

Online supplementary material

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1. Some Definitions

Here we will give a brief description of some popular circular and spherical probability distributions. We also included the definition of circular trimmed mean.

1.1 Circular and Spherical Distributions

The most popular symmetric unimodal distribution used for modeling circular data is the circular normal distribution (a.k.a. von-Mises distribution) which has the probability density function (p.d.f)

$$f(\theta; \mu, \kappa) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(\theta - \mu)}, 0 \leq \theta < 2\pi, 0 \leq \mu < 2\pi, \kappa > 0$$

where $I_0(\kappa)$ is the modified Bessel function of order zero, the parameters μ and κ are respectively called the mean direction and the concentration parameter. We will denote this distribution as $CN(\mu, \kappa)$. Besides circular normal distribution, another popular symmetric unimodal distribution used for modeling circular data is the wrapped normal distribution obtained by wrapping $N(\mu, \sigma^2)$ on to the circle and having the p.d.f.

$$f(\theta; \mu, \rho) = \frac{1}{2\pi} \left\{ 1 + 2 \sum_{p=1}^{\infty} (\rho)^{p^2} \cos p(\theta - \mu) \right\}, 0 \leq \theta < 2\pi, 0 \leq \mu < 2\pi, 0 < \rho < 1.$$

The parameters μ and ρ are respectively called the mean direction and the concentration parameter. We will denote this distribution as $WN(\mu, \rho)$.

One of the most popular symmetric distributions used for modeling spherical data is the von-Mises-Fisher distribution. The random vector $\tilde{\mathbf{X}}$ is said to follow the von-Mises-Fisher distribution $M(\tilde{\boldsymbol{\mu}}, \kappa)$ if it has the following p.d.f

$$f((\theta, \phi); (\alpha, \beta), \kappa) = \frac{\kappa \sin \theta}{4\pi \sinh \kappa} \exp(\cos \theta \cos \alpha + \sin \theta \sin(\phi - \beta)) \text{ where}$$

$\tilde{\mathbf{X}} = (\sin \Theta \cos \Phi, \sin \Theta \sin \Phi, \cos \Theta)^T$, $\tilde{\boldsymbol{\mu}} = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha)^T$, $0 \leq \theta, \alpha \leq \pi$, $0 \leq \phi, \beta \leq 2\pi$ and $\kappa > 0$. The parameters $\tilde{\boldsymbol{\mu}}$ and κ are called ‘mean direction’ and ‘concentration parameter’ respectively.

1.2 The circular trimmed mean

Suppose Θ is a circular random variable with p.d.f $f(\theta)$ and $0 < \gamma \leq 0.5$ is fixed. Let α, β be two points on the unit circle satisfying

$$(i) \int_{\beta}^{\alpha} f(\theta) d\theta = 1-2\gamma \text{ and}$$

$$(ii) d_1(\alpha, \beta) \leq d_1(\mu, \nu) \text{ for all } \mu, \nu \text{ satisfying } \int_{\nu}^{\mu} f(\theta) d\theta = 1-2\gamma \text{ where } d_1(\phi, \xi) \text{ is the length of}$$

the arc starting from ξ and ending at ϕ traversed in the anticlockwise direction. Then

the γ -circular trimmed mean (γ -CTM) is defined as $\mu_{\gamma} = \arg \left[\frac{1}{(1-2\gamma)} \int_{\beta}^{\alpha} e^{i\theta} f(\theta) d\theta \right]$ where

γ is the trimming proportion.

2. Proofs of theorems and lemmas

Proof of Theorem 2.1: Let F_μ denote the $CN(\mu, \kappa)$ distribution and δ_x denote the point mass at x . Also let $G_\varepsilon = (1-\varepsilon)F_0 + \varepsilon\delta_x$. We note that, $E_{(1-\varepsilon)F_0 + \varepsilon\delta_x}(\cos \Theta) = y$ and $E_{(1-\varepsilon)F_0 + \varepsilon\delta_x}(\sin \Theta) = \varepsilon \sin x$. Using these we get

$$W(G_\varepsilon) = \sqrt{y^2 + \varepsilon^2 \sin^2 x} - y. \quad \dots (2.1)$$

Again, an easy computation yields $W(F_\mu) = c$ (2.2)

Now using (2.1) and (2.2) we have

$$\varepsilon_\mu^{**}(W) = \inf \left\{ \varepsilon > 0 : \sqrt{y^2 + \varepsilon^2 \sin^2 x} = y + c \text{ for some } 0 \leq x < 2\pi \right\}.$$

Hence the theorem is established.

Proof of Theorem 2.2: Let F_μ and δ_x be as in the proof of theorem 2.1. Also let $G_\varepsilon = (1-\varepsilon)F_\mu + \varepsilon\delta_x$. It is easy to check from the definition of $W(F)$ that

$$W(G_\varepsilon) = \sqrt{\rho^2(1-\varepsilon)^2 + 2\rho\varepsilon(1-\varepsilon)\cos(x-\mu) + \varepsilon^2} - (\rho(1-\varepsilon)\cos\mu + \varepsilon\cos x)$$

and $W(F_0) = 0$. Then straightforward calculations yield the PBF of W as

$$\varepsilon_\mu^*(W) = \inf \left\{ \varepsilon > 0 : \varepsilon \sin x + \rho(1-\varepsilon)\sin\mu = 0 \text{ for some } x, 0 \leq x < 2\pi \right\}.$$

Since $\varepsilon \sin x + \rho(1-\varepsilon)\sin\mu = 0$ has a solution in $x \in [0, 2\pi)$ if and only if $\frac{\rho|\sin\mu|}{1+\rho|\sin\mu|} < \varepsilon$, we

have, $\varepsilon_\mu^*(W) = \frac{\rho|\sin\mu|}{1+\rho|\sin\mu|}$. Further, the PBP is $\varepsilon^* = \sup_\mu (\varepsilon_\mu^*(W)) = \frac{\rho}{1+\rho}$.

Hence the theorem is established.

Proof of Theorem 3.1: Let F_μ , G_ε and δ_x be as in the proof of theorem 2.1. It is straightforward to check that $W_1(G_\varepsilon) = \arctan^* \left(\frac{\varepsilon \sin x}{\rho(1-\varepsilon) + \varepsilon \cos x} \right)$ and $W_1(F_\mu) = \mu$. Since $\arctan^* \left(\frac{\varepsilon \sin x}{\rho(1-\varepsilon) + \varepsilon \cos x} \right) = \mu$ has a solution in $x \in [0, 2\pi)$ if and only if $\frac{\rho|\sin \mu|}{1 + \rho|\sin \mu|} < \varepsilon$, we have the LBF of W_1 as $\varepsilon_\mu^{**}(W_1) = \frac{\rho|\sin \mu|}{1 + \rho|\sin \mu|}$. The LBP of W_1 can be easily computed to be $\varepsilon^{**} = \sup_\mu (\varepsilon_\mu^{**}(W_1)) = \frac{\rho}{1 + \rho}$.

Hence the theorem is established.

Proof of Theorem 3.2: Let F_μ and δ_x be as in the proof of theorem 2.1. Let G_ε be as in the proof of theorem 2.2. Note that it is straightforward to check that

$$W_1(G_\varepsilon) = \arctan^* \left(\frac{\rho(1-\varepsilon)\sin \mu + \varepsilon \sin x}{\rho(1-\varepsilon)\cos \mu + \varepsilon \cos x} \right) \text{ and } W_1(F_0) = 0.$$

Since $\arctan^* \left(\frac{\rho(1-\varepsilon)\sin \mu + \varepsilon \sin x}{\rho(1-\varepsilon)\cos \mu + \varepsilon \cos x} \right) = 0$ has a solution in $x \in [0, 2\pi)$ if and only if $\frac{\rho|\sin \mu|}{1 + \rho|\sin \mu|} < \varepsilon$, we have the PBF of W_1 as $\varepsilon_\mu^*(W_1) = \frac{\rho|\sin \mu|}{1 + \rho|\sin \mu|}$. Further, the PBP is

$$\varepsilon^* = \sup_\mu (\varepsilon_\mu^*(W_1)) = \frac{\rho}{1 + \rho}.$$

Hence the theorem is established.

Proof of Theorem 4.1: a) Let F_μ and δ_x be as in the proof of theorem 2.1. Let $G_\varepsilon = (1-\varepsilon)F_0 + \varepsilon\delta_x$, $x \in [-\pi, \pi)$ and $0 \leq \gamma < 0.5$. Then we can write

$$G_{\varepsilon}(\theta) = \begin{cases} (1-\varepsilon)F_0(\theta) & \text{if } -\pi \leq \theta < x \\ (1-\varepsilon)F_0(\theta) + \varepsilon & \text{if } \theta \leq x < \pi \end{cases}$$

where $F_0(\theta) = \frac{1}{2\pi I_0(\kappa)} \int_{-\pi}^{\theta} e^{\kappa \cos \varphi} d\varphi$. Suppose $\theta_1 = F_0^{-1}\left(\frac{\gamma}{1-\varepsilon}\right)$, $\theta_2 = F_0^{-1}\left(1 - \frac{\gamma}{1-\varepsilon}\right)$ and also note that since F_0 is symmetric about zero we have $\theta_1 = -\theta_2$.

Case 1: When $\theta_1 < x < \theta_2$

In this case we have, $E_{\gamma, (1-\varepsilon)F_0 + \varepsilon \delta_x}(\sin \Theta) = \frac{(1-\varepsilon)}{(1-2\gamma)} \int_{\theta_1}^{\theta_2} \sin \theta dF_0 + \frac{\varepsilon}{1-2\gamma} \sin x$ and

$E_{\gamma, (1-\varepsilon)F_0 + \varepsilon \delta_x}(\cos \Theta) = \frac{(1-\varepsilon)}{(1-2\gamma)} \int_{\theta_1}^{\theta_2} \cos \theta dF_0 + \frac{\varepsilon}{1-2\gamma} \cos x$. Using the above and after some

simplifications we get

$$W_{\gamma}(G_{\varepsilon}) = \arctan^* \left[\frac{\vartheta_{\gamma,0}(1-\varepsilon) \tan \mu_{\gamma,0} + \varepsilon \sin x}{\vartheta_{\gamma,0}(1-\varepsilon) + \varepsilon \cos x} \right].$$

Let $\varepsilon < \min(\gamma, 1-\gamma)$, $\lambda = F_0^{-1}\left(\frac{\gamma-\varepsilon}{1-\varepsilon}\right)$, and $\psi = F_0^{-1}\left(\frac{1-\gamma}{1-\varepsilon}\right)$. We define the following,

$$\tilde{E}_{\gamma, F_0}(\sin \Theta) = \frac{1}{1-2\gamma} \int_{\lambda}^{\theta_2} \sin \theta dF_0 \text{ and } \check{E}_{\gamma, F_0}(\sin \Theta) = \frac{1}{1-2\gamma} \int_{\theta_1}^{\psi} \sin \theta dF_0.$$

Case 2: When $x \leq \theta_1$

In this case we have, $E_{\gamma, (1-\varepsilon)F_0 + \varepsilon \delta_x}(\sin \Theta) = (1-\varepsilon)\tilde{E}_{\gamma, F_0}(\sin \Theta) = \frac{1-\varepsilon}{1-2\gamma} \int_{\lambda}^{\theta_2} \sin \theta dF_0$ and

$E_{\gamma, (1-\varepsilon)F_0 + \varepsilon \delta_x}(\cos \Theta) = (1-\varepsilon)\tilde{E}_{\gamma, F_0}(\cos \Theta) = \frac{1-\varepsilon}{1-2\gamma} \int_{\lambda}^{\theta_2} \cos \theta dF_0$. Therefore, we get

$$W_{\gamma}(G_{\varepsilon}) = \arctan^* \left[\frac{\tilde{E}_{\gamma, F_0}(\sin \Theta)}{\tilde{E}_{\gamma, F_0}(\cos \Theta)} \right].$$

Case 3: When $x \geq \theta_2$

In this case we have, $E_{\gamma, (1-\varepsilon)F_0 + \varepsilon \delta_x}(\sin \Theta) = (1-\varepsilon)\tilde{E}_{\gamma, F_0}(\sin \Theta) = \frac{1-\varepsilon}{1-2\gamma} \int_{\theta_1}^{\Psi} \sin \theta dF_0$ and

$E_{\gamma, (1-\varepsilon)F_0 + \varepsilon \delta_x}(\cos \Theta) = (1-\varepsilon)\tilde{E}_{\gamma, F_0}(\cos \Theta) = \frac{1-\varepsilon}{1-2\gamma} \int_{\theta_1}^{\Psi} \cos \theta dF_0$. Therefore, we get

$$W_{\gamma}(G_{\varepsilon}) = \arctan^* \left[\frac{\tilde{E}_{\gamma, F_0}(\sin \Theta)}{\tilde{E}_{\gamma, F_0}(\cos \Theta)} \right].$$

Combining the above three cases we have

$$W_{\gamma}(G_{\varepsilon}) = \begin{cases} \arctan^* \left[\frac{\vartheta_{\gamma, 0}(1-\varepsilon) \tan \mu_{\gamma, 0} + \varepsilon \sin x}{\vartheta_{\gamma, 0}(1-\varepsilon) + \varepsilon \cos x} \right], & \theta_1 < x < \theta_2 \\ \arctan^* \left[\frac{\tilde{E}_{\gamma, F_0}(\sin \Theta)}{\tilde{E}_{\gamma, F_0}(\cos \Theta)} \right], & x \leq \theta_1 \\ \arctan^* \left[\frac{\tilde{E}_{\gamma, F_0}(\sin \Theta)}{\tilde{E}_{\gamma, F_0}(\cos \Theta)} \right], & x \geq \theta_2. \end{cases}$$

Using Lemma 1 and Lemma 2 we get

$$\begin{aligned} \varepsilon_{\mu, \gamma}^{**}(W_{\gamma}) &= \inf \left\{ \varepsilon > 0 : W_{\gamma}(G_{\varepsilon}) = \mu \text{ for some } x \in [-\pi, \pi) \right\} \\ &= \inf \left\{ \varepsilon > 0 : \frac{\varepsilon \sin x}{\vartheta_{\gamma, 0}(1-\varepsilon) + \varepsilon \cos x} = \tan \mu \text{ for some } x \in (\theta_1, \theta_2) \right\} \quad \dots (4.1) \\ &= \inf \left\{ \varepsilon > 0 : \varepsilon = \frac{\vartheta_{\gamma, 0}(1-\varepsilon) \sin \mu}{\sin(x-\mu) + \vartheta_{\gamma, 0} \sin \mu} \text{ for some } x \in (\theta_1, \theta_2) \right\}. \end{aligned}$$

Since $\sin(x-\mu) \leq 1$, we have $\varepsilon_{\mu, \gamma}^{**}(W_{\gamma}) \geq \frac{\vartheta_{\gamma, 0} |\sin \mu|}{1 + \vartheta_{\gamma, 0} |\sin \mu|}$ for $0 \leq \mu \leq \pi$.

b) Now, from (4.1) we get $x = \mu + \sin^{-1}(\Lambda)$ where $\Lambda = \varepsilon^{-1} \vartheta_{\gamma,0} (1 - \varepsilon) \sin \mu$ has a solution in $x \in (\theta_1, \theta_2)$ if and only if $|\Lambda| < 1$. We define the following quantities:

$k_1(\mu, \varepsilon) = \sup\{\sin(x - \mu) : x \in (\theta_1, \theta_2)\} < 1$ and $k_2(\mu, \varepsilon) = \inf\{\sin(x - \mu) : x \in (\theta_1, \theta_2)\} > -1$ such that $k_2(\mu, \varepsilon) < k_1(\mu, \varepsilon)$. Let $\tau_1 = k_1(\mu, \varepsilon) + \vartheta_{\gamma,0} |\sin \mu|$ and $\tau_2 = k_2(\mu, \varepsilon) + \vartheta_{\gamma,0} |\sin \mu|$.

Then $x = \mu + \sin^{-1}(\Lambda)$ has a solution in $x \in (\theta_1, \theta_2)$ if and only if

$$\frac{\vartheta_{\gamma,0} |\sin \mu|}{\tau_1} < \varepsilon < \frac{\vartheta_{\gamma,0} |\sin \mu|}{\tau_2} \text{ when } \tau_1, \tau_2 > 0,$$

$$\frac{\vartheta_{\gamma,0} |\sin \mu|}{\tau_1} < \varepsilon \text{ and } \varepsilon < \frac{-\vartheta_{\gamma,0} |\sin \mu|}{\tau_2} \text{ when } \tau_1 > 0, \tau_2 < 0 \text{ and}$$

$$\frac{\vartheta_{\gamma,0} |\sin \mu|}{\tau_2} < \varepsilon < \frac{\vartheta_{\gamma,0} |\sin \mu|}{\tau_1} \text{ when } \tau_1, \tau_2 < 0.$$

Therefore, $\varepsilon_{\mu,\gamma}^{**}(W_\gamma) \geq \frac{\vartheta_{\gamma,0} |\sin \mu|}{k_1(\mu) + \vartheta_{\gamma,0} |\sin \mu|}$.

c) Further, the LBP of W_γ satisfies $\varepsilon^{**} \geq \frac{\vartheta_{\gamma,0}}{1 + \vartheta_{\gamma,0}}$.

Hence the theorem is established.

Proof of Theorem 4.2: Let F_μ and δ_x be as in the proof of theorem 2.1. Further, for $0 \leq \gamma < 0.5$ let $G_\varepsilon = (1 - \varepsilon)F_\mu + \varepsilon\delta_x$, $x \in [\mu - \pi, \mu + \pi)$. Then

$$G_\varepsilon(\theta) = \begin{cases} (1 - \varepsilon)F_\mu(\theta) & \text{if } \mu - \pi \leq \theta < x \\ (1 - \varepsilon)F_\mu(\theta) + \varepsilon & \text{if } \theta \leq x < \mu + \pi \end{cases}$$

where $F_\mu(\theta) = \frac{1}{2\pi I_0(\kappa)} \int_{\mu-\pi}^{\theta} e^{\kappa \cos \varphi} d\varphi = F_0(\theta - \mu)$. Now, when $c_1(\mu) < x < c_2(\mu)$, we have

$$W_\gamma(G_\varepsilon) = \arctan^* \left[\frac{E_{\gamma, G_\varepsilon}(\sin \Theta)}{E_{\gamma, G_\varepsilon}(\cos \Theta)} \right] \text{ and } W_\gamma(F_\mu) = \mu.$$

Note that $\mu = 0$ under H_0 giving $W_\gamma(F_0) = 0$ and hence

$$W_\gamma(G_\varepsilon) = 0 \Rightarrow \int_{c_1(\mu)}^{c_2(\mu)} \sin \theta dG_\varepsilon(\theta) = 0 \Rightarrow (1-\varepsilon)S_\gamma + \varepsilon \sin x = 0$$

$$\text{where } S_\gamma = \int_{c_1(\mu)}^{c_2(\mu)} \sin \theta f_\mu(\theta) d\theta = \cos \mu \int_{v_1}^{v_2} \sin v f_0(v) dv + \sin \mu \int_{v_1}^{v_2} \cos v f_0(v) dv, v = \theta - \mu,$$

$v_1 = F_0^{-1}(\gamma)$ and $v_2 = F_0^{-1}(1-\gamma)$. Since f_0 is symmetric about zero, $v_1 = -v_2$, and $\sin \theta$ is odd function we have

$$S_\gamma = 2 \sin \mu \int_0^{v_2} \cos v f_0(v) dv = 2 C_\gamma \sin \mu = \lambda_\mu.$$

By Lemma 3 we have $C_\gamma > 0$. Therefore, we get

$$\begin{aligned} \varepsilon_{\mu, \gamma}^*(W_\gamma) &= \inf \{ \varepsilon > 0 : W_\gamma(G_\varepsilon) = 0 \text{ for some } x \in (c_1(\mu), c_2(\mu)) \} \\ &= \inf \{ \varepsilon > 0 : (1-\varepsilon)\lambda_\mu + \varepsilon \sin x = 0 \text{ for some } x \in (c_1(\mu), c_2(\mu)) \}. \end{aligned}$$

Now, $(1-\varepsilon)\lambda_\mu + \varepsilon \sin x = 0 \Rightarrow x = \sin^{-1} \left(\frac{-\lambda_\mu(1-\varepsilon)}{\varepsilon} \right) = \sin^{-1}(\Delta)$ where $\Delta = \frac{-\lambda_\mu(1-\varepsilon)}{\varepsilon}$. Then

the equation has a solution in x if and only if $\psi_\mu \leq \Delta \leq \phi_\mu$. This yield:

$$\begin{aligned} \max(0, \min\{\zeta_\mu, 1\}) &\leq \varepsilon \leq \max(0, \min\{\xi_\mu, 1\}) \text{ if } \lambda_\mu - \phi_\mu > 0 \\ \min(1, \max\{\zeta_\mu, \xi_\mu, 0\}) &\leq \varepsilon \text{ if } \lambda_\mu - \phi_\mu < 0 \text{ and } \lambda_\mu - \psi_\mu > 0 \\ \max(0, \min\{\xi_\mu, 1\}) &\leq \varepsilon \leq \max(0, \min\{\zeta_\mu, 1\}) \text{ if } \lambda_\mu - \psi < 0. \end{aligned}$$

Now, when $x < c_1(\mu)$ and $x > c_2(\mu)$, we have $W_\gamma(G_\varepsilon) = (1-\varepsilon)\lambda_\mu \neq 0$. Since W_γ does not involve x , for any ε , there exist no solution for which $x < c_1(\mu)$ and $x > c_2(\mu)$.

Noting that $-2C_\gamma \leq \lambda_\mu \leq 2C_\gamma$, we have

$$\varepsilon_{\mu,\gamma}^*(W_\gamma) = \begin{cases} \max(0, \min\{\xi_\mu, 1\}) & \text{if } 2C_\gamma \geq \lambda_\mu \geq \phi_\mu \\ \min(1, \max\{\xi_\mu, \xi_\mu, 0\}) & \text{if } \psi_\mu \leq \lambda_\mu \leq \phi_\mu \\ \max(0, \min\{\xi_\mu, 1\}) & \text{if } -2C_\gamma \leq \lambda_\mu \leq \psi_\mu. \end{cases}$$

Hence the theorem is established.

Proof of Theorem 6.1: a) Let F_μ denote the $WN(\mu, \rho)$ distribution and δ_x denote the point mass at x . Let G_ε be as in the proof of theorem 2.1. We note that, $E_{(1-\varepsilon)F_0 + \varepsilon\delta_x}(\cos \Theta) = y$ and $E_{(1-\varepsilon)F_0 + \varepsilon\delta_x}(\sin \Theta) = \varepsilon \sin x$. Using these we get

$$W(G_\varepsilon) = \sqrt{y^2 + \varepsilon^2 \sin^2 x} - y.$$

Again, an easy computation yields $W(F_\mu) = c$. The rest of the proof is similar to theorem 2.1.

b) Let F_μ and δ_x be as in the proof of theorem 6.1. Let G_ε be as in the proof of theorem 2.2. It is easy to check from the definition of $W(F)$ that

$$W(G_\varepsilon) = \sqrt{\rho^2(1-\varepsilon)^2 + 2\rho\varepsilon(1-\varepsilon)\cos(x-\mu) + \varepsilon^2} - (\rho(1-\varepsilon)\cos\mu + \varepsilon\cos x)$$

and $W(F_0) = 0$. The rest of the proof is similar to theorem 2.2.

Proof of Theorem 6.2: a) Let F_μ and δ_x be as in the proof of theorem 6.1. Let G_ε be as in the proof of theorem 2.1. It is straightforward to check that