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Foundations of Transmathematics: Set Membership

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Abstract

We introduce a total set theory with unlimited comprehension. We begin by adopting a base logical language comprised of first-order predicate calculus with the equality operator. As usual, this language is composed of atoms, atomic formulas, and well-formed formulas. We also deal with badly-formed formulas. We define that nullity is an atom and that atoms have transcardinality nullity. We define set membership semantically, using a predicate. An arbitrary object is a member of a defined set if and only if the predicate is True, but it is not a member of the set if the predicate is False or a Contradiction or a Gap. We find that the Russell Set and the Russell Class are both sets. The Russell Set is not a member of itself, the Russell Class is not a member of itself, the Russell Set is a member of the Russell Class, and the Russell Class is a member of the Russell Set. This indirect, mutual recursion, between the Russell Set and the Russell Class, avoids the direct, self-recursion that is forbidden by the Russell Paradox. We define the Universal Set and conjecture that it is the class of all classes. We define that the transordinal number nullity is the atom nullity and that the transordinal number infinity is the set of all ordinal numbers. We dissolve the Burali-Forti Paradox by observing that some sets are transordinals, not ordinals. We show that both the transordinal number infinity and the Universal Set have cardinality infinity. We define transorder type so that the transorder of nullity is nullity, the transorder of infinity is infinity, and the transorder of the ordinals is, as usual, the least ordinal that is greater than the given ordinal. We discuss some philosophical aspects of the new set theory and propose a programme of future work.

1 Introduction

In previous work [3] we used a variety of methods from Computer Science to dissolve the Russell Paradox [9]. We constructed the transordinal numbers but without accounting for the Burali-Forti Paradox [15] which establishes that the set of all ordinals does not exist in some circumstances. We defined that nullity, Φ, is the smallest unordered set but mistakenly wrote this with three pairs of nested set brackets instead of four, when nullity is unordered with respect to the von Neumann ordinals [17]. Our mistake was that we failed to observe the the von Neumann ordinal $1 = \{\{\}\}$ is a member of $\{\{\{\}\}\}$. In the current paper, we define that nullity is an atom. The usual Boolean or classical two-valued logic has truth values T, F but we discussed a four-valued logic with the truth values: True, $\mathcal{T} = \{T\}$; False, $\mathcal{F} = \{F\}$; Contradiction, $\mathcal{C} = \{T, F\}$; Gap, $\mathcal{G} = \{\}$. We introduced antinomies as first-class objects. Thus we proposed a set theory with two kinds of objects: sets and antinomies. All of this is effective at dissolving the Russell Paradox but it is a very large departure from the usual set theories.

We now develop a total set theory which uses the usual two-valued Boolean or classical logic and which uses two kinds of objects: sets and atoms. We adopt the usual set operators, including set complement. We call this new set theory Foundations of Transmathematical Logic (FTL) and observe that its membership axiom could be adopted by any set theory.

Let us motivate our development of a total set theory. In the usual arithmetics, division is partial because n/d is not allowed to have $d = 0$. Transarithmetics, along with various other non-standard arithmetics, totalise division by allowing $d = 0$. But this does not make the new arithmetics total. For example, we can ask for the number x such that x is both negative, $x < 0$, and positive, $x > 0$. There is no such number so our putative arithmetics are partial. However, we can fix this by asking for the set $\{x \,|\, x < 0 < x\}$. This is the empty set, which does exist, but this is not enough to totalise the new mathematics because we could ask for the Russell Set, $\{x \,|\, \phi(x)\}$, where $\phi(x) = x \notin x$, which set is ordinarily said not to exist. In order to totalise mathematics, the set $\{x \,|\, \phi(x)\}$ must exist for all $\phi(x)$, where $\phi(x)$ is a formula in our base logical language. That is to say, we must have a total set theory with unlimited comprehension. We obtain one such set theory by requiring that $\phi(x)$ produces no member in the set when it cannot be evaluated; this may occur when $\phi(x)$ is contradictory, when it is unprovable, or when it is a badly-formed formula. If we are unable to determine which one of these negative circumstances applies, we can settle the matter with an axiom or leave the question open, totalising it with the artifice of an epsilon totalisation, pending the occurrence of some other resolution. This definition of set membership simultaneously totalises our base logic and our set theory, which justifies the name *Foundations of Transmathematical Logic (FTL)* and invites future work to dissolve all paradoxes of set theory and logic.

We begin our development of FTL with a discussion of some mathematical background concerning epsilon-totalised functions and, as this paper appears in a multidisciplinary journal, we discuss philosophical issues alongside technical ones.

2 Epsilon Totalised Functions

Computer Science makes widespread and essential use of the empty symbol. The use of an empty symbol dates back, at least, to the reading of a paper by Turing in 1936 [14]. Turing allowed an abstract machine to read and write blank squares on its data tape. This machine later became known as the Turing Machine and is now recognised as one of the foundations of Computer Science.

At some point it became the convention to refer to an empty symbol by the Greek letter epsilon, ϵ. This has the merit that it is possible to write syntactic formulas that involve the absence of a symbol. For example, if $f(x_i) = y_i$ is a procedure called f, with input x_i and output y_i then any of x_i, y_i may be an empty symbol. Thus we may write $f(\epsilon)$ $= \epsilon$ to describe the case where f has no input and no output, as may be required of a procedure whose only purpose is to introduce a delay into a computation. Empty symbols are used very widely in compilers, as described, for example, in [8].

In mathematics, a total function $f(x_i) = y_i$ maps each x_i in the function's domain to exactly one y_i in its image. If a function does not apply to some x_j inside the function's domain, it is said to be a partial function. We may totalise any function by defining $f(x_k) = \epsilon$ for all x_k outside the pre-image of a function. If we are concerned with inverse functions or other special mappings, we may restrict our discussion to only the non-epsilon parts of a function so that we obtain the same results as before we totalised it.

Totalising functions, in this way, is effective but it is not usually very informative. Suppose we totalise division by defining $x/0 = \epsilon$ for all x. This means that division by zero is defined but it produces no result. This totalisation is different from defining $x/0 = \perp$, where $\perp$ is an error value. The difference is that $\perp$ is a value but ϵ is not a value – it is the absence of a value. This is different, too, from the more informative transreal definitions $-\infty = -k/0 = -1/0$, $\Phi = 0/0$, $\infty = k/0 = 1/0$ for all positive k. Notice that Φ is different from $\perp$ because the numerator and denominator of Φ are both zero, whereas the numerator and denominator of $\perp$ are both $\perp$. A recent survey [4] discusses various totalisations of division, though it does not consider epsilon totalisations.

Epsilon totalisations are always available to us but we might prefer to totalise functions with an error value or, where possible, with a substantive value. In the next section, we find that the epsilon totalisation of set membership is non-trivial.

3 Epsilon Totalised Set Membership

Computer Science has developed very many ways to describe sets. A popular method is to use a bag. Bags are described in many computer language manuals, including [13]. Notations vary between languages but, following Pop11 [2], let us write a bag as $\{\%f(x)\%\}$. Of course we are free to arrange that $x = \epsilon$ so that we actually call $f()$. The percentage brackets, % and %, indicate that $f(x)$ is not a member of the bag but is to be executed. When executed, it returns all of the members of the bag. The members of a bag may be duplicated but the duplicates can then be removed to model a set.

Bags may employ the most general algorithm, called *Generate and Test.* A procedure $f(x)$ first calls a procedure $g(x)$ to generate all possible candidate members, x_g, and then calls a procedure $t(x_g)$ to test each candidate to determine whether or not it is admissible. If it is admissible then $t(x_g)$ and, ultimately, $f(x)$ return the candidate x_g, which is collected in the bag. But, otherwise, the candidate is not admissible and these procedures return ϵ, which is to say that the do not return anything at all. Thus inadmissible candidates, x_g, are not collected in the bag.

Thus bag membership and, hence, set membership, is an example of epsilon totalisation: an entire universe of candidate members is generated but, in general, some of the generated candidates are not allowed through testing so, in these cases, ϵ is passed through testing, which is to say no element is passed through testing. All of the approaches to set membership, discussed in this section, are epsilon totalised in a similar way.

Practical computer programs can process only a finite number of elements but we may take inspiration from bags to redefine the usual mathematical notation, $\{x \mid \phi(x)\}$, to mean that $\{\%\phi(x)\%\}$ is a predicate which tests all x_g in a universe of members and evaluates to exactly one of the truth value $\mathcal{T}$, $\mathcal{F}$, $\mathcal{C}$, $\mathcal{G}$ to indicate wether or not x_g is a member of the set. The candidate x_g is admitted into the set if and only if the truth value is $\mathcal{T}$, otherwise, the truth values $\mathcal{F}$, $\mathcal{C}$, $\mathcal{G}$ indicate that the candidate is not admitted into the set. We may write this as follows, for a given set S and arbitrary x chosen from a universe.

$$x \in S = \left\{ \begin{array}{lcl} T & : & \{\%\phi(x)\%\} = \mathcal{T} \\ F & : & \{\%\phi(x)\%\} = \mathcal{F}, \mathcal{C}, \mathcal{G} \end{array} \right. \tag{1}$$

Of course we need to define $\{\%\phi(x)\%\}$ and equality over $\mathcal{T}, \mathcal{F}, \mathcal{C}, \mathcal{G}$ as bases, to avoid the implied recursion on set membership. But, even so, how are we to determine the truth value of $\phi(x)$? We suppose $\phi(x)$ is a formula in a base logical language. If the logical system is consistent then any one proof path that establishes $\phi(x) = T$ also establishes that $\{\%\phi(x)\%\}$ is not any of $\mathcal{F}, \mathcal{C}, \mathcal{G}$. Similarly any one proof path that establishes $\phi(x) = F$ also establishes that $\{\%\phi(x)\%\}$ is not any of $\mathcal{T}, \mathcal{C}, \mathcal{G}$. But if one proof path establishes $\Phi(x) = T$ and a different proof path establishes $\Phi(x) = F$ then the system is a Contradiction and the usual base logical languages blow up, making all theorems true, including $\{\%\phi(x)\%\} = \mathcal{T}, \mathcal{F}, \mathcal{C}, \mathcal{G}$. Alternatively, if there are no proof paths then $\{\%\phi(x)\%\} = \mathcal{G}$. If we are to handle this, we need some kind of paraconsistent logic, but we are reluctant to develop an entire paraconsistent logic here, because that would be a large departure from the usual base logics of set theory.

Let us re-write Equation 1 in logical notation. We intend that $P \vee \neg P$ excludes a Gap and that $\neg(P \,\&\, \neg P)$ excludes a Contradiction.

$$x \in S = \phi(x) \,\&(\phi(x) \vee \neg\phi(x)) \,\&\, \neg(\phi(x) \,\&\, \neg\phi(x)) \tag{2}$$

This logical definition fixes the behaviour of set membership but has the defect that if $\phi(x)$ is a Contradiction or Gap then $\phi(x)$ contaminates all theorems. We need some method of restricting the scope of a Contradiction and Gap so that they do not contaminate theorems outside the scope. The usual hypothetical reasoning provides a suitable scope.

We use a schema for hypotheses to reason about set membership. When we want to know if a given object is a member of a given set, we hypothesise that the object is a member of the set. If it can be proved that it is True that the object is a member of the set, then we discharge the hypothesis and conclude that the object is a member of the set. But if it is a Gap or can be proved False or Contradictory that the object is a member of the set, then we discharge the hypothesis and conclude that the object is not a member of the set. If we are unable to establish which one of these cases holds, we may leave the matter open, with an epsilon totalisation, or else adopt a suitable axiom to settle the matter. Thus, for every possible definition of a set, every object either is or else is not a member of the set. This simultaneously totalises our base logical language and set theory because every sentence in our base language and set theory corresponds to a set.

Let us make clear how hypothetical reasoning prevents Contradictions and Gaps blowing up and contaminating all theorems in our set theory. We start from a consistent base logical language and set theory. Then we introduce a hypothesis, that is we start the scope of a hypothesis, in which we hypothesise some formula $\phi(x)$ in our base logical language. If this formula causes a Contradiction or Gap then we discharge the hypothesis, that is we end the scope of the hypothesis, and conclude that the formula does not add a member to the set, so our set theory remains consistent. Similarly, we do not admit the formula into the true theorems of the base logical language, thereby preserving the consistency of that language.

We call this semantic definition of set membership, as a schema for hypothetical reasoning, *FTL Membership*. It plays a critical role in the set theory FTL which we now introduce.

4 FTL

In this section we develop a set theory called Foundations of Transmathematical Logic (FTL). Like most set theories, FTL uses first-order predicate calculus, with equality, as its base logical language. Most set theories use a syntactic definition of set membership or a type system to avoid the paradoxes of set theory, including the Russell Paradox, but FTL uses a semantic set membership, called FTL Membership. This has profound practical and theoretical consequences.

Zermelo-Fraenkel set theory (ZF) and its popular variant with the Axiom of Choice (ZFC) are well-founded set theories, which means they build up all sets from the empty set. This avoids some paradoxes of set theory, such as the Russell Paradox, but introduces others, such as the Burali-Forti Paradox, and it limits the depth of nesting of sets. If pure sets are used, with no atoms, then the depth limit implies a limit on set size. In particular, there is no such set large enough to be a Universal Set.

FTL uses a semantic set membership, which avoids all paradoxes, and is non-well-founded so it admits a Universal Set. Other non-well-founded set theories admit Universal Sets but these are sometimes not as general as the class of all classes. We conjecture that FTL's Universal Set is the class of all classes. If so, it is as total as it is possible to be. FTL's Universal Set corresponds to a greatest number – it has transcardinality and transordinality infinity.

Some set theories admit objects which are not sets and have no elements. These objects are called atoms and must be specially distinguished from each other and the empty set. FTL admits the atoms of the base logical language into its set theory, including the nullity atom, but uses a polymorphic equality operator to distinguish objects. Sets, S, are defined syntactically and at the point where they are defined the fact that they are sets is asserted in the predicate $\mathrm{IsSet}(S)$. Polymorphic equality uses IsSet to distinguish atoms from sets; atoms are distinguished from each other with the equality operator of predicate calculus; and sets are distinguished from each other by the usual extentionality of set theory.

FTL atoms do not have any elements but they do not have cardinality zero, instead they have transcardinality, transordinality, and transorder type nullity. This provides a second way to distinguish atoms and sets, by checking their transcardinals or transordinals. The transcardinals are the usual cardinals, plus nullity and infinity; similarly the transordinals are the usual ordinals, plus nullity and infinity.

All of FTL's set operations apply to atoms. For example the usual set-builder notation can be used to construct sets that contain atoms as elements. The usual set operations, such as set union, apply only to sets, but the usual definitions apply equally to atoms. Thus the set union of two atoms is the empty set and the set union of an atom and a set is just the set itself. Thus the word "set" in "set union" does not mean that the union of sets is formed, it means that the union of objects is a set. Hence we prefer to say the "union" operation forms the "union set" of objects, regardless of whether the objects are atoms or sets. We describe the other operations of set theory similarly.

The usual order type is extended to a transorder type by defining that the transorder type of nullity is nullity and the transorder type of infinity is infinity. Thus the transordinals find a natural representation in FTL. This is not surprising because FTL was created in order to allow a natural representation of transarithmetics in a set theory.

4.1 Base Logical Language

First-order predicate calculus is described in many places, including [12]. The calculus comprises: logical symbols, such as the logical operators '$\neg$',

‘&’, ‘$\vee$’, ‘$\Rightarrow$’, ‘$\Leftrightarrow$’; the logical quantifies ‘$\forall$’, ‘$\exists$’; and brackets ‘(’, ‘)’; as well as the non-logical symbols for predicate names, constant names, and variables. In a slight departure from the usual conventions, we define that all of the symbols of predicate calculus are atoms of the calculus. This is the first departure we make from the usual propositional calculus.

All of the symbols of predicate calculus occur in a single name space. This has a number of consequences. Firstly, all symbols are distinct. Secondly, the symbols we introduce later, to describe set theory, share the same name space so, for example, ‘{’, ‘|’, ‘}’ become predicate calculus atoms. This shared name space means that predicate calculus can employ predicates from set theory and set theory can employ predicates from the calculus. The usual set-builder notation, $\{x \mid \phi(x)\}$, means that x is admitted into the set if and only if the predicate $\phi(x)$ is True; but the predicate may use symbols from the calculus, such as ‘&’, allowing $\{x \mid 0 < x \;\&\; x < 1\}$ and may use symbols from set theory, such as ‘$\in$’, allowing $\{x \mid x \notin x\}$. The relationship between first-order predicate calculus and set theory is extremely close. This is not surprising because first-order predicate calculus was developed in order to support set theory.

Symbols in predicate calculus have no structure other than their distinct identities. They acquire their meaning only by axioms and definitions that say how they are used. For example, the Boolean symbols, T and F, could be interchanged, so that T means False and F means True. A deep problem in philosophy is arranging that the symbol T does represent what we ordinarily call true.

It is convenient to partition the constant names so that there are reserved constant names and user constant names. The reserved constant names are employed in the predicate calculus and the set theory themselves. Thus we define that the Boolean symbols T and F are reserved constant names. If we want the totalised truth values $\mathcal{T}$, $\mathcal{F}$, $\mathcal{C}$, $\mathcal{G}$ or the nullity symbol, Φ, then we also define that these are reserved constant names. This avoids any name clash we might have with user constant names that may be used for other purposes. Thus user constant names play the role of user defined constants in programming languages.

Unlike atoms, sets are distinguished by their structure so if nullity is a set, it makes sense to ask which set is it? We have given the answer: nullity is the least unordered set with respect to a given ordering, such as the ordering of von Neumann ordinals. This is slightly problematical because we could choose any unordered set and it is deeply problematical because every set has a cardinality but nullity does not. Defining that nullity is an atom avoids the cardinality problem and avoids the question of which atom is nullity. Atoms are interchangeable so nullity is whichever atom has the properties of nullity. In fact we will define that the symbol Φ is an atom of our set theory, which automatically makes it an atom of predicate calculus, and then we will arrange that it has the properties of nullity.

In predicate calculus, a formula is a finite sequence of the calculus's symbols or, as we prefer to say, a finite sequence of predicate calculus's atoms. An atomic formula is a predicate name followed by zero or more constant names. Thus an atomic formula is not an atom, unless it is just a single predicate name with no constant names. A well-formed formula (wff) obeys the following four, recursive rules:

1. Any atomic formula is a wff.
2. If ϕ is a wff, so is $\neg\phi$.
3. If ϕ and ψ are wffs, so are $(\phi \,\&\, \psi)$, $(\phi \vee \psi)$, $(\phi \Rightarrow \psi)$, $(\phi \Leftrightarrow \psi)$.
4. If ϕ is a wff containing a constant name α, then any formula of the form $\forall\beta\phi^{\beta/\alpha}$ or $\exists\beta\phi^{\beta/\alpha}$ is a wff, where $\phi^{\beta/\alpha}$ is the result of replacing one or more occurrences of α in ϕ by some variable β not already in ϕ.

Notice that if there are a finite number of constant names then all of them could occur in a quantified well-formed formula but then it would be impossible to use Rule 4 in this case. Rule 4 is essential to the predicate calculus so a total predicate calculus, using formulas composed of finite sequences of atoms, must have an infinite number of constant names, that is, it must have an infinite number of atoms. The question is, how is a suitable infinitude of atoms to be made available?

We could, simply, use the least, in the usual sense, transfinite number, $\aleph_0$, of atoms. This would provide all the atoms we need for the purposes of predicate calculus itself; but let us explore this question a little further.

Turing [14] uses a finite number of abstract symbols in his computing machine but, in a footnote on page 249, he considers a set of physical symbols with cardinality greater than $\aleph_0$. He allows symbols to be conditionally compact, which means they may have cardinality up to the continuum, c.

> If we regard a symbol as literally printed on a square we may suppose that the square is $0 \leq x \leq 1$, $0 \leq y \leq 1$. The symbol is defined as a set of points in this square, viz. the set occupied by printer's ink. If these sets are restricted to be measurable, we can define the "distance" between two symbols as the cost of transforming one symbol into the other if the cost of moving unit area of printer's ink unit distance is unity, and there is an infinite supply of ink at $x = 2, y = 0$. With this topology the symbols form a conditionally compact space.

However, any of the usual transfinite numbers of atoms is not enough for some purposes in set theory. For example, we can specify that there are Kuratowski two-tuples that, together, instantiate a bijection between the elements of the Universal Set and distinct atoms. FTL Membership will satisfy this specification somehow – and will do so in the obvious way if there are as many atoms as there are elements of the Universal Set. This recursively defined bijection, of itself, ensures that the Universal Set has very many elements. In due course, we will find that the Universal Set has transcardinality and transordinality infinity, ∞, so we need the transnumber ∞ atoms to be available.

Let us give a mental model of how to create infinity atoms. We will use two ideal stamps to print a symbol or atom onto paper. The first stamp is a point stamp, it has a single point at the centre of its face. It prints an atom that represents the point at nullity. The second stamp is a linear stamp. It prints a line segment in different orientations from horizontal to vertical. We construct the linear stamp as follows. We lay off a line segment of unit length and say that this line segment is horizontal. We lay off a second line segment of unit length, with one endpoint copunctal with the mid point of the horizontal line segment. We take this second line segment normal to the horizontal line segment and call it vertical. Now the free end of the vertical line lies at unit height, that is at height one. Next we screw the horizontal line segment so that it is translated along the whole length of the vertical line segment, or axis, and is rotated by a quarter turn. Now let α be an atom. We can stamp the nullity'th atom, α_Φ, onto paper with the point stamp. We can stamp the zero'th atom, α_0, and the one'th atom, α_1, with the linear stamp. If we want to stamp the i'th atom, with $0 < i < 1$, then we grind the linear stamp, removing all material less than i and use the ground face to stamp the paper. If we regard the linear stamp as encoding arctangents, running from zero to one, then the corresponding tangents run from zero to infinity. If we grind the stamp using a Dedekind Cut, and use a third stamp to print a minus sign, then we can stamp any and all of the atoms in the transreal number line. If we use a generalised Dedekind Cut, such as employed in the construction of surreal numbers [5], then the surreal infinitesimals correspond to atoms that lie below all rational numbers greater than zero on the linear stamp and the surreal infinities correspond to points that lie above all of the rational numbers less than one on the linear stamp. If we use a very high density of atoms then we may have ∞ atoms.

This mental model describes how atoms might be laid out on a point and a line. Its construction suggests we can have a well-ordering of atoms. We may also have a well-ordering of atoms if sets are well-ordered and we have a bijection with distinct atoms, as we discussed above.

Holmes discuss the role of physical and abstract atoms in the set theory NFU [7].

> **Axiom of Extensionality.** *If A and B are sets, and for each x, x is an element of A if and only if x is an element of B, then $A = B$.*
>
> The Axiom of Extensionality can be paraphrased in more colloquial English: "Sets with the same elements are the same".
>
> Not all objects in our universe are sets. Objects which are not sets are called "atoms". You can think of ordinary physical objects, for instance, as being atoms. We certainly do not think of them as sets! Atoms have no elements, since they are not sets:
>
> **Axiom of Atoms.** *If x is an atom, then for all y, $y \notin x$ (read "y" is not an element of "x").*
>
> An advantage of the presence of atoms is that we can suppose that the objects of any theory (or the objects of the usual

> physical universe) are available for discussion, even if we do not know how to describe them as sets or do not believe they are sets. It turns out that our axioms will allow us to prove the existence of atoms, which is a rather surprising result.

For Holmes, and many other authors, atoms are defined by the property of not having elements. Anything that has no elements is an atom but it is not clear whether all atoms are part of the set theory NFU. It is not clear what Holmes means by saying physical atoms are "available for discussion." Does he mean that physical atoms are part of the set theory, which is a significant claim, or does he mean that the set theory can be used to describe physical objects, as is usual for any language? If he means that physical atoms are atoms of the set theory then some philosophical issues arise.

Most logics and set theories, including NFU, are atemporal in the sense that, if a proposition is true, then it is true for all time, if it is false, then it is false for all time. But modern physics teaches that there were no atoms for, approximately, the first five hundred million years of our physical universe so physical atoms are not atemporal. The most Holmes might legitimately hope for is that a physical atom is part of some machine that instantiates a part of NFU and that the atom exists for the lifetime of the machine. But there is a further problem to do with identity. If the atom loses an electron and becomes an ion, is it still an atom? If it gains a different electron, is it the same atom? The problem of identity runs deeper. Modern physics teaches that all fundamental particles interact with infinite fields, that all fields contain energy, that all energy has relativistic mass, and that all mass affects spacetime. Thus everything that exists in the physical universe is interlinked so we may legitimately doubt that the universe has parts. We might maintain that atoms or any other supposed part of the physical universe is just a human construct, a way of talking about an interlinked universe that is convenient for us. Alternatively, if Holmes means only that NFU can be used to talk about physical atoms, then none of these problems arise.

We adopt the realistic view that everything that exists is physical. We further suppose that the universe is an interlinked whole, without objective parts, unless, perhaps, it has an unlinked nullity part. We suppose that FTL is instantiated in physical machines, which may be the brains of mathematicians or the circuitry of electronic computers. In particular, we localise FTL's machine in the FTL Membership operation, in much the same way that all computation is localised in the *eval* function in the computer language Lisp [11]. Thus, it suits us to say, that the FTL machine is made up of physical atoms that instantiate all of the FTL atoms and sets, and operations on them, that a specific machine supports. We allow the possibility that a physical atom might describe an FTL atom but we do not require that physical atoms are treated this way. We allow that the physical universe might operate on an FTL machine in a way that causes it to malfunction and stop.

In a second departure from some forms of predicate calculus, we allow formulas to be a sequence of infinitely many atoms so that they can describe high cardinality sets. Once we are equipped with the transordinals, we will see that such a sequence may have a last atom at position ∞ in the sequence.

In a third departure from predicate calculus, we allow badly-formed formulas. There are two ways a formula can be badly formed. Firstly, it can contain an illegal configuration of symbols. In this case we define that it evaluates to a Gap. Secondly, it can contain a physical object that is not a symbol. For example, if we throw an actual spanner into a physical FTL machine, we expect the machine to malfunction or stop. Thus a badly-formed formula is anything that is not a well-formed formula, including being something that is not ordinarily considered to be a formula at all.

In preparation for introducing a set theory, in the next subsection, we adopt a new Rule for well-formed formulas, augmented with an action. The action concerns a predicate $\mathrm{IsSet}(x)$ that evaluates to False, F, unless the existence of x has been asserted.

5. If χ is a formula, then $\{\chi\}$ is a wff and we assert $\mathrm{IsSet}(\{\chi\})$.

Notice that this Rule 5 is not an example of the bracketing in Rule 3 because it applies to any formula, regardless of whether it is well or badly formed. This gives us the freedom to totalise sets by arranging that every formula gives rise to a set.

To be absolutely clear, even when χ is a badly formed formula, $\{\chi\}$ is a well formed formula. That is, the set brackets provide a scope on a badly-formed formula so that it does not contaminate the whole of the logical language. This is effective because the set theory is total, it creates a set for every formula, and maps badly-formed formulas onto the empty set so that they do not contaminate the set theory.

Now that we have a view on what FTL atoms and well-formed formulas are, we can consider the equality operator that is commonly added to first-order predicate calculus. Firstly, the well-formed formula rules are extended to admit a logical symbol representing equality, we use the atom '=='. Axioms are then asserted which make this atom into an operator that applies to atoms, such that $\alpha_i == \alpha_j$ if and only if α_i is identical to α_j. As usual, this is sufficient to distinguish all atoms from each other. We then define that $\neq\neq$ is the negation of $==$.

The equality operators in first-order predicate calculi differ in technical detail. In addition to providing equality over atoms, they usually also provide equality over predicates. We introduce a separate equality operator, $===$, and its negations, $\neq\neq\neq$, to distinguish predicates; but we make no use of it in the set theory developed in the subsection below, because it would admit predicates, as first-class objects, into the set theory, whereas we want only atoms and sets as first-class objects.

In the next subsection, we develop an equality operator, $====$, and its negation, $\neq\neq\neq\neq$, that apply to sets using extensionality. This operator distinguishes sets. We then develop a polymorphic equality, $=$, and its negation, $\neq$, that uses IsSet to distinguish atoms from sets; which uses $==$ to distinguish atoms; and which uses $====$ to distinguish sets.

In future work, we might examine the boundary between equality in first-order predicate calculus and equality in set theory. It is conceivable that a second-order predicate calculus would provide a natural embodiment of sets, by taking the opening set-bracket, '{', as a predicate that consumes its arguments up to the matching closing set-bracket '}'.

4.2 Set Theory

In this subsection we make heavy use of the base logical language introduced in the previous subsection. We begin by defining a set as a well-formed formula of the base logical language. This makes the concept of a set formal, as a linguistic entity. Our definition of set employs FTL Membership.

Axiom 1 *Axiom of Set Membership. The well-formed formula* $\{x \mid \phi(x)\}$ *is a set.*

Here ϕ *is a predicate and* x *is a variable that ranges over all formulas,* χ_i*. For each formula,* χ_i*, bound to* x*, we evaluate all proof paths that establish* $\phi(x)$*. If there are no proof paths or any one proof path is False,* F*, then we say that* x *is not a member of the set,* $x \notin \{x \mid \phi(x)\}$*, otherwise all proof paths are True,* T*, and we say that* x *is a member of the set,* $x \in \{x \mid \phi(x)\}$*.*

Any of the successive atoms 'x*', '*$|$*', '*ϕ*', '(', '*x*', ')' may be the epsilon symbol,* ϵ*, so that they do not occur in the formula; but the occurrence of the sequence of atoms '{', '}' is obligatory.*

After we have introduced the transordinal numbers, we will see that an infinite sequence can have a last, ∞'th, element so requiring a closing bracket, }, at the end of an infinite formula is unproblematic.

We introduce the usual Axiom of Extensionality, except that we do not require that its arguments are sets. Compare with [7], quoted in Section 4.1 above. Thus both the atomic equality, $==$, and the extensional equality, $====$, are total but unguarded. In future we might guard all equality operators.

Axiom 2 *Axiom of Extensionality. For each* x*, if* x *is a member of* A *if and only if* x *is a member of* B*, then* $A ==== B$*. If* A *and* B *are not extensionally equal,* $\neg(A ==== B)$*, we say* $A \neq\neq\neq\neq B$*.*

We now introduce a polymorphic equality that applies to atoms and sets.

Axiom 3 *Axiom of Equality. If neither* x *nor* y *are sets, their equality is decided by atomic equality,* $==$*. If both* x *and* y *are sets, their equality is decided by extensional equality,* $====$*. Otherwise* x *and* y *are not equal. Whenever* x *and* y *are not equal,* $\neg(x = y)$*, we say* $x \neq y$*.*

$$x = y := \begin{cases} x == y & : \quad \neg(\mathrm{IsSet}(x) \vee \mathrm{IsSet}(y)), \\ x ==== y & : \quad \mathrm{IsSet}(x) \,\&\, \mathrm{IsSet}(y), \\ F & : \quad \textit{otherwise}. \end{cases}$$

Notice that in order to present sets as arguments, x and y, to polymorphic equality, $=$, the sets x and y must have been constructed earlier, so they have been asserted in IsSet before IsSet tests them.

We now assert that the Universal Set exists, using an axiom. This brings into existence all possible atoms and sets. Thereafter we do not need any further axioms to make atoms and sets exist. However we do need to axiomatise the properties of atoms, and we may optionally define the names and prove the properties of sets.

Axiom 4 *Axiom of Universal Set. The Universal Set* $\mathbb{V} = \{x \mid x = x\}$ *exists.*

Theorem 5 *The Universal Set contains all and only atoms and sets.*

Proof 6 *The Universal Set is defined for each* x*. Polymorphic equality,* $=$*, admits all atoms because each atom is equal to itself by atomic equality,* $x == x$*. Similarly polymorphic equality admits all sets by extensional equality,* $x ==== y$*. All other objects compare unequal, via the 'otherwise' clause of polymorphic equality, so no other object is a member of the Universal Set. Similarly, no set contains an object that is not an atom or a set.*

Theorem 7 *Empty Set. The Empty Set,* $\{\}$*, exists and has no members.*

Proof 8 $\{\} = \{x \mid \phi(x)\}$ *has epsilon '*x*', '*$|$*', '*ϕ*', '*$($*', '*x*', '*$)$*', whence* $x \mid \phi(x)$ *is epsilon totalised, hence* $\{\}$ *has no members.*

The usual proof that the Empty Set exists as the complement of the Universal Set will be available to us after we have defined set complement but it is convenient to take the above direct proof.

Definition 9 *Singleton Sets. The set-builder notation for a singleton set, that is a set containing exactly one element, is defined by* $\{x\} := \{y \mid y = x\}$*.*

Singleton sets provide a convenient way to make sets of predicate names, logical operators, and brackets. For example: the singleton set, {IsSet}, contains the name of the base logical language's predicate, 'IsSet' that identifies sets; the singleton set, $\{=\}$, contains the name of the polymorphic equality operator, '='; and the singleton set $\{(\}$, contains the opening bracket '('. In some applications, it may be convenient to define variables $singleton\_\{ = \{x \,|(x = \{)\}$ and $singleton\_\} = \{x \,|(x =\})\}$.

Singleton sets can be used, in combination with Kuratowski tuples, to produce a tuple of the sequence of atoms that occur in a badly-formed formula. Hence we can talk about these degenerate objects without having them contaminate the set theory.

For the purposes of the present paper, we are content to use the usual variations of the set-builder notation, without defining them explicitly.

The operations of the usual set theories are guarded, so that they apply only to sets, but FTL Membership is total so we can dispense with the guards. We allow the operations to apply to any objects. For example, the union operator applies to atoms and to sets.

Definition 10 *Union. The set* $A \cup B = \{x \mid x \in A \vee x \in B\}$ *is called the 'union' or 'union set' of arbitrary* A *and* B*.*

When A and B are sets, their union is the usual union. When exactly one of A or B is an atom and the other is a set, their union is the empty set. When both of A and B are atoms, their union is the empty set. Similarly if A or B are badly formed formulas then their union is the empty set. For example, the union of chalk, that we may write with, and cheese, that we may eat, is the empty set.

Many other operations of the usual set theories are similarly unguarded as FTL operations so, for example, the Power Set of an atom is the empty set, but sometimes we must take care to account for the different properties of atoms and sets.

Definition 11 *Complement Set. The 'complement' or 'complement set', x^c, of an arbitrary object, x, is defined such that the complement of a set, S, is the Universal Set, excluding all of the elements of S; and the complement of an atom, α, is the Universal Set, excluding α.*

$$x^c := \begin{cases} \{y \mid y \notin x\} & : \ \mathrm{IsSet}(x), \\ \{y \mid y \neq x\} & : \ \textit{otherwise}. \end{cases}$$

Thus sets have their usual complement. For example, the complement of the Empty Set is the Universal Set and the complement of the Universal Set is the Empty Set. The complement of an atom is the Universal Set, excluding that atom. For example, the complement of nullity is the Universal Set, excluding nullity. Finally the complement of a badly-formed formula, such as a physical spanner, is the empty set.

For the purposes of the present paper, we are content to use the usual set operations, without defining them further.

We end this subsection with a conjecture that FTL's Universal Set is as total as it is possible to be.

Conjecture 12 *The Universal Set is the class of all classes.*

5 Consistency

By construction, FTL Membership guarantees that all of the members of an arbitrary set are chosen consistently, hence all sets are chosen consistently.

In the extreme case where $\phi(x)$ is inconsistent, the base logical language blows up, making all theorems in the base language True, T. Despite this plethora of True theorems, the theorems tell us nothing. They have no information content. But, in this inconsistent case, FTL Membership ensures $\{x \mid \phi(x)\} = \{\}$ for all $\phi(x)$ so that our set theory language is composed of just the empty set. This makes it explicit that an inconsistent base language tells us nothing.

We can use FTL Membership to construct a Universal Set that contains models of all other set theories, which is to say that FTL Membership is consistent with all set theories, even when these set theories are mutually or even internally inconsistent.

We allow sets structured as Kuratowski tuples $\langle N, A, P, L\rangle$. Here N is the Name of a set theory. We allow that the Name may be an atom or a set chosen from a very high cardinality set that provides a unique identifier for each instantiation of a set theory; A is the set of Axioms in the set theory, which are theorems that are held to be True without proof; P is the set of Productions or Proof rules in the set theory which are used to make derivations; and L is the Language comprising all of the derived True theorems of the set theory. These tuples may describe any set theory and are themselves sets so all set theories occur in a set theory that uses FTL Membership. To be clear, every set theory occurs as an L set in any set theory that uses FTL Membership,

As FTL Membership applies to any formula, $\phi(x)$, any set theory employing FTL Membership has a Universal Set and, as we have just seen, all set theories occur in this Universal Set.

It is important to recognise that FTL Membership is a semantic definition, not a syntactic one. The usual set theories involve well-foundedness or type systems, applied syntactically, to avoid paradoxes but this risks producing set theories that are too restrictive so that they become partial.

In the next section, we see how FTL Membership dissolves the Russell Paradox.

6 Russell Paradox Disolved

The Russell Paradox is extremely well known. It is discussed in many places, including [9] which gives excerpts of Russell's original description of the paradox.

The Russell Set is defined to be the set of all sets that are not members of themselves, $R_s = \{x \mid x \notin x\}$, but this creates the paradox that the Russell Set is a member of itself if and only if it is not a member of itself, $x \in x \Leftrightarrow x \notin x$. Russell concludes that this set does not exist but we can construct R_s using FTL Membership.

Given $R_s = \{x \mid x \notin x\}$, we know R_s exists because FTL Membership guarantees that every formula gives rise to a set.

We know R_s is non-empty because, for example, the empty set has no members, so it is not a member of itself, therefore it is a member of the Russell Set, $\{\} \in R_s$.

We know R_s has many members because, for example, the von Neumann ordinals are constructed so that they do not contain themselves, therefore each von Neumann ordinal, o_i, is a member of the Russell Set, $o_i \in R_s$. Furthermore, In Section 10 *Burali-Forti Paradox Dissolved,* we see that the set of all ordinals, $\mathbb{O}$, exists and in Section 8 *Transordinal Numbers,* we see that the set of all ordinals is not a member of itself. Therefore the set of all ordinals is a member of the Russell Set, $\mathbb{O} \in R_s$. All of which is to say that R_s has very many members.

We know the Russell Set is not a member of itself, $R_s \notin R_s$, because its membership formula, $x \notin x$, implies a Contradiction $x \in x \Leftrightarrow x \notin x$ and FTL Membership does not admit Contradictory members.

Thus FTL Membership constructs the Russell Set, R_s. By definition, all of the members of the Russell Set, $r_i \in R_s$, do not contain themselves. By construction with FTL Membership, the Russell Set, R_s, is not a member of itself and is therefore distinct from all of its members, $R_s \neq r_i$. Thus the Russell Class, R_c, of all sets that are not members of themselves is the Russell Set, together with all of its members, $R_c = R_s \cup \{R_s\}$. By construction, $R_s \in R_c$.

Notice that the Russell Class is a set. It is constructed by the two step process: $R_s = \{x \,|\, x \notin x\}$, $R_c = R_s \cup \{R_s\}$. We have no preconception of whether the Russell Class, as defined here, is or is not a member of itself. Let us settle this question.

Employing FTL Membership, we hypothesise that $R_c \in R_c$. This means $R_c \in R_s \cup \{R_s\}$ but $R_c \neq R_s$ so $R_c = r_i \in R_s$ for some specific r_i. But by construction, $r_i \notin r_i$, which is to say $R_c \notin R_c$. This is a Contradiction so we discharge the hypothesis and conclude $R_c \notin R_c$, which implies $R_c \in R_s$.

Gathering all of this together, we have a non-paradoxical construction of the Russell Set in one step and the Russell Class in two steps: $R_s = \{x \,|\, x \notin x\}$, $R_c = R_s \cup \{R_s\}$. This non-paradoxical construction leads to several non-paradoxical conclusions, including: $R_s \notin R_s$, $R_c \notin R_c$, $R_s \in R_c$, $R_c \in R_s$.

7 Transnatural Numbers

In earlier work [3] we defined that nullity is the smallest unordered set but mistakenly wrote this with three pairs of nested set brackets instead of four, when nullity is unordered with respect to the von Neumann ordinals. We could take nullity as $\Phi = \{\{\{\{\}\}\}\}$ but then we would have $|\Phi| = \{\{\}\} = 1$, which is arbitrary. We prefer to take nullity as an atom.

Interestingly, we cannot take nullity as a Quine Atom. We use the von Neumann ordinals [17] as our preferred ordering. The von Neumann ordinals are ordered by set membership so that $x < y$ if and only if $x \in y$, for every ordinal x and y. If we were to take nullity as the Quine Atom, $\Phi = \{\Phi\}$, with a non-finite number of recursively defined set brackets, we would have $\Phi \in \Phi$, see [16], whence we would have $\Phi < \Phi$, which is false in transreal arithmetic. In fact, the Quine Atom is not an atom, it is a reflexive set.

We assume that cardinality is defined, as usual, as a bijection between sets, but is not defined between atoms. This means that no atom has a cardinality, making cardinality partial. We now totalise cardinality by introducing one strictly transcardinal number, nullity. Henceforth the transcardinals are the cardinals, plus nullity. We will later find that the strictly transordinal number infinity, ∞, has a bijection with the Universal Set, so ∞ is one of the usual cardinals, albeit one that was not previously recognised.

Axiom 13 *Axiom of Nullity. The atom nullity has transcardinality nullity,* $|\Phi| = \Phi$.

We have already said that the atom nullity has transcardinality nullity because it is not a set so it does not take part in the bijections between sets that define cardinality. Specifically the atom nullity does not have cardinality $0, 1, 2$, and so on. If we accept bijections between sets, but not between atoms and sets, then the least change we can make to the usual notion of cardinality, to totalise it, is to add one non-cardinal number that is not ordered with respect to the natural numbers. Transreal nullity is unordered with respect to the real numbers so it already has the required property of being unordered with respect to the natural numbers. This is mathematically satisfying but we might want a mental model to give us intuitions about the cardinality nullity.

Consider an enumeration of the transnatural numbers, composed of nullity and the natural numbers: $\Phi, 0, 1, 2, 3$, and so on. Casting this into von Neumann form, we have the enumeration: $\Phi = \Phi, 0 = \{\}, 1 = \{0\}, 2 = \{0, 1\}, 3 = \{0, 1, 2\}$, and so on. Let us make this more concrete by thinking of sets as wooden boxes. The set $0 = \{\}$ has no elements, it is an empty wooden box with no atoms or boxes inside it. The set $1 = \{0\}$ is a box with one element inside it, the box 0. The set $2 = \{0, 1\}$ is a box with two boxes inside it, the boxes 0 and 1. We progress, in this way, employing set-builder notation, until we have enumerated all of the natural numbers. But nullity is different. There is no atom-builder notation. There is no way to place an element inside an atom. Atoms are like solid blocks of wood. There is no possibility of placing anything inside them.

When we apply the usual set membership operator to a set, $s \in S$, the membership operator accesses the set and determines whether or not the element s is in the set S. It can be the case that S is the Empty Set, with no elements in it. But when we apply the usual set membership operator to an atom, $s \in \alpha$, the membership operator fails to access the atom, α, because the atom is not a set. This is a failure or Gap, which is epsilon totalised, in FTL Membership, by returning False, F, for every s. Thus the Empty Set has no elements because it has no elements in it, whereas it does not make sense to use the usual set membership operator to ask if an atom has elements in it. We can, however, use FTL Membership to construct the set of all of the elements in the Empty Set, $\{x \mid x \in \{\}\} = \{\}$, and we can construct the set of all elements in an atom, $\{x \mid x \in \alpha\} = \{\}$. Thus the contents of atoms and the Empty Set are different but the sets of their contents are the same – the Empty Set, $\{\}$.

Let us press our analogy a little harder. We take a wooden block, representing an atom, and attempt to drill a hole in it, so that we can place an element in the cavity; but we find that the block cannot be drilled, it cannot be broken apart, it behaves like an indivisible fundamental particle. This justifies the name 'atom'. But there is another possibility, an atom does have content but the content is absorptive. Absorptive content is different from zero content because, if we add one element to it, the absorptive content is unchanged but the zero content is increased by one. As we continue to add elements, the absorptive content remains unchanged but the zero content enumerates all of the natural numbers. However, we do not have an atom-builder notation so we cannot place any content in an atom. Hence, if an atom does have content, this content is a fundamental property of being an atom. We might try to define that we have a symbol, ϕ, that represents the absorptive content of an atom, and use this in FTL Membership to discover if the absorptive content is in the atom, $\phi \in \alpha$, but if this evaluates to true, then the atom has cardinality one, not zero, as we require. Thus we are free to imagine that an atom has absorptive or nullity content but an atom's content is inaccessible.

Having thus strengthened our intuitions about the cardinality nullity, let us return to the construction of the transnatural numbers.

Axiom 14 *Axiom of Atoms. Each atom,* α*, has cardinality nullity,* $|\alpha| = \Phi$.

Henceforth, we may distinguish atoms from sets by checking their cardinality.

We now formally introduce the transnatural numbers, relative to the von Neumann ordinals, and establish their ordering [1] [6]. In particular we show that nullity is the uniquely unordered number and infinity is the greatest ordered number.

Definition 15 *Natural Numbers. The set of all natural numbers,* $\mathbb{N}$*, is the set of all finite von Neumann ordinals, including zero.*

Definition 16 *Transnatural Infinity. The transnatural number infinity,* ∞*, is given by* $\infty = \mathbb{N}$.

Notice that transnatural infinity is the usual ordinal omega, ω. This is the least non-finite ordinal in the usual mathematics but is the only and greatest non-finite ordinal in the transnatural numbers.

Definition 17 *Transnatural Nullity. The atom nullity,* Φ*, is a transnatural number.*

In the remainder of this section, we refer to transnatural nullity as nullity and transnatural infinity as infinity.

Theorem 18 *Nullity is unordered with respect to the natural numbers and infinity.*

Proof 19 *Firstly nullity is unordered with respect to the natural numbers because, for all* $n \in \mathbb{N}$*, by construction* $n \notin \Phi$ *and* $\Phi \notin n$*. Similarly nullity is unordered with respect to infinity because* $\mathbb{N} \notin \Phi$ *and* $\Phi \notin \mathbb{N}$*. Nullity is not less than or greater than itself because* $\Phi \notin \Phi$*. Finally nullity is equal to itself,* $\Phi = \Phi$*, by atomic equality,* $\Phi == \Phi$.

Theorem 20 *Infinity is greater than every natural number.*

Proof 21 *Let* n *be an arbitrary natural number, that is* $n \in \mathbb{N}$*. But* $\mathbb{N} = \infty$ *so* $n \in \infty$*, whence* $n < \infty$ *for all natural numbers* n.

Notice that in well-founded set theories, such as Zermelo Fraenkel set theory with the Axiom of Choice (ZFC), $\mathbb{N} \notin \mathbb{N}$ by construction so this does not need to be proved specifically.

Theorem 22 *Infinity is not less than or greater than itself.*

Proof 23 *To prove infinity is not less than or greater than itself,* $\infty \not< \infty$*, we must establish* $\infty \notin \infty$ *or, equivalently,* $\mathbb{N} \notin \mathbb{N}$*. We use FTL Membership. We hypothesise* $\infty \in \infty$*, then there is some* $n_i \in \mathbb{N}$ *such that* $n_i = \infty$*. There is no greatest natural number, which is to say there is some* $n_j \in \mathbb{N}$ *such that* $n_j > n_i$*, whence* $n_j \notin n_i$*, but* $n_i = \infty = \mathbb{N}$ *so* $n_j \notin \mathbb{N}$*. This is a Contradiction so we discharge the hypothesis and conclude* $\infty \notin \infty$*, whence* $\infty \not< \infty$*. Finally infinity is equal to itself,* $\infty = \infty$*, by extensional equality,* $\infty ==== \infty$.

We now define the set of all transnatural numbers as in previous transmathematical work.

Definition 24 *Transnatural Numbers. The set of all transnatural numbers,* $\mathbb{N}^T$*, is given by* $\mathbb{N}^T = \mathbb{N} \cup \{\infty, \Phi\}$.

Thus we have totalised cardinality as transcardinality, by including the atom nullity as the only strictly transcardinal number, and we have introduced the transnatural numbers as the natural numbers, plus the set infinity and the atom nullity. This is uncontroversial. In the next section, we take the controversial step of introducing the transordinal numbers.

8 Transordinal Numbers

In this section we define the transordinal numbers in terms of the set of all ordinal numbers. The set of all ordinal numbers does not usually exist but in Section 10 *Burali-Forti Paradox Dissolved* we find that the properties of the transordinals and their transorder type dissolve the Burali-Forti Paradox, which means there is no objection to the existence of the set of all ordinals.

Definition 25 *Ordinal Numbers. The set of all ordinal numbers,* $\mathbb{O}$*, is the set of all von Neumann ordinals, including zero.*

Definition 26 *Transordinal Infinity. The transordinal number infinity,* ∞*, is given by* $\infty = \mathbb{O}$.

Note that, in the usual mathematics, there is no number as large as transordinal infinity. Transordinal infinity is the greatest transordinal.

Definition 27 *Transordinal Nullity. The atom nullity,* Φ*, is a transordinal number.*

All of the theorems and proofs in the above Section 7 *Transnatural Numbers* hold when we replace the natural numbers, $\mathbb{N}$, with the ordinal numbers $\mathbb{O}$.

We now define the set of all transordinal numbers as in earlier transmathematical work.

Definition 28 *Transordinal Numbers. The set of all transordinal numbers,* $\mathbb{O}^T$*, is given by* $\mathbb{O}^T = \mathbb{O} \cup \{\infty, \Phi\}$.

9 Transorder Type

The Burali-Forti Paradox depends heavily on the order type of an ordinal. The usual definition of order type is partial, it applies only to well-ordered sets that have a greater well-ordered set. We now totalise the definition of order type relative to the transordinals. We are then free to compute the order type of any object by establishing an order preserving bijection with the argument of the transorder function.

Definition 29 *Transorder Type. The transorder type,* $\text{transord}(x) = y$*, maps a transordinal,* x*, to a transordinal,* y*, as follows, where* $\mathbb{O}$ *is the set of all ordinal numbers.*

$$\text{transord}(x) = \begin{cases} \Phi & : \quad x \notin \mathbb{O} \ \& \ x \neq \mathbb{O}, \\ \infty & : \quad x = \mathbb{O}, \\ x \cup \{x\} & : \quad \textit{otherwise.} \end{cases}$$

10 Burali-Forti Paradox Dissolved

The Burali-Forti Paradox is discussed in many places, including Weisstein [15] who cites Burali-Forti's original work. The following excerpt is from Weisstein who gives the paradox in a modern form. In the excerpt, 'transfinite' is used in the usual sense, not in any sense related to transmathematics.

> In the theory of transfinite ordinal numbers,
>
> 1. Every well ordered set has a unique ordinal number,
> 2. Every segment of ordinals (i.e., any set of ordinals arranged in natural order which contains all the predecessors of each of its members) has an ordinal number which is greater than any ordinal in the segment, and
> 3. The set B of all ordinals in natural order is well ordered.
>
> Then by statements (3) and (1), B has an ordinal β. Since β is in B, it follows that $\beta < \beta$ by (2), which is a contradiction.

If propositions (1)-(3) are true then there is a contradiction, from which we may infer that the set of all ordinals does not exist. However there are several circumstances in which these propositions are not simultaneously true.

The set theory New Foundations with Urelements (NFU) employs a stratified comprehension in which (2) is not true [7]. Hence the Burali-Forti Paradox does not exist in NFU.

In any set theory with transordinals and well-ordering, (1) is not true because transordinal ∞ is not an ordinal number, precisely because it does not have a greater ordinal number.

In conclusion, the usual mathematics has three kinds of ordinals: zero, which has no predecessors; successor ordinals; and limit ordinals. The transordinals are the ordinals, plus the greatest transordinal, infinity, and an unordered transordinal, nullity. Thus there are five kinds of transordinals.

11 Cardinality of Infinity

Theorem 30 *The Universal Set has Cardinality Infinity.*

Proof 31 *The transordinals obey quadrachotomy [1] but we know* $\infty \neq \Phi$ *so infinity obeys trichotomy, with respect to the cardinal sets. In order to show that* $|\infty| = |\mathbb{V}|$*, we show that* $|\infty| \not> |\mathbb{V}|$ *and* $|\infty| \not< |\mathbb{V}|$*. Firstly we know that infinity itself,* $\infty = \mathbb{O}$*, and every member of infinity,* $m_i \in \mathbb{O}$*,* *is a member of the Universal Set, $m_i, \infty \in \mathbb{V}$, so the cardinality of infinity is not greater than the cardinality of the Universal Set, $|\infty| \not> |\mathbb{V}|$. This is not surprising, because the Universal Set is the greatest cardinal! Secondly, by well ordering, for every cardinal C_i, there is at least one ordinal, O_i, with $|O_i| = C_i$. Let $C_i = |\infty|$. Suppose $C_i < |\mathbb{V}|$ then there is some cardinal $C_j > C_i$ and some ordinal, O_j, such that $|O_j| = C_j$ but then $O_j > O_i = \infty$, which is impossible. We have now established $|\infty| \not> |\mathbb{V}|$ and $|\infty| \not< |\mathbb{V}|$, whence $|\infty| = |\mathbb{V}|$.*

12 Discussion

Transmathematics is intended to be a total system of mathematics. It is intended to apply to everything, including things that are beyond the scope of the usual mathematics. We adopt a philosophy of realism in which the signs and operations of transmathematics are effected by physical machines, such as people or computers, composed of physical atoms. We think of these physical signs and physical operations being mapped onto abstract transmathematical objects via the operation of FTL Membership.

Consider the set $\{x \mid \phi(x)\}$. If $\phi(x)$ is a well-formed formula, in some base logical language, such as first-order predicate calculus with equality, then the bag $\{\%\phi(x)\%\}$ may evaluate to exactly one of True ($\mathcal{T}$), False ($\mathcal{F}$), Contradiction ($\mathcal{C}$), Gap ($\mathcal{G}$). The first three of these truth values – True, False, Contradiction – occur in the usual way and a Gap can occur if $\phi(x)$ is incomputable or unprovable. FTL membership then admits x into the set if and only if $\phi(x)$ is True, but does not admit x into the set if $\phi(x)$ is any of False, Contradiction, Gap. This is a wider reading of "if and only if" than is usual and, as we have seen, this definition of set membership has profound consequences.

It can happen that we are unable to determine whether or not x is a member of the set. In this case, FTL Membership asserts that x is or else is not a member of the set, leaving us free to adopt an axiom to settle the matter. In some circumstances transmathematics might bifurcate so that in one family of transmathematics, x is a member of the set and, in another family, x is not a member of the set. But recall that transmathematics is intended to be total. What are we to do if $\phi(x)$ is not a well-formed formula? There are two ways this can happen. Firstly, $\phi(x)$ might be composed of signs in an illegal configuration, including the epsilon configuration where $\phi(x)$ is not present, or $\phi(x)$ might not be a formula at all, it might be some physical object that does not correspond to any of the signs in the base logical language we are using. In this case $\phi(x)$ is a Gap, whence x is not a member of the set . For example, if $\phi(x)$ is an actual spanner that we throw into the works of FTL membership then the spanner adds nothing to the set – though it may break the machine and cause it to malfunction or stop! Thus FTL membership is totalised over all $\phi(x)$, where $\phi(x)$ is any region of the physical universe, regardless of whether $\phi(x)$ is a formula in our base logical language or not.

In the case that we cannot decide $\phi(x)$, for some set $S = \{x \mid \phi(x)\}$, we may choose not to assert an axiom to settle the truth or falsity of

$\phi(x)$; instead, we may leave the membership of x as an open question by asserting $x \in S = \epsilon$.

It seems to us that the hypothetical reasoning in FTL membership is just the usual metalogical reasoning that is employed to show that an object does not exist; but this leads us into a collision with the usual mathematics. We find that the set of all ordinals does exist but ZFC finds that it does not exist. We could add to ZFC the true theorem, that the set of all ordinals exists, but we might prefer to say only that the set of well-founded ordinals exists and then take this set equal to the transnumber infinity. We find that the Russell Set exists and that the Russell Class is a set but the usual set theories deny this. In future work, we might more closely examine the relationship between FTL and the usual set theories.

A trivial consequence of the fact that the set $\{x \mid \phi(x)\}$ exists for all $\phi(x)$ is that FTL's Universal Set is specified so that is contains all $\phi(x)$, which is to say $\mathbb{V}$ is defined to be the class of all classes. However, we might doubt that all classes are members of the Universal Set because, we might imagine, some classes will lead to Contradictions or Gaps so that FTL membership excludes them from the Universal Set. Our experience with the Russell Paradox does not support such pessimism but we would like a proof that $\mathbb{V}$ is, in fact, the class of all classes. This is a question that might be taken up in future work.

A subtlety of our dissolution of the Russell Paradox is that there is no single set that dissolves the paradox; instead there are two sets, the Russell Set and the Russell Class. These both occur in the Universal Set and both are mutually recursive, that is, the Russell Set does not contain itself but does contain the Russell Class and the Russell Class does not contain itself but does contain the Russell Set. This recursion continues indefinitely, limited only by the last recursion at infinity. But what does this last recursion contain? This is a question that might be taken up in future work.

Thus far, we have discussed the ordering properties of the transordinals. These are essential to developing a set theoretical model of transarithmetics but we also need to model the absorptive properties of nullity. The transorder type might be helpful. If we apply the transorder type directly to a set, as $\text{transord}(S)$, without allowing a bijection of S onto the transordinal numbers, then any set containing nullity is not order isomorphic with any ordinal or infinity so its order type is nullity. This might be used to give us the additive and multiplicative absorptivities, $\Phi + x = \Phi$ and $\Phi \times x = \Phi$, but how are we to obtain natural models of $0 \times \infty = \Phi$, $\infty/\infty = \Phi$, and $\infty - \infty = \Phi$? Perhaps the Dedekind Cut [10] might be helpful? Each Dedekind Cut defines one real number and, together, all of the Dedekind Cuts define the set of real numbers. An individual Dedekind Cut is a partition of the set of rational numbers into two parts $\langle L, R \rangle$. The left set, L, and the right set, R, have a number of properties, two important ones of which are ordering, every member of L is less that every member of R, and non-emptiness, L and R are non-empty. Non-emptiness makes the Dedekind Cuts partial, which inevitably leads to exceptions when using real numbers. However, if we totalise the Dedekind Cut, by allowing emptiness, we have $-\infty = \langle \{\}, \mathbb{Q} \rangle$, $\Phi = \langle \{\}, \{\} \rangle$, $\infty = \langle \mathbb{Q}, \{\} \rangle$. Several questions come to mind. Firstly what arithmetic arrises when the usual operations of Dedekind Cuts operate on the transnumbers $-\infty, \Phi, \infty$? Secondly, is this a total arithmetic? Thirdly what total arithmetics are compatible with it? Fourthly, is there a natural way to express transreal arithmetic in totalised Dedekind Cuts?

Computer Science commonly adopts the heuristic of using top-down development when the development path is foreseeable and bottom-up development when it is not. We have started with a bottom-up development of the transordinals but, in future, perhaps we should try a top-down development from the totalised Dedekind Cuts? If successful, the top-down development would show us the detail of what we needs to be achieved in the bottom-up development from elementary sets.

Regardless of how FTL develops, we can already say how it might influence the usual set theories. Firstly we are free to add the atom nullity to any set theory, even ZFC. We are free to add the set of accessible ordinal numbers, whence we may have an accessible transnumber infinity. Thus we may have the accessible transordinals in any set theory. In any non-well-founded set theory, with a universal set, we may have the transordinal numbers in their entirety, though some of the operations of FTL and those set theories might produce different results. For example, in NFU, the cardinality of the Universal Set's Power Set is less than the cardinality of the Universal Set but, in FTL, we expect the Universal Set's Power Set to be the Universal Set. We may add FTL membership to any of the usual set theories. On the one hand, this would make them arbitrarily extensible, as required of any total system by Gödel's incompleteness theorems, but on the other hand, this would radically alter them.

13 Conclusion

We develop methods for totalising mathematics. Epsilon is the empty symbol of computer language and machine theory. Epsilon totalisations of functions are always effective but are usually trivial. However, the epsilon totalisation of set membership is non-trivial.

We begin by taking nullity as an atom, whence nullity has identical transcardinality, transordinality and transorder type nullity. We introduce infinity as the set of all ordinals and dissolve the Burali-Forti Paradox which, otherwise, might forbid the existence of this set. Hence infinity is simultaneously the greatest cardinal, transordinal, and transorder type. In particular, infinity and the Universal Set have the same cardinality.

We derive various conclusions from our definitions of the Russell Set and the Russell Class, specifically: the Russell Set is not a member of itself, the Russell Class is not a member of itself, the Russell Set is a member of the Russell Class, the Russell Class is a member of the Russell Set. Note that this indirect, mutual recursion, between the Russell Set and the Russell Class, avoids the direct, self recursion that is forbidden by the Russell Paradox.

We propose a number of areas for future work. The totalisation of the Dedekind Cut might play a key role in the development of set theoretical models of transarithmetics. FTL Membership allows us to specify that the Universal Set is the class of all classes but we would like to settle the question of whether or not FTL Membership does allow every member of the class of all classes to enter the Universal Set. FTL Membership trivially enforces consistency but we might want to show, in detail, how each of the paradoxes of logic and set theory is dissolved. We can add FTL theorems to any set theory but it might be helpful to explore non-well-founded set theories that have a Universal Set.

Acknowledgement

Many years ago, Walter Gomide asked the seminal question, which set is nullity? This question finds a negative technical answers in the body of this text, nullity is an atom, but we recommend the following definition to him. Nullity is the order type of a banana.

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