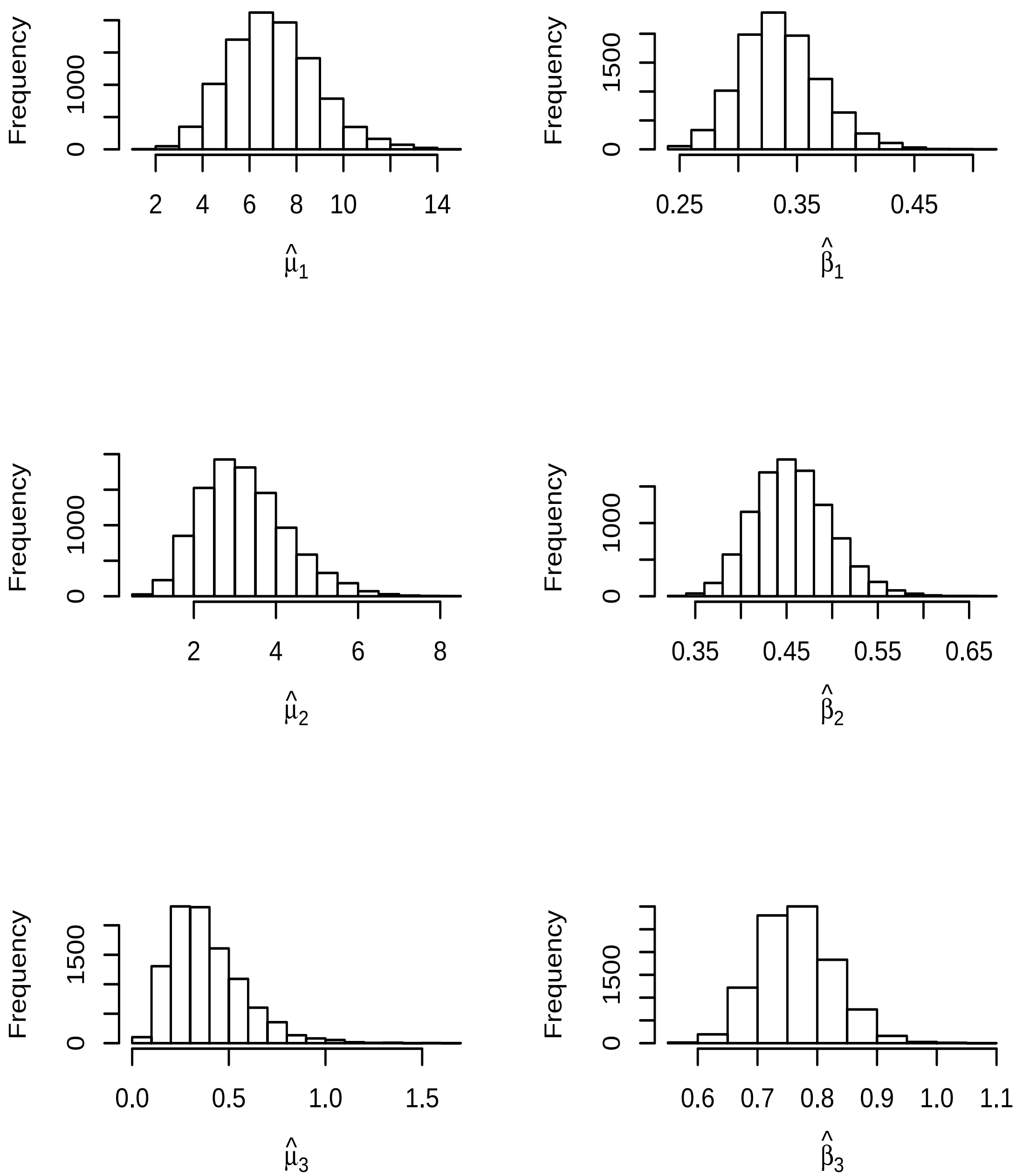
MLE	Data No.									
	1	2	3	4	5	6	7	8	9	10
$\hat{\mu}_1$	7.098	7.079	7.132	7.045	7.096	7.100	7.055	7.131	7.078	7.056
$\hat{\beta}_1$	0.330	0.330	0.329	0.330	0.330	0.330	0.330	0.329	0.330	0.330
$\hat{\mu}_2$	3.147	3.151	3.182	3.154	3.120	3.165	3.133	3.155	3.171	3.164
$\hat{\beta}_2$	0.453	0.453	0.452	0.453	0.454	0.452	0.454	0.453	0.452	0.453
$\hat{\mu}_3$	0.371	0.369	0.370	0.371	0.371	0.372	0.370	0.372	0.372	0.372
$\hat{\beta}_3$	0.755	0.755	0.755	0.755	0.755	0.754	0.755	0.754	0.754	0.754



Supplementary Figure 2: Histograms of the relative deviations of the MLEs for $\mu_1 = 4, \mu_2 = 2, \beta_1 = 0.5, \beta_2 = 2.5$ with sample size n = 100 and 25% masking



Supplementary Figure 3: Plots of mileage versus failure count in log-log scale for the original labor codes (LC) in the warranty data



Supplementary Figure 4: Histogram of the MLEs of the parameters for all three failure modes based on the 10,000 bootstrap samples