DECIDABLE CLASSES OF
NUMBER THEORETIC SENTENCES

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Thesis for Ph.D. - Leicester, 1968.

UMI Number: U309686

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Preface

The decision problem for number theory, formulated by Hilbert, consists in finding a decision procedure (i.e. a completely mechanical finite procedure) for deciding every number theoretic sentence. In 1936, Church showed this to be impossible, and since then mathematicians have pursued the task of finding interesting decidable classes of number theoretic sentences, where by a decidable class we mean a class with a decision procedure. Attempts have been made along many different lines which have yielded many highly varied decidable classes.

In [1], Goodstein and Lee have given a decidable class of equations, and in [2], Rousseau has shown the same class to be decidable by quite different means (leaving open the question of the relationship between the two different approaches).

In Part I, we give a general method for constructing decidable classes, and in Part II, we apply this method to construct particular decidable classes. In the first chapter of the second part we construct a decidable class, which is an extension of the class of Goodstein and Lee, and the connection between the two cited papers is made apparent.

I would like to thank my thesis advisor, Professor R. L. Goodstein, for his generous assistance and kind encouragement during the preparation of this thesis.

PART I

METHOD OF CONSTRUCTION

Chapter 1

LIMITED CONSTITUENTS

§1. Terminology

Let N denote the class of natural numbers $0, 1, 2, \dots$.

We say that $P(x_1, \dots, x_n)$ is a predicate if $P: N^n \rightarrow \{0, 1\}$

and that $f(x_1, \dots, x_n)$ is a function if $f: N^n \rightarrow N$.

(N.B. For any $a_1, \dots, a_n$, we say that $P(a_1, \dots, a_n)$ is true if $P(a_1, \dots, a_n) = 0$, and that $P(a_1, \dots, a_n)$ is false if $P(a_1, \dots, a_n) = 1$.)

We find it expedient to treat predicates and functions together under the single term constituents. We usually denote a set of constituents in the form: $P, Q, R, \dots; f, g, h, \dots$, where

$P, Q, R, \dots$ are the predicates in the set

and $f, g, h, \dots$ are the functions in the set.

Suppose for $i = 0, 1, \dots, k$, that $A_i(x_1, \dots, x_n)$ is a map with domain N^n , the range of which is not restricted to N , but may be a class of sets of even constituents. Then, we denote the class of all $A_0(x_1, \dots, x_n), \dots, A_k(x_1, \dots, x_n)$, such that $x_1, \dots, x_n$ satisfy the predicate $P(x_1, \dots, x_n)$, by

$$\{A_0(x_1, \dots, x_n), \dots, A_k(x_1, \dots, x_n) : P(x_1, \dots, x_n)\}.$$

§2. Limiters and their Representative Functions

Let $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \dots$ be an increasing sequence of finite Boolean algebras of subsets of N , such that, for all $\ell > 0$, we have $\{\ell-1\} \in \mathcal{B}_\ell$. The $\mathcal{B}_\ell$ are called limiters. Let $B_\ell x$ denote $\bigcap \{B \in \mathcal{B}_\ell : x \in B\}$,

and let $\beta_\ell x$ be any function which selects a unique element of $B_\ell x$, i.e. for all x and y , we have

$$\beta_\ell x \in B_\ell x \quad \text{and} \quad B_\ell x = B_\ell y \rightarrow \beta_\ell x = \beta_\ell y.$$

Then, $\beta_\ell x$ is called a representative function for ℓ .

In particular, if $\min A$ denotes the least member of A , then $\min B_\ell x$ is a representative function for B_ℓ .

Thus, if $B \in B_\ell$, we have $x \in B \rightarrow \beta_\ell x \in B$, and, since $-B$ (i.e. the complement of B in N) belongs to B_ℓ , we also have $x \in -B \rightarrow \beta_\ell x \in -B$, and so $x \in B \leftrightarrow \beta_\ell x \in B$. Hence, $B_\ell x = B_\ell \beta_\ell x$, giving $B_\ell x = B_\ell y \leftrightarrow \beta_\ell x = \beta_\ell y$. Therefore, $\beta_\ell^{-1} \beta_\ell x = \{y : \beta_\ell x = \beta_\ell y\} = \{y : B_\ell x = B_\ell y\} = B_\ell x$, and so $\beta_\ell^{-1} : N \rightarrow B_\ell$, giving $B = \beta_\ell^{-1} B \rightarrow B \in B_\ell$. Thus,

$$B \in B_\ell \text{ if and only if } \forall x (x \in B \leftrightarrow \beta_\ell x \in B).$$

Theorem $\beta_i \beta_j = \beta_i$ if $i \leq j$.

Proof

$$\begin{aligned} B_i \subset B_j &\rightarrow (\forall B \in B_i)(B \in B_j) \\ &\rightarrow (\forall B \in B_i) \forall x (\beta_j x \in B \leftrightarrow x \in B) \\ &\rightarrow \forall x (\forall B \in B_i) (\beta_j x \in B \leftrightarrow x \in B) \\ &\rightarrow \forall x (B_i \beta_j x = B_i x) \\ &\rightarrow \forall x (\beta_i \beta_j x = \beta_i x), \end{aligned}$$

which completes the proof.

Corollary 1 β_i is a projection, i.e. $\beta_i \beta_i = \beta_i$.

Corollary 2. $\beta_i \beta_j \beta_i = \beta_i \beta_j$.

Let $\alpha_\ell x$ denote $\min(\ell, x)$, then $\alpha_\ell \beta_\ell = \alpha_\ell$, since

$$\beta_\ell x = x \text{ if } x < \ell, \text{ and } \beta_\ell x \geq \ell \text{ otherwise.}$$

Let b_ℓ denote $\max_x \min B_\ell x$ (this exists since $B_\ell x$ has a finite range).

§3. Limited Predicates

For any predicate $P(x_1, \dots, x_n)$ we let $\bar{P}$ denote

$$\{(x_1, \dots, x_n) : P(x_1, \dots, x_n)\}.$$

We say that ℓ limits Px if and only if $\bar{P} \in \mathcal{B}_\ell$.

Thus, ℓ limits Px if and only if $\forall x (Px \leftrightarrow P\beta_\ell x)$, and so

$$\ell \text{ limits } Px \text{ if and only if } \forall x, y (\beta_\ell x = \beta_\ell y \rightarrow Px \leftrightarrow Py),$$

since β_ℓ is a projection. Hence, if ℓ limits Px , then any k such that $k \geq \ell$, limits Px .

If $P(x_1, \dots, x_n)$ is a predicate, then the components of P are

$$\{P(x, a_2, a_3, \dots, a_n), P(a_1, x, a_3, a_4, \dots, a_n), \dots, P(a_1, a_2, \dots, a_{n-1}, x) : a_1, \dots, a_n \in \mathbb{N}\}.$$

We say that ℓ limits $P(x_1, \dots, x_n)$ if and only if ℓ limits every

component of P . Thus, ℓ limits $P(x_1, \dots, x_n)$ if and only if

$$\forall x_1, \dots, x_n (P(x_1, \dots, x_n) \leftrightarrow P(\beta_\ell x_1, \dots, \beta_\ell x_n)).$$

Hence, if ℓ limits $P(x_1, \dots, x_n)$, then any k , such that $k \geq \ell$, limits $P(x_1, \dots, x_n)$.

Let $\mathcal{B}_\ell^n$ be the Boolean algebra generated by $\{B_1 \times \dots \times B_n : B_1, \dots, B_n \in \mathcal{B}_\ell\}$.

(N.B. $\mathcal{B}_\ell^1 = \mathcal{B}_\ell$, for all ℓ .)

Theorem ℓ limits $P(x_1, \dots, x_n)$ if and only if $\bar{P} \in \mathcal{B}_\ell^n$.

Proof Suppose ℓ limits $P(x_1, \dots, x_n)$. Then

$$\begin{aligned}\bar{P} &= \{(x_1, \dots, x_n) : P(\beta_\ell x_1, \dots, \beta_\ell x_n)\} \\ &= \cup \{ \beta_\ell^{-1} x_1 \times \dots \times \beta_\ell^{-1} x_n : x_1, \dots, x_n \leq b_\ell, \text{ and } P(x_1, \dots, x_n) \}.\end{aligned}$$

Thus, $\bar{P}$ belongs to the Boolean algebra generated by

$$\{ \beta_\ell^{-1} x_1 \times \dots \times \beta_\ell^{-1} x_n : x_1, \dots, x_n \leq b_\ell \}.$$

Now, $\beta_\ell^{-1} x \in \mathcal{B}_\ell$, since β_ℓ is a projection, and so $\bar{P} \in \mathcal{B}_\ell^n$.

Conversely, we suppose $\bar{P} \in \mathcal{B}_\ell^n$. Then, $\{x : P(x, a_2, a_3, \dots, a_n)\} \in \mathcal{B}_\ell$, by induction on the construction of $\bar{P}$, since

$$\bar{P} = B_1 \times \dots \times B_n \rightarrow \{x : P(x, a_2, a_3, \dots, a_n)\} = B_1 \text{ if } a_2 \in B_2, \dots, a_n \in B_n,$$

otherwise it is the empty set,

$$\bar{P} = \bar{Q} \cap \bar{R} \rightarrow \{x : P(x, a_2, a_3, \dots, a_n)\} = \{x : Q(x, a_2, a_3, \dots, a_n)\} \cap \{x : R(x, a_2, a_3, \dots, a_n)\}$$

and

$$\bar{P} = \neg \bar{Q} \rightarrow \{x : P(x, a_2, a_3, \dots, a_n)\} = \neg \{x : Q(x, a_2, a_3, \dots, a_n)\}.$$

Thus, ℓ limits $P(x, a_2, a_3, \dots, a_n)$. Similarly, ℓ limits every component of P , and so ℓ limits $P(x_1, \dots, x_n)$. Thus, we have the theorem.

Corollary $P(x_1, \dots, x_n)$ is limited if and only if $\bar{P} \in \mathcal{B}^n$, where

$$\mathcal{B}^n \text{ denotes } \bigcup_{r \geq 0} \mathcal{B}_r^n$$

Remark: It is clear from the proof of the theorem that $\mathcal{B}_\ell^n$ is the class of finite unions of $\{B_1 \times \dots \times B_n : B_1, \dots, B_n \in \mathcal{B}_\ell\}$.

§4. Limited Functions

We say that a function p_i limits $f x$ if and only if

$$f^{-1} : \mathcal{B}_i \rightarrow \mathcal{B}_{p_i} \text{ for all } i.$$

Theorem π_i limits fx if and only if $\forall x(\beta_i fx = \beta_i f\beta_{\pi_i} x)$ for all i .

Proof π_i limits $fx \iff f^{-1} : \mathcal{B}_i \rightarrow \mathcal{B}_{\pi_i}$
 $\iff (\forall B \in \mathcal{B}_i)(f^{-1} B \in \mathcal{B}_{\pi_i})$
 $\iff (\forall B \in \mathcal{B}_i) \forall x(x \in f^{-1} B \iff \beta_{\pi_i} x \in f^{-1} B)$
 $\iff (\forall B \in \mathcal{B}_i) \forall x(fx \in B \iff f\beta_{\pi_i} x \in B)$
 $\iff \forall x(\forall B \in \mathcal{B}_i)(fx \in B \iff f\beta_{\pi_i} x \in B)$
 $\iff \forall x(\beta_i fx = \beta_i f\beta_{\pi_i} x),$

which complete the proof.

Corollary. π_i limits fx if and only if $\forall x, y(\beta_{\pi_i} x = \beta_{\pi_i} y \rightarrow \beta_i fx = \beta_i fy)$ for all i .

Hence, if π_i limits fx , then any function q_i , such that $q_i \geq \pi_i$ for all i , limits fx .

For a function $f(x_1, \dots, x_n)$ the components of f are

$$\{f(x, a_2, a_3, \dots, a_n), f(a_1, x, a_3, a_4, \dots, a_n), \dots, f(a_1, a_2, \dots, a_{n-1}, x) : a_1, \dots, a_n \in N\}.$$

We say that π_i limits $f(x_1, \dots, x_n)$ if and only if π_i limits every component of f .

Thus, π_i limits $f(x_1, \dots, x_n)$ if and only if

$$\forall x_1, \dots, x_n(\beta_i f(x_1, \dots, x_n) = \beta_i f(\beta_{\pi_i} x_1, \dots, \beta_{\pi_i} x_n)) \text{ for all } i.$$

Hence, if π_i limits $f(x_1, \dots, x_n)$, then any q_i , such that $q_i \geq \pi_i$ for all i , limits $f(x_1, \dots, x_n)$.

Theorem f pi limits $f(x_1, \dots, x_n)$ if and only if
 $f^{-1} : \mathcal{B}_l \rightarrow \mathcal{B}_{pl}^n$ for all l .

Proof

Now, pi limits $f(x_1, a_2, a_3, \dots, a_n)$

$$\leftrightarrow (\forall B \in \mathcal{B}_l)(\text{pi limits}(f(x_1, a_2, a_3, \dots, a_n) \in B))$$

Thus, pi limits $f(x_1, \dots, x_n) \leftrightarrow (\forall B \in \mathcal{B}_l)(\text{pi limits}(f(x_1, \dots, x_n) \in B))$ etc.

$$\leftrightarrow (\forall B \in \mathcal{B}_l)(f^{-1} B \in \mathcal{B}_{pl}^n)$$

$$\leftrightarrow f^{-1} : \mathcal{B}_l \rightarrow \mathcal{B}_{pl}^n,$$

which completes the proof.

Corollary 1. $f(x_1, \dots, x_n)$ is limited if and only if
 $f^{-1} : \mathcal{B} \rightarrow \mathcal{B}^n$, where $\mathcal{B}$ denotes $\bigcup_{r \geq 0} \mathcal{B}_r$.

Proof $f(x_1, \dots, x_n)$ is limited $\leftrightarrow \forall l \exists m (f^{-1} : \mathcal{B}_l \rightarrow \mathcal{B}_m^n)$

$$\leftrightarrow \forall l (f^{-1} : \mathcal{B}_l \rightarrow \mathcal{B}^n), \text{ since } \mathcal{B}_l \text{ is finite.}$$

$$\leftrightarrow f^{-1} : \mathcal{B} \rightarrow \mathcal{B}^n,$$

which completes the proof.

Corollary 2. If ℓ limits the predicate $(f(x_1, \dots, x_n) = a)$, for all a ,
then the constant function ℓ limits $f(x_1, \dots, x_n)$.

Proof The corollary follows by the theorem, since

$$\text{if } B \in \mathcal{B}_\ell, \text{ then } f^{-1} B = \bigcup_{a \in B} f^{-1} a \in \mathcal{B}_\ell^n.$$

§5. Closure Properties of the Class of Limited Constituents

Theorem The class of limited constituents is closed under substitution.

Proof For all $j=1, \dots, n$, i limits $U_j^n(x_1, \dots, x_n)$, where $U_j^n(x_1, \dots, x_n)$ denotes the projection function with value x_j , since β_i is a projection. If a limits $P(x_1, \dots, x_n)$, and q_i limits $g_0(y_1, \dots, y_m), \dots, g_n(y_1, \dots, y_m)$, then q_a limits $P(g_0(y_1, \dots, y_m), \dots, g_n(y_1, \dots, y_m))$. Thus a substitution of a limited function in a limited predicate yields a limited predicate. If p_i limits $f(x_1, \dots, x_n)$, and q_i limits $g_0(y_1, \dots, y_m), \dots, g_n(y_1, \dots, y_m)$, then q_{p_i} limits $f(g_0(y_1, \dots, y_m), \dots, g_n(y_1, \dots, y_m))$. Thus, substitution of a limited function in a limited function yields a limited function, and the theorem follows.

Theorem The class of limited predicates is closed under the usual logical operations.

Proof If a limits $P(x_1, \dots, x_n)$, and b limits $Q(y_1, \dots, y_m)$, then

$$\max(a, b) \text{ limits } P(x_1, \dots, x_n) \wedge Q(y_1, \dots, y_m),$$

$$\max(a, b) \text{ limits } P(x_1, \dots, x_n) \vee Q(y_1, \dots, y_m),$$

and a limits $\neg P(x_1, \dots, x_n), \forall x_1 P(x_1, \dots, x_n), \exists x_1 P(x_1, \dots, x_n)$, and the theorem follows.

Theorem The class of limited predicates is closed under bounded quantification, i.e. under $(\exists x \leq y)$ and $(\forall x \leq y)$, where

$(\exists x_1 \leq y)P(x_1, x_2, \dots, x_n)$ denotes $\exists x_1 (x_1 \leq y \wedge P(x_1, x_2, \dots, x_n))$
and $(\forall x_1 \leq y)P(x_1, x_2, \dots, x_n)$ denotes $\forall x_1 (x_1 \leq y \rightarrow P(x_1, x_2, \dots, x_n))$

Proof Suppose a limits $P(x_1, \dots, x_n)$.

Then, $P(x_1, \dots, x_n) \leftrightarrow P(\beta_a x_1, \dots, \beta_a x_n)$

and so, $(\exists x_1 \leq y)P(x_1, \dots, x_n) \leftrightarrow (\exists x_1 \leq \alpha_{b_a} y)P(\beta_a x_1, \dots, \beta_a x_n)$.

Thus, $(\exists x_1 \leq y)P(x_1, \dots, x_n) \leftrightarrow (\exists x_1 \leq \beta_{b_a} y)P(\beta_a x_1, \dots, \beta_a x_n)$,

$$\text{since } \alpha_{b_a} = \alpha_{b_a} \beta_{b_a},$$

and so, $(\exists x_1 \leq y)P(x_1, \dots, x_n) \leftrightarrow (\exists x_1 \leq \beta_{b_a} y)P(x_1, \dots, x_n)$.

Hence, b_a limits $(\exists x_1 \leq y)P(x_1, \dots, x_n)$, since $a \leq b_a$.

Similarly, b_a limits $(\forall x_1 \leq y)P(x_1, \dots, x_n)$, and the theorem follows.

Theorem The class of limited constituents is closed under bounded

and unbounded μ -operations, i.e. under $(\mu x \leq y)$ and μx , where

$\mu x_1 P(x_1, x_2, \dots, x_n)$ denotes the least x_1 such that $P(x_1, x_2, \dots, x_n)$

if there exists such an x_1 , otherwise it denotes 0,

and $(\mu x_1 \leq y)P(x_1, x_2, \dots, x_n)$ denotes $\mu x_1 (x_1 \leq y \wedge P(x_1, x_2, \dots, x_n))$.

Proof If a limits $P(x_1, \dots, x_n)$, then the constant function

a limits $\mu x_1 P(x_1, \dots, x_n)$.

Thus, the limited constituents are closed under the unbounded μ -operation.

Moreover, if a limits $P(x_1, \dots, x_n)$, then as above

$$(\exists x_1 \leq y)P(x_1, \dots, x_n) \leftrightarrow (\exists x_1 \leq \beta_{b_a} y)P(x_1, \dots, x_n)$$

and so, $(\mu x_1 \leq y)P(x_1, \dots, x_n) \leftrightarrow (\mu x_1 \leq \beta_{b_a} y)P(x_1, \dots, x_n)$.

Hence, the constant function b_a limits $(\mu x_1 \leq y)P(x_1, \dots, x_n)$, since $a \leq b_a$.

Thus, the limited constituents are closed under the bounded μ -operation,
which completes the proof.

§6. Constituents which are Always Limited

For any limiters, we have $\{0\}, \dots, \{l-1\} \in \mathcal{B}_\ell$, and so,
for all $a \in \mathbb{N}$, we have:

$a + 1$ limits the predicates $x < a$, $x = a$, $x > a$, and $x|a$

i limits the identity function x , since β_i is

a projection

the constant function 0 limits the constant function a

$a + i$ limits $\max(a, x)$, since $\beta_i \beta_{i+a} = \beta_i$

and

the constant function a limits $\min(a, x)$, $a \dot{-} x$, since $\alpha_a \beta_a = \alpha_a$, where

$x \dot{-} y$ denotes $\max\{x - y, 0\}$.

Thus, the constituents $\{x < a, x = a, x > a, x|a, a, x, \min(a, x),$
 $\max(a, x), a \dot{-} x : a \in \mathbb{N}\}$ are always limited.

§7. Constituents which are Never Limited

Clearly, any predicate with an essentially infinite set of components
cannot be limited, because otherwise some $\mathcal{B}_\ell$ would be infinite.

Examples of predicates with an essentially infinite set of components
are $x = y$, $x \leq y$, $x < y$, $x|y$, and "x and y are coprime". An example
of a predicate of more than one variable with an essentially finite set
of components is $x \equiv y \pmod{a}$, for any $a \in \mathbb{N}$.

Moreover, functions which yield predicates with an infinite set of components on substitution in limited predicates, such as $x = 0$, are clearly unlimited. Examples of such functions are $|x, y|$, $x \div y$, $x \cap y$, and $R_x y$, where

$x \cap y$ denotes the highest common factor of x and y ,
and $R_x y$ denotes the remainder of y upon division by x .

It will emerge later that $x \cup y$, i.e. the least common multiple of x and y , may be limited. Thus, we have the rather interesting fact that $x \cap y$ yields a predicate with an infinite set of components on substitution in some limited predicate, whereas $x \cup y$ does not.

Chapter 2

DECIDABILITY OF THE CLASS OF SENTENCES WITH A GIVEN EFFECTIVELY LIMITED SET OF RECURSIVE CONSTITUENTS, IF

b_{ℓ} IS RECURSIVE

§1. Terminology

We say that ξ effectively limits a set of constituents $\mathcal{L}$ if and only if ξ is a computable map with domain $\mathcal{L}$, such that for every constituent C in $\mathcal{L}$, we have $\xi(C)$ limits C , and if C is a function then $\xi(C)$ is recursive.

If $\mathcal{L}$ is a set of constituents, we let $\mathcal{L}^*$ denote the class of all constituents, constants, and sentences, generated from $\mathcal{L} \cup \{x = 0, x = 1, x = 2, \dots\}$, by substitution, the propositional operations, bounded and unbounded quantification, and the bounded and unbounded μ -operations. More precisely, $\mathcal{L}^*$ denotes the smallest class containing $\mathcal{L} \cup \{x = 0, x = 1, x = 2, \dots\} \cup \{U_j^n : j, n \in \mathbb{N}, \text{ and } 1 \leq j \leq n\}$, where $U_j^n(x_1, \dots, x_n)$ denotes the projection function with value x_j , such that, if the predicates $P(x_1, \dots, x_n), Q(y_1, \dots, y_m)$, and the functions $f(x_1, \dots, x_n), g_1(y_1, \dots, y_m), \dots, g_n(y_1, \dots, y_m)$ belong to $\mathcal{L}^*$, then the following constituents belong to $\mathcal{L}^*$:

$$P(g_1(y_1, \dots, y_m), \dots, g_n(y_1, \dots, y_m))$$

$$f(g_1(y_1, \dots, y_m), \dots, g_n(y_1, \dots, y_m))$$

$$P(x_1, \dots, x_n) \wedge Q(y_1, \dots, y_m)$$

$$P(x_1, \dots, x_n) \vee Q(y_1, \dots, y_m)$$

$$\neg P(x_1, \dots, x_n)$$

$$\begin{aligned}
& \exists x_1 P(x_1, x_2, \dots, x_n) \\
& \forall x_1 P(x_1, x_2, \dots, x_n) \\
& \mu x_1 P(x_1, x_2, \dots, x_n) \\
& (\exists x_1 \leq y) P(x_1, x_2, \dots, x_n) \\
& (\forall x_1 \leq y) P(x_1, x_2, \dots, x_n) \\
& \text{and } (\mu x_1 \leq y) P(x_1, x_2, \dots, x_n).
\end{aligned}$$

We call the sentences in $\mathcal{L}^*$, sentences with constituents $\mathcal{L}$.

§2. Decision Procedure

Suppose that $\mathcal{L}$ is a given set of recursive constituents and that ξ effectively limits $\mathcal{L}$. Below, we define by induction, a map ψ , which effectively limits the constituents in $\mathcal{L}^*$.

$$\psi(C) = \xi(C), \text{ for all } C \in \mathcal{L}.$$

$$\psi(x = a) = a + 1, \text{ for all } a.$$

$$\psi(U_j^n(x_1, \dots, x_n)) = 1, \text{ for all } n \text{ and } j, \text{ such that } j \leq n.$$

For all predicates $P(x_1, \dots, x_n), Q(y_1, \dots, y_m)$, and all functions $f(x_1, \dots, x_n), g_1(y_1, \dots, y_m), \dots, g_n(y_1, \dots, y_m)$ in $\mathcal{L}^*$, we let

$$\psi(P(g_1, \dots, g_n)) = \max(\psi(g_1)(\psi(P)), \dots, \psi(g_n)(\psi(P)))$$

$$\psi(f(g_1, \dots, g_n)) = \max(\psi(g_1)(\psi(f)(1)), \dots, \psi(g_n)(\psi(f)(1)))$$

$$\psi(P(x_1, \dots, x_n) \wedge Q(y_1, \dots, y_m)) = \max(\psi(P(x_1, \dots, x_n)), \psi(Q(y_1, \dots, y_m)))$$

$$\psi(P(x_1, \dots, x_n) \vee Q(y_1, \dots, y_m)) = \max(\psi(P(x_1, \dots, x_n)), \psi(Q(y_1, \dots, y_m)))$$

$$\psi(\neg P(x_1, \dots, x_n)) = \psi(P(x_1, \dots, x_n))$$

$$\psi(\exists x_1 P(x_1, \dots, x_n)) = \psi(P(x_1, \dots, x_n))$$

$$\psi(\forall x_1 P(x_1, \dots, x_n)) = \psi(P(x_1, \dots, x_n))$$

$\psi(\mu x_1 P(x_1, \dots, x_n))$ = the constant function $\psi(P(x_1, \dots, x_n))$

$\psi((\exists x_1 \leq y)P(x_1, \dots, x_n)) = b_\psi(P(x_1, \dots, x_n))$

$\psi((\forall x_1 \leq y)P(x_1, \dots, x_n)) = b_\psi(P(x_1, \dots, x_n))$

and

$\psi((\mu x_1 \leq y)P(x_1, \dots, x_n))$ = the constant function $b_\psi(P(x_1, \dots, x_n))$.

Clearly, the function values of ψ are recursive, and so ψ is computable if b_ℓ is recursive (as a function of ℓ). Moreover, for every constituent C in $\mathcal{L}^*$, we have $\psi(C)$ limits C , by induction on the construction of C since, if C is the resultant of any permissible operation with subjects $C_1, \dots, C_n$, then we may easily verify, from the closure properties given in §5 of Chapter 1, that $\psi(C)$ limits C if $\psi(C_1)$ limits $C_1, \psi(C_2)$ limits $C_2, \dots, \psi(C_n)$ limits C_n . Thus, ψ effectively limits the constituents in $\mathcal{L}^*$.

For every sentence S , with constituents $\mathcal{L}$, we let

$\theta(S)$ denote $\max\{\psi(P) : P \text{ is a subpredicate of } S\}$.

Then, θ is a computable map, and $\theta(S)$ limits every subpredicate of S .

Let $\phi(S, y)$ denote the result of replacing, in S , all unbounded $\exists x$ by $(\exists x \leq y)$, all unbounded $\forall x$ by $(\forall x \leq y)$, and all unbounded μx by $(\mu x \leq y)$. Then, $\phi(S, b_{\theta(S)})$ is equivalent to S , since, if a limits $P(x_1, \dots, x_n)$, then

$$\exists x_1 P(x_1, \dots, x_n) \leftrightarrow (\exists x_1 \leq b_a) P(x_1, \dots, x_n)$$

$$\forall x_1 P(x_1, \dots, x_n) \leftrightarrow (\forall x_1 \leq b_a) P(x_1, \dots, x_n)$$

$$\text{and } \mu x_1 P(x_1, \dots, x_n) = (\mu x_1 \leq b_a) P(x_1, \dots, x_n).$$

Now, ϕ is a computable map, since the constituents in $\mathcal{L}$ are recursive. Thus, if b_ℓ is recursive, then $\phi(S, b_\ell(S))$ is a computable map, and hence is a decision procedure for deciding sentences with constituents $\mathcal{L}$. Hence, we have the following Theorem. If $\mathcal{L}$ is an effectively limited set of recursive constituents, and if b_ℓ is recursive, then the class of sentences with constituents $\mathcal{L}$ is decidable.

Chapter 3

LIMITING THE CONSTITUENTS : $P_0x, P_1x, P_2x, \dots; fx$

§1. Suitable Limiters

Suppose we are given the single variable predicates $P_0x, P_1x, P_2x, \dots$, and the single variable function fx . Let $\mathcal{B}_\ell$ be the Boolean algebra generated by $\{f^{-b}a, f^{-b}\bar{P}_a : a, b < \ell\}$.

Then, $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \dots$ are limiters, such that $a+1$ limits P_ax , for all $a \in \mathbb{N}$, and $i+1$ limits fx .

Thus, we have effectively limited the constituents.

If $\forall x(fx=x)$, then $\mathcal{B}_\ell$ is the Boolean algebra generated by $\{\{a\}, \bar{P}_a : a < \ell\}$.

If $\forall a, x P_ax$, then $\mathcal{B}_\ell$ is the Boolean algebra generated by $\{f^{-b}a : a, b < \ell\}$.

§2. A Bound for b_ℓ

$$\begin{aligned}
 \text{We have } B_\ell x &= \bigcap_{a, b < \ell} \{y : P_a f^b x \leftrightarrow P_a f^b y \wedge f^b x = a \leftrightarrow f^b y = a\} \\
 &= C_\ell x \cap \bigcap_{b < \ell} \{y : (\forall a < \ell)(f^b x = a \leftrightarrow f^b y = a)\} \\
 &\quad \text{where, } C_\ell x \text{ denotes } \bigcap_{a, b < \ell} \{y : P_a f^b x \leftrightarrow P_a f^b y\}. \\
 &= C_\ell x \cap \bigcap_{b < \ell} \{y : f^b y \geq \ell \text{ if } f^b x \geq \ell, \text{ and } f^b y = f^b x \text{ otherwise}\} \\
 &= \begin{cases} C_\ell x \cap I_\ell^\ell & \text{if } (\forall b < \ell)(f^b x \geq \ell) \\ C_\ell x \cap I_\ell^n \cap f^{-n} f^n x & , \text{ otherwise.} \end{cases}
 \end{aligned}$$

where, n denotes the least b such that $f^b x < \ell$,

and I_ℓ^n denotes $\{z : z, fz, \dots, f^{n-1} z \geq \ell\}$.

Thus, $\min B_\ell x \leq \max\{\mu(C_\ell x \cap I_\ell \ell), \mu(C_\ell x \cap I_\ell^n \cap f^{-n} m) : m, n < \ell\}$, where

μA denotes $\mu z(z \in A)$, for every subset A of N , and so

$b_\ell \leq \max\{\mu(C_\ell x \cap I_\ell \ell), \mu(C_\ell x \cap I_\ell^n \cap f^{-n} m) : m, n < \ell, \text{ and } x \in N\}$.

Let $-'$ denote $-$, and let $-^2$ denote $- -$.

Then, $z \in C_\ell x$ if and only if $\bigwedge_{a, b < \ell} -^{v(a, b)}_{Pa f^b z}$, where

$v(a, b)$ denotes 2 if $Pa f^b x$, and $v(a, b)$ denotes 1, otherwise.

Thus, $b_\ell \leq \max\{\mu z(z, fz, \dots, f^{\ell-1} z \geq \ell \wedge \bigwedge_{a, b < \ell} -^{v(a, b)}_{Pa f^b z})$,

$\mu z(z, fz, \dots, f^{n-1} z \geq \ell \wedge f^n z = m \wedge \bigwedge_{a, b < \ell} -^{v(a, b)}_{Pa f^b z}) : m, n < \ell, \text{ and } v : N^2 \rightarrow \{1, 2\}\}$.

Let b'_ℓ denote $\max\{F_\ell, \mu z(f^{l-1} z > F_\ell \wedge f^l z = y \wedge \bigwedge_{a, b < \ell} -^{v(a, b)}_{Pa f^b z})$,

$\mu z(f^{l-1} z > F_\ell \wedge \bigwedge_{a, b < \ell} -^{v(a, b)}_{Pa f^b z}) : i < \ell, y \leq F_\ell, \text{ and } v : N^2 \rightarrow \{1, 2\}\}$, where

F_ℓ denotes $\max\{f^l j : i, j < \ell\}$.

Theorem $b_\ell \leq b'_\ell$.

Proof Suppose $n \leq \ell$, then $z, fz, \dots, f^{n-1} z \geq \ell$ if and only if

$F_\ell \geq z, fz, \dots, f^{n-1} z \geq \ell \vee (\exists i < n)(f^{l-1} z > F_\ell \geq f^l z, \dots, f^{n-1} z \geq \ell) \vee f^{n-1} z > F_\ell$

since, $f^{l-1} z > F_\ell \rightarrow z, \dots, f^{l-1} z \geq \ell$, if $i < n$.

$$\begin{aligned}
& \text{Thus, } \mu z(z, fz, \dots, f^{\ell-1} z \geq \ell \wedge \bigwedge_{a, b < \ell} \neg \nu(a, b)_{P_a f^b z}) \\
& \leq \max\{F_\ell, \mu z(f^{\ell-1} z > F_\ell \geq f^\ell z, \dots, f^{\ell-1} z \geq \ell \wedge \bigwedge_{a, b < \ell} \neg \nu(a, b)_{P_a f^b z}), \\
& \quad \mu z(f^{\ell-1} z > F_\ell \wedge \bigwedge_{a, b < \ell} \neg \nu(a, b)_{P_a f^b z}) : i < \ell-1\}
\end{aligned}$$

by distribution, since $\mu z(Pz \vee Qz) \leq \max(\mu zPz, \mu zQz)$,

for all predicates Pz and Qz ;

$\leq b'_\ell$.

Similarly, if $m, n < \ell$, then

$$\begin{aligned}
& \mu z(z, fz, \dots, f^{n-1} z \geq \ell \wedge f^n z = m \wedge \bigwedge_{a, b < \ell} \neg \nu(a, b)_{P_a f^b z}) \\
& \leq \max\{F_\ell, \mu z(f^{\ell-1} z > F_\ell \geq f^\ell z, \dots, f^{n-1} z \geq \ell \wedge f^n z = m \wedge \bigwedge_{a, b < \ell} \neg \nu(a, b)_{P_a f^b z}), \\
& \quad \mu z(f^{n-1} z > F_\ell \wedge f^n z = m \wedge \bigwedge_{a, b < \ell} \neg \nu(a, b)_{P_a f^b z}) : i < n-1\}
\end{aligned}$$

$\leq b'_\ell$, since $F_\ell \geq \ell-1$.

Hence, $b_\ell \leq b'_\ell$, which completes the proof.

Let $b_\ell^{(k)}$ denote $\max\{\ell-1, \mu x(fx=y \wedge \bigwedge_{a < \ell} \neg \nu a_{P_a x}) : y < k, \text{ and } \nu : N \rightarrow \{1, 2\}\}$.

Theorem If $\forall x(fx < k)$, then $b_\ell \leq b_\ell^{(k)}$.

Proof For every atom B in $\mathcal{B}_\ell$, there exists a predicate Rx , and a function $\nu : N \rightarrow \{1, 2\}$, such that, for all $x \geq \ell$, we have

$$x \in B \iff Rfx \wedge \bigwedge_{a < \ell} \neg \nu_a P_a x$$

$$\text{and so, } x \in B \iff (\exists y < k)(fx=y \wedge Ry \wedge \bigwedge_{a < \ell} \neg \nu_a P_a x).$$

Thus, $\mu x (x \in B) \leq b_\ell^{(k)}$, which completes the proof.

$$\text{Let } (\mu x \geq y)Px \text{ denote } \begin{cases} \min\{x : x \geq y \wedge Px\} & \text{if } (\exists x \geq y)Px \\ y & \text{, otherwise.} \end{cases}$$

If $\bigvee x(fx=x)$, then $F_\ell = \ell-1$, and so

$$b'_\ell = \max\{\ell-1, (\mu z > \ell-1)(z=y \wedge \bigwedge_{a < \ell} \neg \nu_a P_a z), (\mu z > \ell-1) \bigwedge_{a < \ell} \neg \nu_a P_a z : y \leq \ell-1, \\ \text{and } \nu : N \rightarrow \{1, 2\} \}$$

$$= \max\{(\mu z \geq \ell) \bigwedge_{a < \ell} \neg \nu_a P_a z : \nu : N \rightarrow \{1, 2\} \}.$$

If $\bigvee a, x P_a x$, then

$$b'_\ell = \max\{\mu z(f^{i-1} z > F_\ell \wedge f^i z = y), \mu z(f^{\ell-1} z > F_\ell) : i < \ell, \text{ and } y \leq F_\ell\}, \quad \text{and}$$

$$b_\ell^{(k)} = \max\{\ell-1, \mu x(fx=y) : y < k\}.$$

Moreover, if $\bigvee a, x P_a x$ and f is a projection, then

$$b'_\ell = \max\{(\mu z > F_\ell)(fz=y), \mu z(fz > F_\ell) : y \leq F_\ell\}.$$

We now show that, if $\bigvee a, x P_a x$ and f is onto, we may find a much simpler bound for b_ℓ .

Let $p_j y$ denote $\max_{z \leq y} (\mu x > j)(fx=z)$,

and let b_ℓ^* denote $p_0^{\ell-1} p_{F_\ell} F_\ell$.

Lemma If f is onto, then $p_j y \geq y$.

Proof Suppose not, then $p_j y < y$, for some y .

$$\text{i.e. } \max_{z \leq y} (\mu x > j)(fx=z) < y$$

$$\text{i.e. } (\bigvee z \leq y)((\mu x > j)(fx=z) < y)$$

$$\text{and so } (\bigvee z \leq y)(\mu x(fx=z) < y).$$

Now, f is inverse to $\mu x(fx=z)$, and so it maps some subset of $[0, y)$ onto $[0, y]$, which is impossible, and the lemma follows.

Theorem If $\bigvee a, x p_a x$ and f is onto, then $b_\ell \leq b_\ell^*$.

Proof Let Fx denote $\mu y(fy=x)$, then

$$fFx=x, \text{ giving } f^l F^l x=x,$$

$$\text{and so } \mu z(f^l z > F_\ell \wedge f^{l+1} z = y) \leq F^l (\mu x > F_\ell)(fx=y)$$

$$\text{and } \mu z(f^{\ell-1} z > F_\ell) \leq F^{\ell-1}(F_\ell+1).$$

Thus, $b'_\ell \leq \max\{F^l (\mu x > F_\ell)(fx=y), F^{\ell-1}(F_\ell+1) : l < \ell, \text{ and } y \leq F_\ell\}$.

Now, $Fy \leq (\mu x > 0)(fx=y) \leq p_0 y$

and so, $F^l y \leq p_0^l y$, since p_0 is increasing.

Thus, $b'_\ell \leq \max\{p_0^i p_{F_\ell} y, p_0^{\ell-1} (F_{\ell+1}) : i < \ell, \text{ and } y \leq F_\ell\}$, since p_0^i is increasing.

$$\leq \max\{p_0^i p_{F_\ell} F_\ell : i < \ell\}, \text{ since}$$

p_{F_ℓ} and p_0^i are increasing, and $p_{F_\ell} F_\ell > F_\ell$.

The theorem now follows by the lemma.

We now give an example to show how to use this theorem to calculate a bound for b_ℓ in a particular case.

Example: Suppose $\bigvee a, x p_a x$ and $f = E$, where

$E x$ denotes the least y such that $x-y$ is a square.

Then, if $q_j y$ denotes $(j+y)^2 + j$, and q_y denotes $2y^2$, we have

$b_\ell \leq b_\ell^*$, since E is onto

$$= p_0^{\ell-1} p_{\ell-1}(\ell-1), \quad \text{since } F_\ell = \ell-1$$

$$\leq p_0^{\ell-1} q_{\ell-1}(\ell-1), \quad \text{since } p_j y \leq q_j y$$

$$\leq q_0^{\ell-1} q_{\ell-1}(\ell-1), \quad \text{since } q_0 \text{ is increasing}$$

$$= q_0^{\ell-1} (4(\ell-1)^2 + \ell - 1)$$

$$\leq q_0^{\ell-1} (5(\ell-1)^2)$$

$$\leq q^{\ell-1} (5(\ell-1)^2), \text{ since } q_0 y \leq q_y, \text{ and } q \text{ is increasing}$$

$$\leq 2q^{\ell-1} (5(\ell-1)^2)$$

$$= (10(\ell-1)^2)^{2^{\ell-1}}, \text{ since } 2q^{n+1} y = (2q^n y)^2.$$

We now show that in the even more special case, when $f^{-1}x$ is unbounded for all x , we may find a simple representative function for B_ℓ , for each ℓ , and hence determine an even simpler bound for b_ℓ .

Remark: This set of representative functions is of particular interest because it can be used to examine whether any further given constituents are limited.

We recall that

$$B_\ell x = \begin{cases} I_\ell \ell & \text{if } (\bigvee b < \ell)(f^b x \geq \ell) \\ I_\ell n \cap f^{-n} f^n x & , \text{ otherwise} \end{cases}$$

where, n denotes the least b such that $f^b x < \ell$.

We now define recursively the auxiliary functions $\beta_0(n,y), \beta_1(n,y), \dots$ as follows: $\beta_\ell(0,y) = y$
 $\beta_\ell(n+1,y) = (\mu x \geq \ell)(fx = \beta_\ell(n,y)).$

These auxiliary functions have the important property that $\beta_\ell(n,y) \in I_\ell n \cap f^{-n} y$ which is a consequence of the following theorem.

Theorem $\beta_\ell(n,y) \in I_\ell n.$

Proof by induction on n :

We have $\beta_\ell(n+1,y) \geq \ell$ and $f^{l+1} \beta_\ell(n+1,y) = f^l \beta_\ell(n,y)$

and so, $\beta_\ell(n,y), \dots, f^{n-1} \beta_\ell(n,y) \geq \ell \rightarrow \beta_\ell(n+1,y), f \beta_\ell(n+1,y), \dots, f^n \beta_\ell(n+1,y) \geq \ell$

i.e. $\beta_\ell(n,y) \in I_\ell n \rightarrow \beta_\ell(n+1,y) \in I_\ell(n+1).$

Moreover, $\beta_\ell(0, y) \in I_\ell 0$, since $I_\ell 0 = N$, which completes the induction, and the theorem follows.

We now define the representative functions, as follows:

$$\beta_\ell x \text{ denotes } \begin{cases} \beta_\ell(\ell, 0) & \text{if } (\forall n < \ell)(f^n x \geq \ell) \\ \beta_\ell(n, f^n x) & , \text{ otherwise} \end{cases}$$

where, n denotes the least b such that $f^b x < \ell$.

Clearly, $\beta_\ell x$ is a representative function for $\mathcal{B}_\ell$, since

$$B_\ell x = B_\ell y \rightarrow (\forall b < \ell)(\alpha_\ell f^b x = \alpha_\ell f^b y) \rightarrow \beta_\ell x = \beta_\ell y.$$

We now determine a bound for $\beta_\ell x$, and hence for b_ℓ , for each ℓ .

Let $h_\ell z$ denote $\max_{w \leq z} (\mu x \geq \ell)(fx=w)$. Then

$$\beta_\ell(n+1, y) \leq h_\ell \beta_\ell(n, y)$$

and so, $\beta_\ell(n, y) \leq h_\ell^n y$, since h_ℓ is increasing.

$$\text{Now, } \beta_\ell x \leq \max(\beta_\ell(\ell, 0), \max_{n, y < \ell} \beta_\ell(n, y))$$

$$\leq \max(h_\ell^\ell 0, \max_{n, y < \ell} h_\ell^n y)$$

$$\leq \max_{n \leq \ell} h_\ell^n \ell$$

$$\leq h_\ell^\ell \ell, \quad \text{since } h_\ell y \geq y \text{ (just as } p_j y \geq y).$$

Hence, $b_\ell \leq h_\ell^\ell \ell$, since $\min B_\ell x \leq \beta_\ell x$.

We now give an example to show how to use this theorem to calculate a bound for b_ℓ in a particular case.

Example: Suppose $\bigvee a, x P_a x$ and $f = E$ (i.e. the same example as before). Then, if $q_\ell y$ denotes $(j+y)^2 + j$, and py denotes $5y^2$, we have

$$\begin{aligned} b_\ell &\leq h_\ell^\ell, \quad \text{since } E^{-1}x \text{ is unbounded for all } x. \\ &\leq q_\ell^\ell, \quad \text{since } h_\ell y \leq q_\ell y \text{ and } q_\ell \text{ is increasing.} \\ &\leq p^\ell, \quad \text{since } q_\ell y \leq py, \text{ for all } y \geq \ell, \text{ and } p \text{ is increasing.} \\ &\leq 5p^\ell \\ &= (5\ell)^{2^\ell}, \text{ since } 5p^{n+1}y = (5p^n y)^2. \end{aligned}$$

§3. The Computability of b_ℓ Relative to the Constituents and their Basic Functions

We now prove that b_ℓ is recursive if the constituents and certain functions, constructed from these constituents and determined by them, are recursive. We call these functions the basic functions of the constituents.

Theorem b_ℓ is recursive if the constituents and the basic functions, defined as

$$\left\{ \mu x (f^{l-1} x > z \wedge f^l x = y) \bigwedge_{a, b < \ell} -\nu(a, b) P_a f^b x, \mu x (f^{l-1} x > z \wedge \bigwedge_{a, b < \ell} -\nu(a, b) F_a f^b x) : \right.$$

$i \in \mathbb{N}$, and $\nu : \mathbb{N}^2 \rightarrow \{1, 2\}$ are recursive.

Proof

$$\begin{aligned}
 b_\ell &= \max_x \min B_\ell x \\
 &= \max_{x \leq b'_\ell} \min B_\ell x \\
 &= \max_{x \leq b'_\ell} \min \{y \leq b'_\ell : y \in B_\ell x\}.
 \end{aligned}$$

Thus, b_ℓ is recursive if the constituents and b'_ℓ are recursive, since the predicate $y \in B_\ell x$ is recursive if the constituents are recursive. The theorem follows, since b'_ℓ is recursive if F_ℓ and the basic functions are recursive.

We now show that we may reduce the basic functions by imposing restraints on the constituents.

An immediate corollary to the above theorem is that, if f is a projection, then b_ℓ is recursive if the constituents and the basic functions, defined as

$$\begin{aligned}
 &\left\{ (\mu x > z)(fx = y \wedge \bigwedge_{a < \ell} \neg^{\nu a} P_a x), \mu x(fx > z \wedge \bigwedge_{a < \ell} \neg^{\nu a} P_a x \wedge \bigwedge_{a < \ell} \neg^{\rho a} P_a fx), \right. \\
 &\quad \left. (\mu x > z) \neg^{\nu 0} P_0 x : \nu, \rho : N \rightarrow \{1, 2\} \right\}, \text{ are recursive.}
 \end{aligned}$$

If $\bigvee x(fx < k)$, then, as in the above theorem, b_ℓ is recursive if the constituents and $b_\ell^{(k)}$ are recursive. Now $b_\ell^{(k)}$ is recursive if the basic functions, defined as $\{\mu x(fx = y \wedge \bigwedge_{a < \ell} \neg^{\nu a} P_a x) : \nu : N \rightarrow \{1, 2\}\}$, are recursive. Thus, b_ℓ is recursive if the constituents and their basic functions are recursive.

The following statements follow similarly, from the bounds given for b_ℓ in the preceding section.

If $\forall x(fx=x)$, then b_ℓ is recursive if the constituents and the basic functions, defined as $\{(\mu x > z) \bigwedge_{a < \ell} \neg \nu^a P_a x : \nu : N \rightarrow \{1,2\}\}$, are recursive.

If $\forall a, x P_a x$, then b_ℓ is recursive if the constituents and the basic functions, defined as $\{\mu x(f^{l-1}x > z \wedge f^l x = y), \mu x(f^{l-1}x > z) : l, \ell \in N\}$, are recursive.

If $\forall a, x P_a x$, and f is a projection, then b_ℓ is recursive if the constituents and the basic functions, defined as $\{(\mu x > z)(fx = y), \mu x(fx > z)\}$, are recursive.

If $\forall a, x P_a x$, and f is bounded, then b_ℓ is recursive if the constituents and the basic function, $\mu x(fx = y)$, are recursive.

If $\forall a, x P_a x$, and f is onto, then b_ℓ is recursive if the constituents and the basic function, $(\mu x > z)(fx = y)$, are recursive.

If $\forall a, x P_a x$, and $f^{-1}x$ is unbounded for all x , then b_ℓ is recursive if the constituents are recursive.

§4. The Decidability of the Class of Sentences with the Given Constituents, if the Constituents and their Basic Functions are Recursive

By chapter 2, the class of sentences with these constituents is decidable if b_ℓ and the constituents are recursive, and if the constituents are effectively limited. However, the constituents are automatically effectively limited, by our choice of limiters. Thus, the class of sentences with these constituents is decidable if the constituents and their basic functions are recursive, since then b_ℓ is recursive.

PART II

PARTICULAR CONSTRUCTIONS

Introduction

In each individual chapter of this second part, we define the limiters, as in chapter 3 above, from the set of constituents given in the heading. Hereafter, we refer to the given constituents as the initial constituents. We consider the class of limited constituents, thus defined, and we discover as many simple limited constituents as we find practicable. We also prove that certain key constituents are unlimited, in order to show that search for further limited constituents in these directions would be fruitless. In most chapters, we give a simple set of representative functions, which we use to test whether a constituent is limited. Let $\mathcal{L}$ be the class of all constituents proved to be limited. $\mathcal{L}$ includes the initial constituents, and the constituents, in chapter 1, we showed to be always limited. In each chapter, it is clear that $\mathcal{L}$ is an effectively limited class of recursive constituents, and moreover we are able to show that $b_{\mathcal{L}}$ is recursive. Thus, by chapter 2 above, the class of sentences with constituents $\mathcal{L}$ is decidable.

We have two degrees of freedom in this construction of decidable classes:

1. The choice of initial constituents.
2. The choice of further constituents.

Further ineffective procedures in the construction are:

1. Testing whether these further constituents are limited.
2. Testing whether $\mathcal{L}$ is effectively limited.
3. Testing whether each constituent in $\mathcal{L}$ is recursive.
4. Testing whether $b_{\mathcal{L}}$ is recursive.

Remark: In giving the initial constituents, we omit fx if $\forall x(fx = x)$, and $P_a x$ if $\forall x P_a x$, since such constituents are of little significance in determining the limiters.

Chapter 1

NO INITIAL CONSTITUENTS

Let B_ℓ be the Boolean algebra generated by $\{a : a < \ell\}$.

Then, B is the class of all finite and cofinite subsets of N .

Now, $\min B_\ell x = \alpha_\ell x$, and so $b_\ell = \ell$. Thus, b_ℓ is recursive.

§1. Limited Constituents

Now, ℓ limits Px if and only if $\bigvee x (Px \leftrightarrow Pa_\ell x)$, i.e.
 $(\bigvee x \geq \ell) (Px \leftrightarrow P\ell)$.

Also, π limits fx if and only if $\bigvee x (\alpha_\ell fx = \alpha_\ell fa_{\pi} x)$, i.e.
 $(\bigvee x \geq \pi) (\alpha_\ell fx = \alpha_\ell f\pi)$.

Thus, i limits the function $f(x_1, \dots, x_n)$, if f is strictly increasing, and so

i limits $x+1$, $x+y$, xy , x^y , $\min(x,y)$, $\max(x,y)$, and $x!$.

We may also readily verify that, for each $a \in N$, we have

$a+i$ limits $x \dot{-} a$, and $|x,a|$

$a i$ limits $[x/a]$, where $[x/a]$ denotes $\max\{y : ay \leq x\}$

and i^2 limits Rtx , where Rtx denotes $\max\{y : y^2 \leq x\}$.

§2. Unlimited Constituents

$[a]_b$ is neither finite nor cofinite, and so $x \equiv a \pmod{b}$ is unlimited for all a , and all $b > 1$, where $[a]_b$ denotes $\{x : x \equiv a \pmod{b}\}$.

Also, the set of primes is neither finite nor cofinite, and so $\text{Pr}(x)$ is unlimited, where $\text{Pr}(x)$ denotes " x is a prime".

Theorem $R_b x$ is unlimited, for all $b > 1$.

Proof Suppose there exists $b > 1$, such that $R_b x$ is limited. Then, there exists a function p_i , such that p_i limits $R_b x$, and so, for all $x \geq p_i$, we have

$$\alpha_i R_b x = \alpha_i R_b p_i.$$

Thus, for all $x \geq p_b$, we have

$$R_b x = R_b p_b, \text{ since } R_b x < b.$$

Hence, $R_b(p_b+1) = R_b p_b$, a contradiction.

Thus, $R_b x$ cannot be limited, and we have the theorem.

Theorem E_x is unlimited.

Proof Suppose p_i limits E_x , then

$$(\forall x \geq p_i)(\alpha_i E_x = \alpha_i E p_i)$$

and so, $\alpha_1 E(p_1)^2 = \alpha_1 E((p_1)^2+1)$, a contradiction.

Hence, E_x cannot be limited, and so we have the theorem.

§3. The Ackermann Function

An obvious question to ask is "Is every function in the sequence $x+1, x+y, xy, y^x, \dots$ limited". In fact, they are all limited, and moreover, $i+3$ limits $G(x,y,z)$, where G is a slightly modified Ackermann function, as we now show. First, we define G by double recursion as follows:

$$G(x,y,0) = x+1, G(0,y,1) = y, G(0,y,2) = 0, G(0,y,z+3) = \alpha_1 y,$$

$$G(x+1,y,z+1) = G(G(x,y,z+1),y,z).$$

These defining equations differ from the normal defining equations for the Ackermann function, only at $G(0,0,z+3)$, which is normally 1. Thus, $G(x,y,z)$ coincides with the normal Ackermann function everywhere (provided we let $0^0 = 0$) except at $y=0$ and $z > 3$, where G is 0 and the normal Ackermann function is $R_2(x+1)$. Hence, G differs from the normal Ackermann function, precisely where the normal function is not limited. Moreover, it seems perfectly reasonable to take $G(x,y,0), G(x,y,1), G(x,y,2), \dots$ as the sequence $x+1, x+y, xy, y^x, \dots$, provided we are willing to let $0^0 = 0$.

The rest of this section will be concerned with showing that $i+3$ limits $G(x,y,z)$. We first give the following theorem which holds for all functions $f(x_1, \dots, x_n)$.

Theorem If $\text{pi limits } f(x_1+k, \dots, x_n+k)$, then $\text{pi}+k$ limits $f(x_1, \dots, x_n)$, provided

$$(\forall a < k)(\text{pi}+k \text{ limits } f(a, x_2, \dots, x_n), \dots, f(x_1, \dots, x_{n-1}, a)),$$

for all k .

Proof by induction on n :

The theorem holds for all one-variable functions,

i.e. if $\text{pi limits } f(x+k)$, then $\text{pi}+k$ limits fx , since

$$(\forall x \geq \text{pi}+k)(\alpha_i fx = \alpha_i f((x-k)+k) = \alpha_i f(\alpha_{\text{pi}}(x-k)+k) = \alpha_i f(\text{pi}+k)).$$

We now make the inductive hypothesis that the theorem holds for all $(n-1)$ -variable functions. Then, for any function $f(x_1, \dots, x_n)$, satisfying the above provision, we have

π limits $f(x_1+k, \dots, x_n+k)$

$\rightarrow (\forall a \geq k)(\pi \text{ limits } f(a, x_2+k, \dots, x_n+k), \dots, f(x_1+k, \dots, x_{n-1}+k, a))$

$\rightarrow (\forall a \geq k)(\pi+k \text{ limits } f(a, x_2, \dots, x_n), \dots, f(x_1, \dots, x_{n-1}, a))$

by the inductive hypothesis and the provision.

$\rightarrow \forall a(\pi+k \text{ limits } f(a, x_2, \dots, x_n), \dots, f(x_1, \dots, x_{n-1}, a))$, by the provision.

$\rightarrow \pi+k \text{ limits } f(x_1, \dots, x_n)$.

Thus, the theorem holds for all n -variable functions, and we have completed our induction, and the theorem follows.

Corollary If π limits $f(x_1+k_1, \dots, x_n+k_n)$, then $\pi+k$ limits $f(x_1, \dots, x_n)$, where k denotes $\max(k_1, \dots, k_n)$, provided

$$(\forall a_i < k_i)(\pi+k \text{ limits } f(a_1, x_2, \dots, x_n), \dots, f(x_1, \dots, x_{n-1}, a_n)).$$

Proof If π limits $f(x_1+k_1, \dots, x_n+k_n)$, then π limits $f(x_1+k, \dots, x_n+k)$, by the repeated substitution of x_i+1 for x_i ($i=1, \dots, n$). Thus, the corollary follows by the theorem, since, if the provision is satisfied, then

$$(\forall a < k)(\pi+k \text{ limits } f(a, x_2, \dots, x_n), \dots, f(x_1, \dots, x_{n-1}, a)).$$

We now derive the basic properties of G , which enable us to apply the above corollary.

It is a simple exercise to prove the following for all $z > 2$:

$$\begin{aligned}
G(0,y,z) &= \alpha y, \\
G(1,y,z) &= y, \\
G(x,0,z) &= 0, \\
G(x,1,z) &= 1, \\
\text{and } G(2,2,z) &= 4.
\end{aligned}$$

In the following lemmas, we show that G is strictly increasing essentially everywhere else. More precisely, we prove that $G(x+3,y+3,z+1)$, $G(2,y+3,z+1)$, and $G(x+3,2,z+1)$ are strictly increasing.

Lemma 1 $G(x,b,c)$ is strictly increasing, for all $b > 1$, and all c .

Proof by induction on c :

$$\begin{aligned}
G(x,b,0) &= x+1, \\
G(x,b,1) &= x+b, \\
G(x,b,2) &= xb, \\
\text{and } G(x,b,3) &= b^x.
\end{aligned}$$

Thus, $G(x,b,c)$ is strictly increasing for $c = 0,1,2,3$.

Moreover, if $c > 2$, and if $G(x,b,c)$ is strictly increasing, then

$$\begin{aligned}
&\forall x (G(x,b,c) > x), \text{ since } G(0,b,c) = 1, \text{ and so, for all } x, \\
&G(x+1,b,c+1) = G(G(x,b,c+1),b,c) > G(x,b,c+1),
\end{aligned}$$

i.e. $G(x,b,c+1)$ is strictly increasing, which completes the proof by induction.

Lemma 2 $G(x+1,b,c) > x+1$, for all $b > 1$, and all c .

Proof by induction on x :

The lemma holds at $x=0$, since $G(1,b,c) = b > 1$.

Moreover, if the lemma holds at x , then it holds at $x+1$, by Lemma 1.

This completes the induction, and so the lemma holds for all x .

Lemma 3 $G(x+1, y+1, z) > 0$.

Proof Suppose $G(x+1, 1, z) > 0$, for all x , then, for all x ,

$$G(x, 1, z+1) > 0 \rightarrow G(x+1, 1, z+1) > 0, \text{ since}$$

$$G(x+1, 1, z+1) = G(G(x, 1, z+1), 1, z).$$

Thus, $G(x+1, 1, z+1) > 0$, for all x , by induction on x .

Hence, $G(x+1, 1, z) > 0$, for all x and all z , by induction on z .

The lemma now follows by lemma 2.

Lemma 4 $G(a, y, c)$ is strictly increasing, for all $a, c > 0$.

Proof by induction on c :

The lemma holds for $c=1$, since $G(a, y, 1) = a+y$.

We now make the inductive hypothesis that $G(a, y, c)$ is strictly increasing, for all $a > 0$. Then

$G(a, y, c+1)$ is strictly increasing

$$\rightarrow \bigvee_y (G(a, y+2, c+1) > G(a, y+1, c+1))$$

$$\rightarrow \bigvee_y (G(G(a, y+2, c+1), y+2, c) > G(G(a, y+1, c+1), y+2, c))$$

$$\rightarrow \bigvee_y (G(G(a, y+2, c+1), y+2, c) > G(G(a, y+1, c+1), y+1, c))$$

$$\rightarrow \bigvee_y (G(a+1, y+2, c+1) > G(a+1, y+1, c+1))$$

$$\rightarrow G(a+1, y, c+1) \text{ is strictly increasing, since } G(a+1, 1, c+1) > G(a+1, 0, c+1).$$

Thus, $G(a, y, c+1)$ is strictly increasing, for all $a > 0$, by induction on a .

Hence, $G(a, y, c)$ is strictly increasing, for all $a, c > 0$, by induction on c ,

which completes the proof.

Lemma 5 $G(a,b,z)$ is strictly increasing, for all $a > 1$, and all $b > 2$.

Proof $G(a,b,z+1) = G(G(a-1,b,z+1),b,z) > G(a,b,z)$, since
 $G(a-1,b,z+1) > G(a-1,b,z+1) > a-1$,

which completes the proof.

Lemma 6 $G(x,2,z) > x+1$, for all $x > 1$, and all $z > 0$.

Proof by induction on x :

We make the inductive hypotheses that $G(x,2,z) > x+1$, for all $z > 0$.

Then, $G(G(x,2,z+1),2,z) > G(x,2,z+1) + 1$

and so, $G(x+1,2,z+1) > x+2$.

Thus, $G(x+1,2,z) > x+2$, for all $z > 0$, since

$$G(x+1,2,1) > x+2.$$

Hence, $G(x,2,z) > x+1$, for all $x > 1$ and all $z > 0$, by induction on x ,
since $G(2,2,z) = 4$, which completes the proof.

Lemma 7 $G(a,b,z)$ is strictly increasing for all $a > 2$, and all $b > 1$.

Proof $G(a,2,z+1) = G(G(a-1,2,z+1),2,z) > G(a,2,z)$, by lemma 6.

Thus, the lemma follows, by lemma 5.

Theorem $i+3$ limits $G(x,y,z)$.

Proof We have from the above lemmas, that $G(x+3,y+3,z+3)$ is
strictly increasing. Thus, i limits $G(x+3,y+3,z+3)$, and so the theorem
follows by the preceding theorem, provided

$$(\forall a < 3)(i+3 \text{ limits } G(a,y,z), G(x,a,z), G(x,y,a)).$$

Now, we proved in §2 that $(\forall a < 3) (i \text{ limits } G(x,y,a))$.

Moreover, $G(0,y,z+3) = \alpha y$,

$G(1,y,z+3) = y$,

$G(2,y,z+3)$ is strictly increasing,

$G(x,0,z+3) = 0$,

$G(x,1,z+3) = 1$,

and $G(x,2,z+3)$ is strictly increasing.

Thus, $i \text{ limits } G(a,y,z+3), G(x,a,z+3)$, for all $a < 3$, and so by the corollary to the preceding theorem, $i+3 \text{ limits } G(a,y,z), G(x,a,z)$, for all $a < 3$, since $i+3 \text{ limits } G(x,y,c)$ for $c = 0,1,2$, which completes the proof.

Corollary Not all limited recursive functions are primitive recursive.

§4. Closure of the Limited Functions under $\Sigma, \Pi, \dots$

Let $\phi_k(x,y)$ denote $G(x,y,k)$, for all $k \in \mathbb{N}$.

Then, $\phi_0(x,y), \phi_1(x,y), \phi_2(x,y), \dots$ is the sequence $x+1, x+y, xy, y^x, \dots$.

We define ϕ_k for any number of variables, recursively as follows:

$$\phi_k(a) = a$$

$$\phi_k(a_0, \dots, a_n, a_{n+1}) = \phi_k(\phi_k(a_0, \dots, a_n), a_{n+1}).$$

We may now define the operations $\Sigma, \Pi, \dots$ (i.e. continued addition, continued multiplication, continued exponentiation, and so on) denoted by $\Phi_1, \Phi_2, \dots$.

We define $\Phi_k f(x_0, x_1, \dots, x_n)$ as

$$\phi_k(f(0, x_1, \dots, x_n), f(1, x_1, \dots, x_n), \dots, f(x_0, x_1, \dots, x_n)).$$

Thus, in particular,

$$\Phi_1 f(x_0, x_1, \dots, x_n) = \sum_{x \leq x_0} f(x, x_1, \dots, x_n),$$

and
$$\Phi_2 f(x_0, x_1, \dots, x_n) = \prod_{x \leq x_0} f(x, x_1, \dots, x_n).$$

Lemma 1 $\phi_{k+2}(a+1, b) = \phi_{k+1}(b, b^{(a)})$, where
 $\phi_k(a, b^{(n)})$ denotes $\phi_k(a, b, \dots, b)$, with b occurring n times.

Proof by induction on a :

The lemma holds for $a = 0$, since $\phi_{k+2}(1, b) = b = \phi_{k+1}(b)$.

Moreover, $\phi_{k+2}(a+1, b) = \phi_{k+1}(\phi_{k+2}(a, b), b)$

and so, if $\phi_{k+2}(a, b) = \phi_{k+1}(b, b^{(a-1)})$, then

$$\phi_{k+2}(a+1, b) = \phi_{k+1}(b, b^{(a)}), \text{ which completes the induction.}$$

Thus, we have the lemma, for all a .

Lemma 2 $\alpha_l \phi_k(a, b^{(i+x)}) = \alpha_l \phi_k(a, b^{(i)})$.

Proof The lemma holds for $b = 0$, since

$$\phi_0(a, 0^{(i)}) = a + i \geq i$$

$$\phi_1(a, 0^{(i)}) = a$$

and $\phi_{k+2}(a, 0^{(i)}) = 0$, since $\phi_{k+2}(a, 0) = 0$, for all $i > 0$.

The lemma holds for $b = 1$, since

$$\phi_0(a, 1^{(i)}) = a + i \geq i.$$

$$\phi_1(a, 1^{(i)}) = a + i \geq i$$

$$\phi_2(a, 1^{(i)}) = a$$

and $\phi_{k+3}(a, 1^{(i)}) = 1$, since $\phi_{k+3}(a, 1) = 1$, for all $i > 0$.

Finally, suppose $b > 1$.

Then, $\phi_k(a, b^{(l)}) \geq 1$, unless $k=2$ and $a=0$, since

$$\phi_0(a, b^{(l)}) = a + 1$$

$$\phi_1(a, b^{(l)}) = a + b1$$

$$\phi_2(a, b^{(l)}) = ab^l$$

$$\phi_k(a, b^{(0)}) \geq 0$$

$$\phi_k(a, b^{(1)}) = \phi_k(a, b) \geq \phi_k(0, b) = 1$$

$$\phi_k(a, b^{(2)}) = \phi_k(\phi_k(a, b), b) \geq \phi_k(1, b) \geq 2$$

and, if $i, k > 2$,

$$\begin{aligned} \text{then } \phi_k(a, b^{(l)}) &\geq \phi_k(0, b^{(l)}) \\ &= \phi_k(1, b^{(l-1)}), \text{ since } \phi_k(0, b) = 1 \\ &= \phi_k(b, b^{(l-2)}), \text{ since } \phi_k(1, b) = b \\ &= \phi_{k+1}(1-1, b), \text{ by lemma 1.} \\ &\geq 1, \text{ by lemma 2, §3.} \end{aligned}$$

Thus, the lemma holds for $b > 1$, and so the lemma holds, for all b .

Theorem If p_i limits fx , then $p(i+3) + i$ limits $\Phi_k f x$.

$$\begin{aligned} \text{Proof } &\alpha_i \Phi_k f(p(i+3) + i + x) \\ &= \alpha_i \phi_k(\Phi_k f(p(i+3)), f(p(i+3)+1), f(p(i+3)+2), \dots, f(p(i+3)+i+x)) \\ &= \alpha_i \phi_k(\Phi_k f(p(i+3)), \alpha_{i+3} f(p(i+3)+1), \alpha_{i+3} f(p(i+3)+2), \dots, \alpha_{i+3} f(p(i+3)+i+x)) \\ &= \alpha_i \phi_k(\Phi_k f(p(i+3)), f(p(i+3)))^{(i+x)} \\ &= \alpha_i \phi_k(\Phi_k f(p(i+3)), f(p(i+3)))^{(i)}, \text{ by lemma 2.} \\ &= \alpha_i \Phi_k f(p(i+3)+i), \end{aligned}$$

which completes the proof.

Corollary If p_i limits $f(x_0, x_1, \dots, x_n)$, then $p(i+3)+1$ limits $\Phi_k f(x_0, x_1, \dots, x_n)$.

Proof $p(i+3)+1$ limits $\Phi_k f(x_0, a_1, \dots, a_n)$, for all $a_1, \dots, a_n$, by the theorem. Moreover, $p(i+3)$ limits $\Phi_k f(a_0, x_1, \dots, x_n)$, for all a_0 , since $i+3$ limits $\phi_k(y_0, \dots, y_{a_0})$, which completes the proof.

Hence, both the limited functions and the recursively limited functions (i.e. functions with recursive limits) are closed under $\Sigma, \Pi, \dots$.

§5. A decidable Class

Let $\mathcal{F}$ denote the class of functions generated by $\Sigma, \Pi, \dots$ from $\{a, x, x+1, x+y, xy, x^y, G(x, y, z), \min(x, y), \max(x, y), a \dot{-} x, x \dot{-} a, |a, x|, x!, \lfloor x/a \rfloor, \text{Rtx} : a \in \mathbb{N}\}$.

Let $\mathcal{L}$ denote $\{x > a, x = a, x < a, x|a : a \in \mathbb{N}\} \cup \mathcal{F}$

Clearly, $\mathcal{L}$ is effectively limited, since the recursively limited functions are closed under $\Sigma, \Pi, \dots$, and moreover, each constituent in $\mathcal{L}$ is recursive. Thus, the class of sentences with constituents $\mathcal{L}$ is decidable.

§6. The Connection with papers [1] and [2]

In [2], Rousseau calls a function $f(x_1, \dots, x_n)$ definite if and only if $f^{-1} : \mathcal{B} \rightarrow \mathcal{B}^n$. Thus, a function is definite if and only if it is limited. Hence, our general theory of limited constituents is a natural extension of Rousseau's concept of definite.

An alternative description of a limited function $f(x_1, \dots, x_n)$ is

$$\forall i \exists m \forall x_1, \dots, x_n (\alpha_i f(x_1, \dots, x_n) = \alpha_i f(\alpha_m x_1, \dots, \alpha_m x_n)).$$

This is a natural generalisation of the property, held by the class C of functions of Goodstein and Lee, which they use to decide the class $\{f(x_1, \dots, x_n) = a : f \in C, \text{ and } a \in \mathbb{N}\}$. Thus, the limited functions are a natural extension of C , i.e. the definite functions are a natural extension of C , and we see the connection with the two cited papers. Moreover, the decidable class given above is considerably more extensive than the decidable class given in [1].

Chapter 2

Ex

Let $\mathcal{B}_\ell$ be the Boolean algebra generated by $\{E^{-b}a : a, b < \ell\}$. Now, $E^{-1}x$ is unbounded for all x , and so we may define a simple representative function for $\mathcal{B}_\ell$, for each ℓ , as in part I, chapter 2, §2. However, in this particular case, we may find an even simpler representative function for $\mathcal{B}_\ell$.

$$\text{Let } \beta_\ell x \text{ denote } \begin{cases} \sum_{n=1}^{\ell} \ell^{2^n} & \text{if } (\forall n < \ell)(E^n x \geq \ell) \\ E^m x + \sum_{n=1}^m \ell^{2^n} & , \text{ otherwise} \end{cases}$$

where, m denotes the least n such that $E^n x < \ell$.

Lemma If $\ell > 1$, then $\sum_{n=0}^m \ell^{2^n} \leq 2\ell^{2^m}$.

Proof by induction on m :

The lemma holds for $m = 0$, since $\sum_{n=0}^0 \ell^{2^n} = \ell \leq 2\ell = 2\ell^{2^0}$.

Moreover, if $\sum_{n=0}^m \ell^{2^n} \leq 2\ell^{2^m}$, then $\sum_{n=0}^{m+1} \ell^{2^n} \leq 2\ell^{2^m} + \ell^{2^{m+1}}$

$$\leq 2\ell^{2^{m+1}}, \text{ since } 2j \leq j^2 \text{ if } 1 < j.$$

This completes the induction, and so, we have the lemma, for all m .

Theorem If $\ell > 1$, then $\beta_\ell x = B_\ell x$.

Proof Suppose $y < \ell$. Then, $E(y + \sum_{n=1}^{m+1} \ell^{2^n}) = y + \sum_{n=1}^m \ell^{2^n}$, since

$y + \sum_{n=1}^m \ell^{2^n} < 2\ell^{2^m}$, by the lemma, and since $E(v+u^2) = v$ if $v \leq 2u$.

Thus, $E^m(y + \sum_{n=1}^m \ell^{2^n}) = y$, and so

$y + \sum_{n=1}^m \ell^{2^n} \in I_\ell^m \cap E^{-m}y$, by induction on m , since $y + \sum_{n=1}^{m+1} \ell^{2^n} \geq \ell$.

Hence, $\sum_{n=1}^{\ell} \ell^{2^n} \in I_\ell^\ell$, and $E^m x + \sum_{n=1}^m \ell^{2^n} \in I_\ell^m \cap E^{-m}E^m x$ if $E^m x < \ell$.

i.e. $\beta_\ell x \in \begin{cases} I_\ell^\ell & \text{if } (\forall n < \ell)(E^n x \geq \ell) \\ I_\ell^m \cap E^{-m}E^m x & , \text{ otherwise} \end{cases}$

where, m denotes the least n such that $E^n x < \ell$,

which completes the proof.

Corollary $\beta_\ell x \in B_\ell x$, for all ℓ .

Proof $\beta_1 x \in B_1 x$, since $\beta_1 x = \alpha_1 x$ and $\mathcal{B}_1$ is the Boolean algebra generated by $\{\{0\}\}$.

The corollary now follows from the theorem.

Thus, $\beta_\ell x$ is representative, since

$$B_\ell x = B_\ell y \rightarrow \bigvee b(\alpha_\ell E^b x = \alpha_\ell E^b y) \rightarrow \beta_\ell x = \beta_\ell y.$$

Finally, we give a bound for $\beta_\ell x$.

$$\beta_\ell x < \sum_{n=0}^{\ell} \ell^{2^n} \leq 2\ell^{2^\ell}.$$

Thus, $b_\ell < 2\ell^{2^\ell}$, and so b_ℓ is recursive, since the constituents and $2\ell^{2^\ell}$ are recursive.

§1. Limited Constituents

ℓ limits Px if and only if $\forall x (Px \leftrightarrow P\beta_\ell x)$, i.e. for all x

$$\left\{ \begin{array}{ll} Px \leftrightarrow P \sum_{n=1}^{\ell} \ell^{2^n} & \text{if } (\forall n < \ell) (E^n x \geq \ell) \\ \text{and } Px \leftrightarrow P(E^m x + \sum_{n=1}^{\ell} \ell^{2^n}) & , \text{ otherwise} \end{array} \right.$$

where, m denotes the least n such that $E^n x < \ell$.

p_i limits fx if and only if $\forall x (\beta_i fx = \beta_i f\beta_{p_i} x)$, i.e. for all x

$$\left\{ \begin{array}{ll} \beta_i fx = \beta_i f \sum_{n=1}^{\ell} (p_i)^{2^n} & \text{if } (\forall n < p_i) (E^n x \geq p_i) \\ \beta_i fx = \beta_i f(E^m x + \sum_{n=1}^m (p_i)^{2^n}), & \text{ otherwise} \end{array} \right.$$

where, m denotes the least n such that $E^n x < \ell$.

Let $Sq(x)$ denote " x is a square".

Clearly, 2 limits $Sq(x)$, since $\overline{Sq} = E^{-1}0$.

By construction, $i+1$ limits Ex .

Theorem i limits x^{2a} , for all a .

Proof

$$\beta_i (y^2) = \left\{ \begin{array}{ll} y^2 & \text{if } y^2 < i \\ i^2 & , \text{ otherwise,} \end{array} \right.$$

and so, $\beta_i x^{2a} = \beta_i \left(\sum_{n=1}^{\ell} i^{2^n} \right)^{2a}$ if $(\forall n < i) (E^n x \geq i)$

and, $\beta_i x^{2a} = \beta_i \left(E^m x + \sum_{n=1}^m i^{2^n} \right)^{2a}$, otherwise,

where m denotes the least n such that $E^n x < i$,

which completes the proof.

§2. Unlimited Constituents

Theorem $x \equiv a \pmod{b}$ is unlimited, for all a , and all $b > 1$.

Proof Suppose $\ell > b > 1$ and ℓ limits $x \equiv a \pmod{b}$.

Then, $x \equiv a \pmod{b} \leftrightarrow \sum_{n=1}^{\ell} \ell^{2^n} \equiv a \pmod{b}$ if $(\forall n < \ell)(E^n x \geq \ell)$.

Putting $x = a_c$, where a_c denotes $c + \sum_{n=1}^{\ell} \ell^{2^n}$, we obtain

$$a_c \equiv a \pmod{b} \leftrightarrow \sum_{n=1}^{\ell} \ell^{2^n} \equiv a \pmod{b} \text{ if } c < b, \text{ since}$$

$$(\forall n < \ell)(E^n a_c \geq \ell) \quad \text{if } c < \ell.$$

This yields a contradiction, and so $x \equiv a \pmod{b}$ cannot be limited.

Theorem $Sq(x+1)$ is unlimited.

Proof Suppose ℓ limits $Sq(x+1)$.

Then, $Sq(x+1) \leftrightarrow Sq(\beta_{\ell} x + 1)$.

Putting $x = c$, where c denotes $\ell^2 + 2\ell$, we obtain $Sq(\beta_{\ell} c + 1)$.

Now, $\beta_{\ell} c$ is either $\sum_{n=1}^{\ell} \ell^{2^n}$, or $E^{m+1} c + \sum_{n=1}^{m+1} \ell^{2^n}$ with $E^{m+1} c < \ell$,

since $c \geq \ell$. Moreover, $y + \sum_{n=1}^{m+1} \ell^{2^n} + 1$ is not a square, if $y < \ell$, since

$$(\ell^{2^m})^2 < y + \sum_{n=1}^{m+1} \ell^{2^n} < \sum_{n=0}^{m+1} \ell^{2^n} \leq \ell^{2^{m+1}} + 2\ell^{2^m} < (\ell^{2^m} + 1)^2.$$

Similarly, $\sum_{n=1}^{\ell} \ell^{2^n} + 1$ is not a square.

Thus, we have a contradiction, and so $Sq(x+1)$ cannot be limited.

Corollary $x+1$ is unlimited.

Proof Suppose $x+1$ were limited. Then, $Sq(x+1)$ would be limited, since $Sq(x)$ is limited. This contradicts the theorem, and so $x+1$ cannot be limited.

Thus, we clearly have $x+y$ is unlimited.

Remark: I have been unable to decide whether $x \dot{-} 1$ is limited or not. If it is limited, then $x \dot{-} a$ is limited, for all a , giving $x+a$ is unlimited for all positive a , since otherwise $x+1$ would be limited.

§3. A Decidable Class

Let $\mathcal{L}$ denote $\{x > a, x=a, x < a, x|a, Sq(x), a, x, x^{2a}, \min(a,x), \max(a,x), a \dot{-} x, \exists x:a \in N\}$. Clearly, $\mathcal{L}$ is an effectively limited class of recursive constituents. Thus, the class of sentences with constituents $\mathcal{L}$ is decidable.

Chapter 3

$\text{Pr}(x)$

Let $\mathcal{B}_\ell$ be the Boolean algebra generated by $\{\{a\}, \{p_1, p_2, \dots\} : a < \ell\}$, where p_i denotes the i -th prime.

The basic functions in this case are $\{(\mu x > z) \neg^n \text{Pr}(x) : n \in \{1, 2\}\}$.

Now, $\text{Pr}(x)$ and each basic function is recursive, and so b_ℓ is recursive.

§1. Limited Constituents

By construction, 0 limits $\text{Pr}(x)$.

Clearly, 1 limits xy , x^y , and $x!$, since each component-wise inverse of $\{p_1, p_2, \dots\}$ is either $\{p_1, p_2, \dots\}$ or the empty set.

§2. Unlimited Constituents

Clearly, $x \equiv a \pmod{b}$ is unlimited, for all a , and all $b > 1$.

We may readily verify that, for all $a > 0$ and all $b > 1$, the following key functions are unlimited:

$$x+a, x+y, x \dot{-} a, |x, a|, [x/b], R_b x, Rtx, \text{ and } \pi x$$

where πx denotes the greatest prime less than or equal to x .

Theorem $\min(x, y)$ is unlimited.

Proof Suppose ℓ limits $\text{Pr}(\min(x, y))$.

Then, ℓ limits $\text{Pr}(\min(p_\ell + 1, y))$

i.e. $\{y : \text{Pr}(\min(p_\ell + 1, y))\} \in \mathcal{B}_\ell$

i.e. $\{p_1, \dots, p_\ell\} \in \mathcal{B}_\ell$, a contradiction since $p_\ell \geq \ell$.

Thus, $\text{Pr}(\min(x,y))$ cannot be limited, and so $\min(x,y)$ cannot be limited.

Theorem $\max(x,y)$ is unlimited.

Proof Suppose ℓ limits $\text{Pr}(\max(x,y))$.

Then, ℓ limits $\text{Pr}(\max(p_\ell+1,y))$,

i.e. $\{y : \text{Pr}(\max(p_\ell+1,y))\} \in \mathcal{B}_\ell$

i.e. $\{p_{\ell+1}, p_{\ell+2}, \dots\} \in \mathcal{B}_\ell$, a contradiction since $p_\ell \geq \ell$.

Thus, $\text{Pr}(\max(x,y))$ cannot be limited, and so $\max(x,y)$ cannot be limited.

§3. A Decidable Class

Let $\mathcal{L}$ denote $\{x > a, x = a, x < a, x|a, \text{Pr}(x), a, x, xy, x^y, \min(a,x), \max(a,x), a : x, x! : a \in \mathbb{N}\}$. Clearly, $\mathcal{L}$ is an effectively limited class of recursive constituents. Thus, the class of sentences with constituents $\mathcal{L}$ is decidable.

Chapter 4

$\text{Pr}(x); \pi x$

Let $\mathcal{B}_\ell$ be the Boolean algebra generated by
 $\{\pi^{-b}a, \pi^{-b}\{p_1, p_2, \dots\} : a, b < \ell\},$

i.e. $\{\{a\}, \{p_1, p_2, \dots\}, \pi^{-1}a : a < \ell\},$ since

π is a projection, and $\pi^{-1}\{p_1, p_2, \dots\} = \{p_1, p_2, \dots\}.$

Thus, $\mathcal{B}_\ell$ is the Boolean algebra generated by

$$\{\{a\}, \{p_1, p_2, \dots\}, \{p_1^{a_1} \dots p_r^{a_r} : a_1, \dots, a_r \in \mathbb{N}\} : a, p_r < \ell\}$$

since, $\pi^{-1}p_r = \{p_1^{a_1} \dots p_r^{a_r} : a_1, \dots, a_r \in \mathbb{N}\} - \{p_1^{a_1} \dots p_{r-1}^{a_{r-1}} : a_1, \dots, a_{r-1} \in \mathbb{N}\},$

$$\text{and } \{p_1^{a_1} \dots p_r^{a_r} : a_1, \dots, a_r \in \mathbb{N}\} = \pi^{-1}p_1 \cup \dots \cup \pi^{-1}p_r.$$

The basic functions in this case are

$$\{(\mu x > z) \neg^n \text{Pr}(x), (\mu x > z)(\pi x = y \wedge \neg^n \text{Pr}(x)), \mu x(\pi x > z \wedge \neg^n \text{Pr}(x)) : n \in \{1, 2\}\},$$

and each basic function is recursive, since

$$(\mu x > z)(\pi x = y \wedge \text{Pr}(x)) = \begin{cases} y & \text{if } y > z \wedge \text{Pr}(y) \\ z+1 & \text{, otherwise.} \end{cases}$$

$$(\mu x > z)(\pi x = y \wedge \neg \text{Pr}(x)) \leq y^{z+1}$$

$$\mu x (\pi x > z \wedge \text{Pr}(x)) = (\mu x > z) \text{Pr}(x)$$

and $\mu x(\pi x > z \wedge \neg \text{Pr}(x)) = 2 \min\{p > z : p \text{ is a prime}\}.$

Thus, $\mathcal{B}_\ell$ is recursive, since the constituents are recursive.

§1. Limited Constituents

By construction, 0 limits $\text{Pr}(x)$.

Theorem i limits xy , and i limits x^y .

Proof Let $M_a x$ denote ax
 $N_a x$ denote x^a
 and $P_a x$ denote a^x .

Then, $M_a^{-1} \{p_1^{a_1} \dots p_r^{a_r} : a_1, \dots, a_r \in N\} = \{p_1^{a_1} \dots p_r^{a_r} : a_1, \dots, a_r \in N\}$

$N_a^{-1} \{p_1^{a_1} \dots p_r^{a_r} : a_1, \dots, a_r \in N\} = \{p_1^{a_1} \dots p_r^{a_r} : a_1, \dots, a_r \in N\}$

and $P_a^{-1} \{p_1^{a_1} \dots p_r^{a_r} : a_1, \dots, a_r \in N\} = \begin{cases} N & \text{if } a \in \{p_1^{a_1} \dots p_r^{a_r} : a_1, \dots, a_r \in N\} \\ \text{the empty set,} & \text{otherwise.} \end{cases}$

Thus, if B belongs to $\mathcal{B}_1$, then each component-wise inverse of B belongs to $\mathcal{B}_1$, and so we have the theorem.

§2. Unlimited Constituents

We do not appear to be able to limit the key functions:

$x+a$, $x+y$, $x \cdot a$, $|x, a|$, $[x/b]$, $R_b x$, Rtx , $x!$ for all $a > 0$, and all $b > 1$.

§3. A Decidable Class

Let $\mathcal{L}$ denote

$\{x > a, x = a, x < a, x|a, \text{Pr}(x), a, x, x^y, \min(a, x), \max(a, x), a \cdot x, \pi x : a \in N\}$.

Clearly, $\mathcal{L}$ is an effectively limited class of recursive constituents.

Thus, the class of sentences with constituents $\mathcal{L}$ is decidable.

Remark: Although this class contains many statements about primes, which we would not normally expect to be able to decide, we, nevertheless, are unable to decide really interesting statements about primes with constituents such as $x+1$. An example of such a statement, with constituents $\text{Pr}(x)$ and $x+1$, is "There is a twin prime greater than the largest known twin prime".

We have not considered the initial constituents " $\text{Pr}(x); x+1$ ", since in this case, we have among our basic functions, the function

$$(\mu x > z)(\text{Pr}(x) \wedge \text{Pr}(x+2))$$

which is just as difficult to show recursive (if indeed it is), as it is to decide whether there is a twin prime greater than the largest known twin prime. Moreover, I have not been able to show that b_ℓ is recursive directly, without recourse to the basic functions. Thus, it appears that " $\text{Pr}(x); x+1$ " are not suitable initial constituents.

Chapter 5

$$\underline{x \equiv a \pmod{b} \quad (a, b \in \mathbb{N})}$$

Let B_ℓ be the Boolean algebra generated by $\{[a]_b : a, b < \ell\}$.

Thus, B is the class of all eventually periodic subsets of $\mathbb{N}$.

We first prove some properties of the highest common factor and least common multiple of two numbers, which we need in the sequel.

Theorem $ab = (a \cap b)(a \cup b)$

$$a(b \cap c) = ab \cap ac$$

$$a(b \cup c) = ab \cup ac$$

$$\text{and } a \cap (b \cup c) = (a \cap b) \cup (a \cap c).$$

Proof Let $a = p_1^{\alpha_1} \dots p_n^{\alpha_n}$

$$b = p_1^{\beta_1} \dots p_n^{\beta_n}$$

$$\text{and } c = p_1^{\gamma_1} \dots p_n^{\gamma_n}.$$

Then, each property follows respectively from the following identities on the exponents.

$$\alpha_i + \beta_i = \min(\alpha_i, \beta_i) + \max(\alpha_i, \beta_i)$$

$$\alpha_i + \min(\beta_i, \gamma_i) = \min(\alpha_i + \beta_i, \alpha_i + \gamma_i)$$

$$\alpha_i + \max(\beta_i, \gamma_i) = \max(\alpha_i + \beta_i, \alpha_i + \gamma_i)$$

$$\text{and } \min(\alpha_i, \max(\beta_i, \gamma_i)) = \max(\min(\alpha_i, \beta_i), \min(\alpha_i, \gamma_i)),$$

which completes the proof.

We now use the relation

$$[a]_b \cap [a]_c = [a]_{b \cup c}$$

to simplify $B_\ell x$.

Clearly, for all $x \geq \ell$, we have

$$\begin{aligned}
 B_\ell x &= \left(\bigcap \{[a]_b : a, b < \ell, \text{ and } x \equiv a \pmod{b}\} - \right. \\
 &\quad \left. \bigcup \{[a]_b : a, b < \ell, \text{ and } x \not\equiv a \pmod{b}\} \right) - [0, \ell) \\
 &= \bigcap \{[a]_b : a, b < \ell, \text{ and } x \equiv a \pmod{b}\} - [0, \ell) \\
 &= \bigcap \{[x]_b : b < \ell\} - [0, \ell) \\
 &= [x]_{L\ell} - [0, \ell)
 \end{aligned}$$

where $L\ell$ denotes the least common multiple of all $b < \ell$ (and $L1 = 1$).

We now define a representative function for B_ℓ .

$$\text{Let } \beta_\ell x \text{ denote } \begin{cases} x & \text{if } x < \ell \\ 2L\ell + R_{L\ell} x & , \text{ otherwise.} \end{cases}$$

The following two theorems show that $\beta_\ell x$ is representative.

Theorem $\beta_\ell x \in B_\ell x$

Proof $L\ell \geq \ell - 1$

and so, $2L\ell \geq 2\ell - 2 \geq \ell$ if $\ell \geq 2$.

Thus, $2L\ell \geq \ell$, for all ℓ .

Hence, $\beta_\ell x \in B_\ell x$ if $x \geq \ell$.

Thus, we have the theorem, for all x .

Theorem $B_\ell x = B_\ell y \rightarrow \beta_\ell x = \beta_\ell y$

Proof
$$\begin{aligned} B_\ell x = B_\ell y &\rightarrow (\forall a, b < \ell)(x \equiv a \pmod{b} \leftrightarrow y \equiv a \pmod{b}) \\ &\rightarrow (\forall b < \ell)(x \equiv y \pmod{b}) \\ &\rightarrow x \equiv y \pmod{L\ell} \\ &\rightarrow (\forall x, y \geq \ell)(\beta_\ell x = \beta_\ell y). \end{aligned}$$

Thus, we have the theorem for all x and y , since

$$B_\ell x = B_\ell y \rightarrow \alpha_\ell x = \alpha_\ell y.$$

Finally, we give a bound for $\beta_\ell x$.

Theorem
$$\beta_\ell x < 3\ell^\ell.$$

Proof
$$\begin{aligned} \beta_\ell x &< 3L\ell, \text{ since } \ell \leq 2L\ell \leq 3L\ell \\ &\leq 3\ell^\ell, \text{ since } L\ell \leq \ell^\ell, \end{aligned}$$

which completes the proof.

Thus, $b_\ell < 3\ell^\ell$, and so b_ℓ is recursive, since the constituents and $3\ell^\ell$ are recursive.

§1. Limited Constituents

a limits Px if and only if $\forall x, y (\beta_a x = \beta_a y \rightarrow Px \leftrightarrow Py)$

$$\text{i.e. } (\forall x, y \geq a)(\beta_a x = \beta_a y \rightarrow Px \leftrightarrow Py)$$

$$\text{i.e. } (\forall x, y \geq a)(x \equiv y \pmod{La} \rightarrow Px \leftrightarrow Py)$$

$$\text{i.e. } (\forall x \geq a)(P(x + La) \leftrightarrow P(x))$$

pi limits fx if and only if $(\forall x, y)(\beta_{pi} x = \beta_{pi} y \rightarrow \beta_l fx = \beta_l fy)$

$$\text{i.e. } (\forall x, y \geq pi)(\beta_{pi} x = \beta_{pi} y \rightarrow \beta_l fx = \beta_l fy)$$

$$\text{i.e. } (\forall x, y \geq pi)(x \equiv y \pmod{pi} \rightarrow \beta_l fx = \beta_l fy)$$

$$\text{i.e. } (\forall x \geq pi)(\beta_l f(x + Lpi) = \beta_l f(x))$$

$$\text{i.e. } (\forall x \geq pi)(\alpha_l f(x + Lpi) = \alpha_l f(x) \wedge f(x + Lpi) \equiv fx \pmod{Li}).$$

By construction, $b + 1$ limits $x \equiv a \pmod{b}$

and so, $b + 1$ limits $x \equiv y \pmod{b}$

In particular, $b + 1$ limits $b \mid x$.

i limits $x + 1$, since

$$(\forall x \geq i)(\alpha_i(x+Li+1) = i = \alpha_i(x+1) \wedge (x+Li+1) \equiv (x+1) \pmod{Li})$$

Thus, i limits $x + y$, since i limits $x + a$, for all a ,

and so, i limits xy , since i limits xa , for all a ,

and, i limits x^a , for all a .

Our next theorem shows that x^y is limited.

Lemma $a^{i+i!} \equiv a^i \pmod{i}$

Proof There exist j and k such that $i = jK$, $a \cap k = 1$, and $j \mid a^i$.

Thus, by Euler's theorem, $a^{\phi k} \equiv 1 \pmod{k}$, where ϕ is Euler's totient function,

and so, $a^{i!} \equiv 1 \pmod{k}$, since $\phi k \mid k!$

Thus, $ja^{i!} \equiv j \pmod{i}$.

Hence, $a^i a^{i!} \equiv a^i \pmod{i}$,

which completes the proof.

Theorem $(Li)! + 1$ limits x^y .

Proof Let p_i denote $(Li)! + 1$.

Then, $(\forall x \geq i)(\alpha_i a^{x+Lp_i} = \alpha_i a^x)$, since $(\forall x, y \geq i)(\alpha_i a^x = \alpha_i a^y)$,

and so, $(\forall x \geq p_i)(\alpha_i a^{x+Lp_i} = \alpha_i a^x)$, since $p_i \geq i$

Moreover, $a^{Li+(Li)!} \equiv a^{Li} \pmod{Li}$, by the lemma,

and so, $(\forall x \geq pi)(a^{x+(Li)!} \equiv a^x \pmod{Li})$, since $pi \geq Li$.
 Therefore, $(\forall x \geq pi)(a^{x+Lpi} \equiv a^x \pmod{Li})$, since $(Li)! \mid Lpi$.
 Thus, pi limits a^x , and so we have the theorem, since $pi \geq i$.

Theorem $i + a$ limits $x \dot{-} a$, for all a .

Proof $(\forall x \geq i + a)(\alpha_i((x + L(i+a)) \dot{-} a) = i = \alpha_i(x \dot{-} a))$
 and $(\forall x \geq i + a)((x + L(i+a)) \dot{-} a \equiv x \dot{-} a \pmod{Li})$, since
 $Li \mid L(i+a)$,

which completes the proof.

Theorem Li limits $x!$

Proof $(\forall x \geq Li)(\alpha_i(x + LLi)! = i = \alpha_i x!)$
 and $(\forall x \geq Li)(x + LLi \equiv 0 \equiv x! \pmod{Li})$,

which completes the proof.

Theorem $a + 1$ limits $R_a x$, for all a .

Proof $(\forall x \geq a + 1)(R_a(x + L(a+1)) = R_a x)$, since $a \mid L(a+1)$,
 which completes the proof.

Theorem $a(Li+1)$ limits $[x/a]$, for all a .

Proof $(\forall x \geq ai)(\alpha_i([(x + L(a(Li+1)))/a]) = i = \alpha_i[x/a])$
 and $\forall x([(x + L(a(Li+1)))/a] \equiv [x/a] \pmod{Li})$, since
 $[(x + aj)/a] = [x/a] + j$
 and $aLi \mid L(a(Li+1))$,

which completes the proof.

Theorem $Li + 1$ limits Lx .

Proof $(\forall x > i)(\alpha_i L(x + L(Li+1))) = i = \alpha_i Lx$,

and $(\forall x \geq i)(L(x + L(Li+1)) \equiv 0 \equiv Lx \pmod{Li})$,

which completes the proof.

Theorem $a + 1$ limits $a \cap x$, for all a .

Proof If $y|a$, then $y|x + L(a+1) \leftrightarrow y|x$, since $a|L(a+1)$,

and so, $a \cap (x + L(a+1)) = a \cap x$,

which completes the proof.

Theorem $i + a + 1$ limits $a \cup x$, for all a .

Proof If $y|a$, then $y|x + L(i+a+1) \leftrightarrow y|x$, since $a|L(i+a+1)$,

and so, $a \cap (x + L(i+a+1)) = a \cap x$.

Now, $\frac{a}{a \cap x} (x + L(i+a+1)) \equiv \frac{a}{a \cap x} x \pmod{Li}$, since

$$Li | L(i+a+1),$$

and so, $\frac{a(x + L(i+a+1))}{a \cap (x + L(i+a+1))} \equiv \frac{ax}{a \cap x} \pmod{Li}$

$$\text{i.e. } a \cup (x + L(i+a+1)) \equiv a \cup x \pmod{Li}.$$

Moreover, $(\forall x \geq i+a+1)(\alpha_i (a \cup (x + L(i+a+1))) = i = \alpha_i (a \cup x))$,

which completes the proof.

Remark: It seems that $x \cup y$ is unlimited, but I have, as yet, been unable to show this.

§2. Unlimited Constituents

Theorem The predicate "x is a power of a" is unlimited,
for all $a > 1$.

Proof Suppose there is an $a > 1$, such that "x is a power of a" is limited. Then, there exists ℓ such that ℓ limits "x is a power of a" and so for all $x \geq \ell$, we have

$$x + L\ell \text{ is a power of } a \iff x \text{ is a power of } a.$$

Putting $x = a^{L\ell}$, we obtain

$$a^{L\ell} + L\ell \text{ is a power of } a, \text{ since } a^{L\ell} > L\ell \geq \ell - 1.$$

This is a contradiction, since $a^{L\ell} + L\ell < a^{L\ell+1}$.

Thus, "x is a power of a" cannot be limited, and so we have the theorem.

Theorem The predicate "x is an a-th power" is unlimited, for
all $a > 1$.

Proof Suppose there is an $a > 1$, such that "x is an a-th power" is limited. Then, there exists ℓ such that ℓ limits "x is an a-th power", and so for all $x \geq \ell$, we have

$$x + L\ell \text{ is an } a\text{-th power} \iff x \text{ is an } a\text{-th power}.$$

Putting $x = (L\ell + 1)^a$, we obtain

$$(L\ell + 1)^a + L\ell \text{ is an } a\text{-th power, since } (L\ell + 1)^a \geq L\ell + 1 \geq \ell.$$

This is a contradiction, since $(L\ell + 1)^a + L\ell < (L\ell + 2)^a$.

Thus, "x is an a-th power" cannot be limited, and so we have the theorem.

Theorem $\min(x,y)$ is unlimited.

Proof Suppose $\min(x,y)$ is limited. Then, there exists m such that, for all $x \geq m$, we have

$$\min(x + lm, y) \equiv \min(x, y) \pmod{L3}$$

$$\text{i.e. } \min(x + lm, y) \equiv \min(x, y) \pmod{2}, \text{ since } L3 = 2.$$

Putting $x = m$, and $y = m+1$, we obtain

$$\min(m + lm, m+1) \equiv \min(m, m+1) \pmod{2}$$

$$\text{i.e. } m + 1 \equiv m \pmod{2}, \text{ a contradiction.}$$

Thus, $\min(x,y)$ cannot be limited, and so we have the theorem.

Theorem $\max(x,y)$ is unlimited.

Proof Suppose $\max(x,y)$ is limited. Then, there exists $m > 2$, such that, for all $x \geq m$, we have

$$\max(x + lm, y) \equiv \max(x, y) \pmod{L3}$$

$$\text{i.e. } \max(x + lm, y) \equiv \max(x, y) \pmod{2}, \text{ since } L3 = 2.$$

Putting $x = m$, and $y = m+1$, we obtain

$$\max(m + lm, m+1) \equiv \max(m, m+1) \pmod{2}$$

$$\text{i.e. } m + lm \equiv m+1 \pmod{2}$$

This is a contradiction, since $2 \nmid lm$.

Thus, $\max(x,y)$ cannot be limited, and so we have the theorem.

Theorem Rtx is unlimited.

Proof Suppose Rtx is limited. Then, there exists $m > 2$, such that, for all $x \geq m$, we have

$$Rt(x + Lm) \equiv Rtx \pmod{L3}$$

i.e. $Rt(x + Lm) \equiv Rtx \pmod{2}$, since $L3 = 2$.

Now, $Rt(m + Lm) > Rtm$, and so there exists $x \geq m$, such that

$$Rt(x + Lm) = Rtx + 1, \text{ which contradicts the above.}$$

Thus, Rtx cannot be limited, and so we have the theorem.

Remark: It would be interesting to try and decide whether the other functions in the sequence $x+1, x+y, xy, y^x, \dots$ are limited. If we could show that the rest of the sequence is unlimited, then we would know that the limited functions are not closed under $\Phi_3, \Phi_4, \Phi_5, \dots$. In the next section we show that the limited functions are closed under Φ_1 and Φ_2 .

§3. Closure of the Limited Functions under Σ and Π

In this section, we show that both the limited functions and the recursively limited functions are closed under Σ and Π .

Theorem If p_i limits f_x , then $p_i + (i+1)L_i$ limits $\Sigma f(x)$.

Proof Let q_i denote $p_i + (i+1)L_i$.

Now, $(\forall x \leq p_i + L_i)(\Sigma f(p_i + iL_i) < i \rightarrow$

$$f(x) + f(x+L_i) + \dots + f(x + (i-1)L_i) < i)$$

and so, $(\forall x \geq p_i)(x \leq p_i + L_i \wedge \Sigma f(p_i + iL_i) < i \rightarrow if(x) < i)$.

Thus, $(\forall x \geq p_i)(x \leq p_i + L_i \wedge \Sigma f(p_i + iL_i) < i \rightarrow f_x = 0)$

and so, $(\forall x \geq p_i)(\Sigma f(p_i + iL_i) < i \rightarrow f_x = 0)$.

Hence, $\alpha_i \Sigma f(p_i + iL_i + x) = \alpha_i \Sigma f(p_i + iL_i)$,

and so, $(\forall x \geq q_i)(\alpha_i \Sigma f(x + L_i) = \alpha_i \Sigma f(x))$, since $q_i \geq p_i + iL_i$.

Now, there exists j such that $Lq_i = jLiLp_i$, since $LiLp_i < q_i$,
 and moreover, $(\forall x \geq q_i)(f(x+Lp_i) \equiv fx \pmod{Li})$, since $q_i \geq p_i$.
 Thus, $\Sigma f(q_i+Lq_i) \equiv \Sigma f(q_i) + jLi(f(q_i+1) + \dots + f(q_i+Lp_i)) \pmod{Li}$
 i.e. $\Sigma f(q_i+Lq_i) \equiv \Sigma f(q_i) \pmod{Li}$
 and so, $(\forall x \geq q_i)(\Sigma f(x+Lq_i) \equiv \Sigma f(x) \pmod{Li})$.
 Hence, q_i limits $\Sigma f(x)$, and so we have the theorem.

Corollary If p_i limits $f(x_0, x_1, \dots, x_n)$, then $p_i + (i+Li)Lp_i$ limits
 $\Sigma f(x_0, x_1, \dots, x_n)$.

Proof q_i limits $\Sigma f(x, a_1, \dots, a_n)$, for all $a_1, \dots, a_n$, by the theorem.

Moreover, p_i limits $\Sigma f(a, x_1, \dots, x_n)$, for all a , since i limits
 $y_0 + \dots + y_a$.

Thus, we have the corollary, since $p_i \leq q_i$.

This corollary shows that both the limited functions and the
 recursively limited functions are closed under Σ .

Theorem If p_i limits fx , then $((Li)! + 1)Lp_i$ limits $\Pi f(x)$.

Proof Let q_i denote $((Li)! + 1)Lp_i$,
 and let a denote $f(p_i + 1) \dots f(p_i + Lp_i)$.

Then, $0 < \Pi f(p_i + iLp_i) < i \rightarrow \Pi f(p_i) a^i < i$

$$\rightarrow a = 1$$

$$\rightarrow (\forall x > p_i)(fx = 1)$$

Thus, $\alpha_i \Pi f(p_i + iLp_i + x) = \alpha_i \Pi f(p_i + iLp_i)$

and so, $(\forall x \geq q_i)(\alpha_i \Pi f(x+Lq_i) = \alpha_i \Pi f(x))$, since $q_i \geq p_i + iLp_i$.

Now, there exists j such that $Lq_i = j(Li)!Lp_i$, since $(Li)!Lp_i < q_i$, and moreover, $(\forall x \geq p_i)(f(x+Lp_i) \equiv fx \pmod{Li})$.

Thus, $\Pi f(p_i + LiLp_i + Lq_i) \equiv \Pi f(p_i) a^{Li+j(Li)!} \pmod{Li}$
 $\equiv \Pi f(p_i) a^{Li} \pmod{Li}$, by the lemma in §2.
 $\equiv \Pi f(p_i + LiLp_i) \pmod{Li}$,

and so, $(\forall x \geq q_i)(\Pi f(x+Lq_i) \equiv \Pi f(x) \pmod{Li})$, since $q_i \geq p_i + LiLp_i$.
Hence, q_i limits $\Pi f(x)$, and so we have the theorem.

Corollary If p_i limits $f(x_0, x_1, \dots, x_n)$, then $((Li)!+1)Lp_i$
limits $\Pi f(x_0, x_1, \dots, x_n)$.

Proof q_i limits $\Pi f(x, a_1, \dots, a_n)$, for all $a_1, \dots, a_n$, by the theorem.
Moreover, p_i limits $\Pi f(a, x_1, \dots, x_n)$, for all a , since i limits
 $y_0, \dots, y_a$.

Thus, we have the corollary, since $p_i \leq q_i$.

§4. A Decidable Class

Let $\mathcal{F}$ denote the class of functions generated by Σ and Π from
 $\{a, x, x+1, x+y, xy, x^y, \min(a, x), \max(a, x), a-x, x-a, x!, Lx, [x/a], R_a x, a \cap x, a \cup x : a \in \mathbb{N}\}$.

Let $\mathcal{L}$ denote $\{x > a, x = a, x < a, x | a, a | x, x \equiv y \pmod{a} : a \in \mathbb{N}\} \cup \mathcal{F}$.

Clearly, $\mathcal{L}$ is effectively limited, since the recursively limited functions are closed under Σ and Π , and moreover, each constituent in $\mathcal{L}$ is recursive. Thus, the class of sentences with constituents $\mathcal{L}$ is decidable.

Chapter 6

$$\underline{a|x \quad (a \in \mathbb{N})}$$

Let $\mathcal{B}_\ell$ be the Boolean algebra generated by $\{\{a\}, [a] : a < \ell\}$, where $[a]$ denotes $\{az : z \in \mathbb{N}\}$.

We now use the relation $[a] \cap [b] = [a \cup b]$ to simplify $B_\ell x$. Clearly, we have, for all x greater than or equal to ℓ ,

$$\begin{aligned} B_\ell x &= \bigcap \{[a] : a < \ell, \text{ and } a|x\} = \bigcup \{[a] : a < \ell, \text{ and } a \nmid x\} = [0, \ell) \\ &= [L\ell \cap x] = \bigcup \{[a] : a < \ell, \text{ and } a \nmid x\} = [0, \ell), \text{ since} \end{aligned}$$

the least common multiple of all a less than ℓ which divide x is $L\ell \cap x$.

We now define a representative function for $\mathcal{B}_\ell$.

$$\text{Let } \beta_\ell x \text{ denote } \begin{cases} x & \text{if } x < \ell \\ (L\ell \cap x)(L\ell + 1) & , \text{ otherwise.} \end{cases}$$

The following two theorems show that $\beta_\ell x$ is representative.

Theorem $\beta_\ell x \in B_\ell x$

Proof $(\forall x \geq \ell)(\beta_\ell x \notin \bigcup \{[a] : a < \ell, \text{ and } a \nmid x\})$, since

$$\begin{aligned} a < \ell &\rightarrow a|L\ell \\ &\rightarrow a \cap (L\ell + 1) = 1 \\ &\rightarrow a|(L\ell \cap x)(L\ell + 1) \rightarrow a|L\ell \cap x \end{aligned}$$

and so, $a < \ell \wedge a \nmid x \rightarrow a \nmid (L\ell \cap x)(L\ell + 1)$.

Moreover, $(\forall x \geq l)(\beta_l x \geq Ll + 1 \geq l)$.

Hence, $(\forall x \geq l)(\beta_l x \in B_l x)$, and so we have the theorem for all x .

Theorem $B_l x = B_l y \rightarrow \beta_l x = \beta_l y$.

Proof $B_l x = B_l y \rightarrow (\forall a < l)(a|x \leftrightarrow a|y)$
 $\rightarrow Ll \cap x = Ll \cap y$
 $\rightarrow (\forall x, y \geq l)(\beta_l x = \beta_l y)$.

Thus, we have the theorem for all x and y , since

$$B_l x = B_l y \rightarrow \alpha_l x = \alpha_l y.$$

Finally, we give a bound for $\beta_l x$.

Theorem $\beta_l x \leq l^{2l}$

Proof $\beta_l x \leq Ll(Ll+1)$, for all $x \geq l$.
 $\leq l^{2l}$, since $Ll < l^l$.

Thus, we have the theorem, since $l \leq l^{2l}$.

Thus, $b_l \leq l^{2l}$, and so b_l is recursive, since the constituents and l^{2l} are recursive.

§1. Limited Constituents

Clearly, the limited predicates are a subset of the limited predicates of the preceding chapter. An obvious question is

"For which a and b are the predicates $x \equiv a \pmod{b}$ limited?"

The following theorem provides the complete answer and thus clarifies the relationship with the preceding chapter.

Theorem $x \equiv a \pmod{b}$ is limited if and only if $b|2a$, for all a, b .

Proof Suppose $x \equiv a \pmod{b}$. Then, there exists ℓ , greater than a , such that ℓ limits $x \equiv a \pmod{b}$.

Thus, for all x greater than or equal to ℓ , we have

$$x \equiv a \pmod{b} \leftrightarrow (L\ell \cap x)(L\ell + 1) \equiv a \pmod{b}$$

and so, there exists k such that

$$x \equiv a \pmod{b} \leftrightarrow (ak \cap x)(ak + 1) \equiv a \pmod{b}, \text{ since } a|L\ell.$$

Thus, for all y greater than or equal to ℓ , we have

$$ay \equiv a \pmod{b} \leftrightarrow (ak \cap ay)(ak + 1) \equiv a \pmod{b}$$

and so, $y \equiv 1 \pmod{\beta} \leftrightarrow (k \cap y)(ak + 1) \equiv 1 \pmod{\beta}$, where β denotes $b/a \cap b$,

since $ax \equiv a \pmod{b} \leftrightarrow x \equiv 1 \pmod{\beta}$.

Thus, $y \equiv 1 \pmod{\beta} \leftrightarrow k \cap y \equiv 1 \pmod{\beta}$, since, if we put $y = k\beta\ell + 1$, we obtain $ak + 1 \equiv 1 \pmod{\beta}$,

and so, $(\ell+1)^{k+1} \equiv 1 \equiv (\ell+1)^k - 1 \pmod{\beta}$, since

$$k \cap ((\ell+1)^k + 1) = 1 = k \cap ((\ell+1)^k - 1)$$

and $(\ell+1)^k - 1 \geq \ell k \geq \ell$.

Hence, $\beta|2$, and so $b|2(a \cap b)$, giving $b|2a$.

Conversely, we suppose $b|2a$.

Then, if $b|a$, we have $x \equiv a \pmod{b} \leftrightarrow b|x$,

and if $b \nmid a$, we have b is even and $b/2|a$, giving

$$x \equiv a \pmod{b} \leftrightarrow b/2|x \wedge b \nmid x.$$

Thus, in either case $x \equiv a \pmod{b}$ is clearly limited, and so we have the theorem.

We have the following interesting corollary to this theorem.

Corollary $[a]_b$ is a Boolean combination of $\{[0]_c : c \in \mathbb{N}\}$ if and only if $b|2a$, for all a, b .

a limits Px if and only if, for all $x \geq a$, we have

$$Px \leftrightarrow P\beta_a x$$

$$\text{i.e. } P(x) \leftrightarrow P((La \cap x)(La + 1))$$

p_i limits fx if and only if, for all $x \geq p_i$, we have

$$\beta_i fx = \beta_i f\beta_{p_i} x$$

$$\text{i.e. } \beta_i fx = \beta_i f((Lp_i \cap x)(Lp_i + 1))$$

$$\text{i.e. } \alpha_i fx = \alpha_i f((Lp_i \cap x)(Lp_i + 1))$$

$$\text{and } Li \cap fx = Li \cap f((Lp_i \cap x)(Lp_i + 1)).$$

We now give some simple examples of limited functions.

Theorem i limits xy .

Proof i limits ax , since

$$\alpha_i ax = i = \alpha_i a(Li \cap x)(Li + 1), \text{ if } ax > 0$$

$$\text{and } Li \cap a(Li \cap x)(Li + 1) = Li \cap a(Li \cap x) = Li \cap ax,$$

which completes the proof

Theorem i limits x^y .

Proof i limits x^a , since

$$\alpha_i (Li \cap x)^a (Li + 1)^a = i = \alpha_i x^a, \text{ for all } x \geq i, \text{ and all } a > 0,$$

$$\text{and } Li \cap (Li \cap x)^a (Li + 1)^a = Li \cap (Li \cap x)^a, \text{ since } Li \text{ is coprime to } Li + 1.$$

$$= Li \cap (Li)^a \cap x^a$$

$$= Li \cap x^a.$$

Also, i limits $f_b x$, if $f_b x$ denotes b^x , since
 $f_b^{-1}[a] = \{x : a|b^x\} \in \mathcal{B}_a$, since $a|b^a \leftrightarrow a|b^{a+x}$,
 and so, $f_b^{-1} : \mathcal{B}_1 \rightarrow \mathcal{B}_1$, and we have proved the theorem.

Clearly, 3 limits $R_2 x$, since $R_2^{-1} 1 = N - [2]$.

Theorem i limits $x!$

Proof Let Fx denote $x!$. Then,

$$F^{-1}[a] \supset [a, \infty)$$

and so $F^{-1}[a] \in \mathcal{B}_a$.

Thus, $F^{-1} : \mathcal{B}_1 \rightarrow \mathcal{B}_1$,

which completes the proof.

Theorem $a + 1$ limits $a \cap x$, for all a .

Proof $a \cap (L(a+1) \cap x)(L(a+1) + 1) = a \cap (L(a+1) \cap x)$, since
 $a|L(a+1)$.

$$= a \cap x,$$

which completes the proof.

Theorem $i + 1$ limits Lx .

Proof For all $x \geq i + 1$, we have

$$\alpha_i L((L(i+1) \cap x)(L(i+1) + 1)) = i = \alpha_i Lx, \text{ since } L(i+1) \geq i$$

and $Li \cap L((L(i+1) \cap x)(L(i+1) + 1)) = Li = Li \cap Lx$,

which completes the proof.

Theorem i limits $x \cup y$.

Proof i limits $a \cup x$, since

$\alpha_i(a \cup (Li \cap x)(Li + 1)) = i = \alpha_i(a \cup x)$, for all $x \geq i$
 and $Li \cap (a \cup (Li \cap x)(Li + 1)) = (Li \cap a) \cup (Li \cap (Li \cap x)(Li + 1))$
 $= (Li \cap a) \cup (Li \cap x)$
 $= Li \cap (a \cup x)$,
 which completes the proof.

§2. Unlimited Constituents

Theorem $x + a$ is unlimited, for all $a > 0$.

Proof Let $f_a x$ denote $x + a$, then, for all $a > 1$, we have

$$f_a^{-1}[a+1] = [1]_{a+1} \notin \mathcal{B}, \text{ by } \S 1.$$

$$\text{and } f_1^{-1}[3] = [2]_3 \notin \mathcal{B}, \text{ by } \S 1.$$

Thus, $x + a$ cannot be limited, if $a > 0$, and so we have the theorem.

Corollary $x + y$ is unlimited.

Theorem $x \dot{-} a$ is unlimited, for all $a > 0$.

Proof Let $f_a x$ denote $x \dot{-} a$, then, for all $a > 1$, we have

$$f_a^{-1}[a+1] = \{0\} \cup [a]_{a+1} \notin \mathcal{B}, \text{ by } \S 1$$

$$\text{and } f_1^{-1}[3] = \{0\} \cup [1]_3 \notin \mathcal{B}, \text{ by } \S 1.$$

Thus, $x \dot{-} a$ cannot be limited, if $a > 0$, and so we have the theorem.

Theorem $R_a x$ is unlimited, for all $a > 2$.

Proof $R_a^{-1}(1) = [1]_a \nmid \mathcal{B}$, by §1.

Thus, $R_a x$ cannot be limited, if $a > 2$, and so we have the theorem.

Theorem $[x/a]$ is unlimited, for all $a > 1$.

Proof Suppose there is an $a > 1$, such that $[x/a]$ is limited.

Then, there exists $p > 2a$, such that, for all $x \geq p$, we have

$$[(Lp \cap x)(Lp + 1)/a] \equiv [x/a] \pmod{L3}, \text{ and so}$$

$$[(Lp \cap x)(Lp + 1)/a] \equiv [x/a] \pmod{2}, \text{ since } L3 = 2.$$

Putting $x = Lp-1$, we obtain

$$[(Lp+1)/a] \equiv [(Lp-1)/a] \pmod{2}, \text{ since } Lp > p \text{ if } p > 3.$$

Now, there exists q such that $Lp = 2aq$, and so

$$[(2aq+1)/a] \equiv [(2aq-1)/a] \pmod{2}$$

$$\text{i.e.} \quad 2q \equiv 2q-1 \pmod{2}, \text{ a contradiction.}$$

Thus, $[x/a]$ cannot be limited, and so we have the theorem.

Theorem $\min(x,y)$ is unlimited.

Proof Suppose $\min(x,y)$ is limited. Then, there exists $p > 3$, such that, for all $x \geq p$, we have

$$\min((Lp \cap x)(Lp+1), y) \equiv \min(x,y) \pmod{L3}$$

$$\text{and so, } \min((Lp \cap x)(Lp+1), y) \equiv \min(x,y) \pmod{2}, \text{ since } L3=2.$$

Putting $x = L_{p-1}$, and $y = L_p$, we obtain

$$\min(L_{p+1}, L_p) \equiv \min(L_{p-1}, L_p) \pmod{2}, \text{ since } L_p > p.$$

$$\text{i.e. } L_p \equiv L_{p-1} \pmod{2}, \text{ a contradiction.}$$

Thus, $\min(x, y)$ cannot be limited, and so we have the theorem.

Theorem $\max(x, y)$ is unlimited.

Proof Suppose $\max(x, y)$ is limited. Then, there exists $p > 3$,

such that, for all $x \geq p$, we have

$$\max((L_p \cap x)(L_{p+1}), y) \equiv \max(x, y) \pmod{L_3}$$

$$\text{and so, } \max((L_p \cap x)(L_{p+1}), y) \equiv \max(x, y) \pmod{2}, \text{ since } L_3 = 2.$$

Putting $x = L_{p-1}$, and $y = L_p$, we obtain

$$\max(L_{p+1}, L_p) \equiv (L_{p-1}, L_p) \pmod{2}, \text{ since } L_p > p.$$

$$\text{i.e. } L_{p+1} \equiv L_p \pmod{2}, \text{ a contradiction.}$$

Thus, $\max(x, y)$ cannot be limited, and so we have the theorem.

Theorem Rtx is unlimited.

Proof Suppose Rtx is limited. Then, there exists $p > 3$, such that, for all $x \geq p$, we have

$$Rt((Lp \cap x)(Lp+1)) \equiv Rtx \pmod{L3}$$

and so, $Rt((Lp \cap x)(Lp+1)) \equiv Rtx \pmod{2}$, since $L3 \equiv 2$.

Putting $x = Lp(Lp-1)$, we obtain

$$Rt(Lp(Lp+1)) \equiv Rt(Lp(Lp-1)) \pmod{2}$$

i.e. $Lp \equiv Lp-1 \pmod{2}$, a contradiction.

Thus, Rtx cannot be limited, and so we have the theorem.

§3. A Decidable Class

Let $\mathcal{L}$ denote $\{x > a, x = a, x < a, x|a, a|x, a, x, xy, x^y, \min(a, x), \max(a, x), a \div x, x!, Lx, R_2x, a \cap x, x \cup y : a \in \mathbb{N}\}$.

Clearly, $\mathcal{L}$ is an effectively limited class of recursive constituents.

Thus, the class of sentences with constituents $\mathcal{L}$ is decidable.

Remark: We see that, apart from containing sentences with $x \cup y$ a constituent, this class is considerably less extensive than that of the preceding chapter.

Chapter 7

x is a power of a ($a \in N$)

Let B_ℓ be the Boolean algebra generated by $\{\{a\}, [a] : a < \ell\}$, where $[a]$ denotes $\{a^z : z \in N\}$.

We now use the relation $[a^b] \cap [a^c] = [a^{b \cup c}]$ to simplify $B_\ell x$.

We first give the following definitions.

Let μx denote the least m such that x is a power of m , and let νx denote the greatest n such that x is an n -th power.

Then, x is a power of a if and only if $\mu a = \mu x$ and $\nu a | \nu x$.

Moreover, $x = \mu x^{\nu x}$.

We say that x is exponent free if $\mu x = x$.

For all $x \geq \ell$, we have

$$\begin{aligned} B_\ell x &= \left(\bigcap \{[a] : a < \ell, \text{ and } x \text{ is a power of } a\} - \right. \\ &\quad \left. \bigcup \{[a] : a < \ell, \text{ and } x \text{ is not a power of } a\} \right) - [0, \ell). \\ &= \begin{cases} (N - \bigcup \{[a] : a < \ell\}) - [0, \ell] & \text{if } \mu x \geq \ell \\ \left([(\mu x)^{(\text{Lk}_\ell \cap \nu x)}] - \bigcup \{[a] : a < \ell, \text{ and } x \text{ is not a power of } a\} \right) \\ \quad - [0, \ell), & \text{otherwise,} \end{cases} \end{aligned}$$

where, k_ℓ denotes the least k such that $(\mu x)^k \geq \ell$,

since, $\{a < \ell : x \text{ is a power of } a\} = \{a < \ell : \mu a = \mu x, \text{ and } \nu a | \nu x\}$

$$= \begin{cases} \text{the empty set} & \text{if } \mu x \geq \ell \\ \{(\mu x)^n : n < k_\ell, \text{ and } n | \nu x\} & \text{, otherwise,} \end{cases}$$

and the least common multiple of all $n < k_\ell$, which divide νx , is equal to

$\text{Lk}_\ell \cap \nu x$.

We now define a representative function for $\mathcal{B}_\ell$.

$$\text{Let } \beta_\ell x \text{ denote } \begin{cases} x & \text{if } x < \ell \\ (\mu x)(\text{Ik}_\ell \cap \nu x)(\ell+1) & \text{if } \mu x < \ell \leq x \\ p_\ell & \text{, otherwise.} \end{cases}$$

The following two theorems show that $\beta_\ell x$ is representative.

Theorem $\beta_\ell x \in B_\ell x$.

Proof We require only to show that

if $x \geq \ell > \mu x$, then $(\mu x)(\text{Ik}_\ell \cap \nu x)(\ell+1) \in B_\ell x$, since

$$p_\ell \in \left(N - \bigcup_{a < \ell} [a] \right) - [0, \ell) .$$

Now, for all $a < \ell$, we have $\nu a \cap (\ell+1)$, giving

$$\nu a | (\text{Ik}_\ell \cap \nu x)(\ell+1) \rightarrow \nu a | \nu x, \text{ and so}$$

$(\mu x)(\text{Ik}_\ell \cap \nu x)(\ell+1)$ is a power of $a \rightarrow x$ is a power of a .

Thus, $(\mu x)(\text{Ik}_\ell \cap \nu x)(\ell+1) \nmid \bigcup \{[a] : a < \ell, \text{ and } x \text{ is not a power of } a\}$

Finally, we claim that $(\mu x)(\text{Ik}_\ell \cap \nu x)(\ell+1) \geq \ell$, for all $x \geq \ell$.

This is clear when $x = 0$, and $x = 1$.

When $x \geq 2$, we have $\mu x \geq 2$, and so

$$(\mu x)(\text{Ik}_\ell \cap \nu x)(\ell+1) \geq 2 (\text{Ik}_\ell \cap \nu x)(\ell+1) \geq 2^{\ell+1} \geq 2^\ell > \ell.$$

Thus, the claim holds, and so we have the theorem.

Theorem $B_\ell x = B_\ell y \rightarrow \beta_\ell x = \beta_\ell y$.

Proof Suppose $B_\ell x = B_\ell y$. Then, for all $a < \ell$, we have

x is a power of $a \iff y$ is a power of a , and so

$$\bigcap \{[a] : a < \ell, \text{ and } x \text{ is a power of } a\} = \bigcap \{[a] : a < \ell, \text{ and } y \text{ is a power of } a\}.$$

Thus, for all x and y , such that $\mu x, \mu y < \ell$, we have

$$\left[(\mu x)^{(\text{Ik}_\ell \cap \nu x)} \right] = \left[(\mu y)^{(\text{Ik}_\ell \cap \nu y)} \right]$$

and so, $(\mu x)^{(\text{Ik}_\ell \cap \nu x)} = (\mu y)^{(\text{Ik}_\ell \cap \nu y)}$

giving, $\beta_\ell x = \beta_\ell y$ if $x, y \geq \ell$.

Thus, we have the theorem for all x and y , since, if

$B_\ell x = B_\ell y$, then

$$\alpha_\ell x = \alpha_\ell y, \text{ and } \alpha_\ell \mu x = \alpha_\ell \mu y.$$

Finally, we give a bound for $\beta_\ell x$.

Theorem $\beta_\ell x \leq \ell^{L\ell(L\ell+1)} + 1$.

Proof Clearly, $k_\ell \leq \ell$, and so

$$(\text{Ik}_\ell \cap \nu x)^{(L\ell+1)} \leq L\ell(L\ell+1).$$

Thus, we have the theorem, since $\ell, p_\ell \leq \ell^{L\ell(L\ell+1)} + 1$.

Thus, $b_\ell \leq \ell^{L\ell(L\ell+1)} + 1$, and so, b_ℓ is recursive, since the constituents and $\ell^{L\ell(L\ell+1)} + 1$ are recursive.

§1. Limited Constituents

ℓ limits Px if and only if, for all $x \geq \ell$, we have

$$P\beta_\ell x \leftrightarrow Px$$

i.e. $P(\mu x)^{(Lk_\ell \cap \nu x)(L\ell+1)} \leftrightarrow Px$ if $\mu x < \ell$, and

$$Pp_\ell \leftrightarrow Px, \text{ otherwise.}$$

p_i limits fx if and only if, for all $x \geq p_i$, we have

$$\beta_i fx = \beta_i f\beta_{p_i} x$$

i.e. $\beta_i fx = \beta_i f(\mu x)^{(Lk_{p_i} \cap \nu x)(Lp_i+1)}$ if $\mu x < p_i$, and

$$\beta_i fx = \beta_i fp_{p_i}, \text{ otherwise.}$$

Clearly, $\beta_i x = \beta_i y$ if and only if (a) $\alpha_i x = \alpha_i y$

$$(b) \alpha_i \mu x = \alpha_i \mu y$$

$$\text{and (c) } Lk_i \cap \nu x = Lk_i \cap \nu y.$$

Thus, p_i limits fx if and only if the following six conditions hold:

1. $\alpha_i fx = \alpha_i f(\mu x)^{(Lk_{p_i} \cap \nu x)(Lp_i+1)}$ if $x \geq p_i > \mu x$.
2. $\alpha_i \mu fx = \alpha_i \mu f(\mu x)^{(Lk_{p_i} \cap \nu x)(Lp_i+1)}$ if $x \geq p_i > \mu x$.
3. $Lk_i \cap \nu fx = Lk_i \cap \nu f(\mu x)^{(Lk_{p_i} \cap \nu x)(Lp_i+1)}$ if $x \geq p_i > \mu x$.
4. $\alpha_i fx = \alpha_i fp_{p_i}$ if $\mu x \geq p_i$.
5. $\alpha_i \mu fx = \alpha_i \mu fp_{p_i}$ if $\mu x \geq p_i$.
6. $Lk_i \cap \nu fx = Lk_i \cap \nu fp_{p_i}$ if $\mu x \geq p_i$.

Note that the final condition holds if $p_i \geq i$, for all i , since then $\mu x \geq i$, and so $N_i = 1$.

Thus, p_i limits fx if $p_i \geq i$, for all i , and if the first five conditions hold.

Theorem p_i limits fx if fx is exponent free and $fx \geq i$, for all $x \geq p_i$.

Proof Condition 1 holds, since $\alpha_i fx = i = \alpha_i f(\mu x)^{(Lk_{p_i} \cap vx)(Lp_i+1)}$
if $x \geq p_i$.

Condition 2 holds, since $\mu fx = fx$ if $x \geq p_i$, and since
condition 1 holds.

Condition 3 holds, since $\nu fx = 1$ if $x \geq p_i$.

Condition 4 holds, since $\alpha_i fx = i = \alpha_i fp_{p_i}$ if $\mu x \geq p_i$.

Condition 5 holds, since $\mu fx = fx$ if $\mu x \geq p_i$, and since
condition 4 holds.

Finally, condition 6 holds, since $\nu fx = fx$ if $\mu x \geq p_i$,

which completes the proof.

Corollary 1 i limits $x!$

Proof $p_{i+1} \leq 2p_i \leq p_i^2$, by Bertrand's theorem,
and so $\pi x \leq x < (\pi x)^2$, giving $\pi x | x!$ and $(\pi x)^2 \nmid x!$

Thus, $x!$ is exponent free.

Moreover, $x! \geq i$, for all $x \geq i$, since $x! \geq x$.

The corollary now follows by the theorem.

Corollary 2 $i + 1$ limits Lx .

Proof As in corollary 1, we have $\pi(x-1) < x \leq (\pi(x-1))^2$, and so

$$\pi(x-1) | Lx \text{ and } (\pi(x-1))^2 \nmid Lx.$$

Thus, Lx is exponent free.

Moreover, $Lx \geq i$, for all $x \geq i+1$, since $L(x+1) \geq x$.

The corollary now follows from the theorem.

Corollary 3 i limits p_x .

Proof p_x is a prime, and so is clearly exponent free.

Moreover, $p_x \geq i$, for all $x \geq i$, since $p_x \geq x$.

The corollary now follows from the theorem.

Corollary 4 $2i$ limits πx .

Proof πx is a prime, and so is clearly exponent free.

Moreover, $\pi x \geq i$, for all $x \geq 2i$, since $\pi 2x \geq x$, by Bertrand's theorem. The corollary now follows from the theorem.

By construction, $a + 1$ limits " x is a power of a ".

Theorem i limits μx .

Proof The first three conditions hold, since $\mu x = \mu(\mu x)^{(Lk_1 \cap \nu x)(Li+1)}$.

Condition 4 holds, since $\alpha_1 \mu x = i = \alpha_1 \mu p_1$ if $\mu x \geq i$.

Condition 5 holds, since $\mu \mu x = \mu x$, and since condition 4 holds,

which completes the proof.

Theorem i limits x^a , for all a .

Proof for all $a > 0$:

Condition 1 holds, since $\alpha_i x^a = i = \alpha_i (\mu x)^{(Lk_i \cap vx)(Li+1)a}$ if $x \geq i$.

Condition 2 holds, since $\mu(x^a) = \mu(\mu x)^{na}$.

Condition 3 holds, since $Lk_i \cap a(Lk_i \cap vx)(Li+1) = Lk_i \cap a(Lk_i \cap vx)$
 $= Lk_i \cap avx$.

(The first equality holds, since $k_i \leq i$, giving $Lk_i \cap (Li+1) = 1$).

Condition 4 holds, since $\alpha_i x^a = i = \alpha_i x^{p_i}$ if $\mu x \geq i$.

Condition 5 holds, since $\alpha_i \mu(x^a) = i = \alpha_i \mu(p_i^a)$ if $\mu x \geq i$,

which completes the proof.

§2. Unlimited Constituents

Theorem $\text{Pr}(x)$ is unlimited.

Proof Suppose ℓ limits $\text{Pr}(x)$. Then, for all x , such that $\mu x \geq \ell$,
 we have $\text{Pr}(p_\ell) \leftrightarrow \text{Pr}(x)$

and we obtain a contradiction by putting $x = p_\ell p_{\ell+1}$.

Thus, $\text{Pr}(x)$ cannot be limited, and so we have the theorem.

Theorem $a|x$ is unlimited, for all $a > 1$.

Proof Suppose there is an $a > 1$, such that some ℓ limits $a|x$.

Then, for all x , such that $\mu x \geq \ell$, we have

$$a|p_\ell \leftrightarrow a|x.$$

Then, either $a|p_\ell$, giving $a|p_{\ell+1}$, a contradiction

or $a \nmid p_\ell$, giving $a \nmid ap_{\ell+a}$, a contradiction.

Thus, $a|x$ cannot be limited, and so we have the theorem.

Theorem The predicate x is an a -th power is unlimited,
for all $a > 1$.

Proof Suppose there is an $a > 1$, such that some ℓ limits the
predicate x is an a -th power.

Then, for all x , such that $\mu x \geq \ell$, we have

$$p_\ell \text{ is an } a\text{-th power} \leftrightarrow x \text{ is an } a\text{-th power}$$

and we obtain a contradiction by putting $x = p_\ell^a$.

Thus, " x is an a -th power" cannot be limited, and so we have the theorem.

§3. A Decidable Class

Let $\mathcal{L}$ denote $\{x > a, x = a, x < a, x|a, x \text{ is a power of } a, a, x, x^a, \min(a, x), \max(a, x), a \dot{-} x, x!, Lx, p_x, \pi x : a \in \mathbb{N}\}$.

Clearly, $\mathcal{L}$ is an effectively limited class of recursive constituents.

Thus, the class of sentences with constituents $\mathcal{L}$ is decidable.

Chapter 8

x is an a -th power ($a \in \mathbb{N}$)

Let B_ℓ be the Boolean algebra generated by $\{\{a\}, [a] : a < \ell\}$, where $[a]$ denotes $\{z^a : z \in \mathbb{N}\}$.

Now, x is an a -th power if and only if $a \mid \nu x$.

We now use the relation " $[a] \cap [b] = [a \cup b]$ " to simplify $B_\ell x$.

For all $x \geq \ell$, we have

$$\begin{aligned} B_\ell x &= \left(\bigcap \{[a] : a < \ell, \text{ and } x \text{ is an } a\text{-th power}\} - \right. \\ &\quad \left. \bigcup \{[a] : a < \ell, \text{ and } x \text{ is not an } a\text{-th power}\} \right) - [0, \ell]. \\ &= \left([L\ell \cap \nu x] - \bigcup \{[a] : a < \ell, \text{ and } x \text{ is not an } a\text{-th power}\} \right) - [0, \ell], \end{aligned}$$

since the least common multiple of all $a < \ell$, which divide νx , is equal to $L\ell \cap \nu x$.

We now define a representative function for ℓ .

$$\text{Let } \beta_\ell x \text{ denote } \begin{cases} x & \text{if } x < \ell \\ p_\ell^{L\ell \cap \nu x} & , \text{ otherwise.} \end{cases}$$

The following two theorems show that $\beta_\ell x$ is representative.

Theorem $\beta_\ell x \in B_\ell x$.

Proof $p_\ell^{L\ell \cap \nu x} \notin \bigcup \{[a] : a < \ell, \text{ and } a \nmid \nu x\}$, since
if $a \mid L\ell \cap \nu x$, then $a \mid \nu x$.

Thus, for all $x \geq \ell$, we have $\beta_\ell x \in B_\ell x$, since $p_\ell^{L\ell \cap \nu x} \geq p_\ell \geq \ell$.

Hence, we have the theorem, for all x .

Theorem $B_\ell x = B_\ell y \rightarrow \beta_\ell x = \beta_\ell y$.

Proof Suppose $B_\ell x = B_\ell y$. Then, for all $a < \ell$, we have
 x is an a -th power $\leftrightarrow y$ is an a -th power, and so
 $a | \nu x \leftrightarrow a | \nu y$.

Thus, the least common multiple of all $a < \ell$, which divide νx , is equal
to the least common multiple of all $a < \ell$, which divide νy .

i.e. $L_\ell \cap \nu x = L_\ell \cap \nu y$.

Thus, for all $x, y \geq \ell$, we have $\beta_\ell x = \beta_\ell y$.

Hence, we have the theorem, for all x and y , since

$$B_\ell x = B_\ell y \rightarrow \alpha_\ell x = \alpha_\ell y.$$

Finally, we give a bound for $\beta_\ell x$.

$$\beta_\ell x \leq p_\ell^{L_\ell}, \text{ since } \ell, p_\ell^{L_\ell \cap \nu x} \leq p_\ell^{L_\ell}.$$

Thus, $b_\ell \leq p_\ell^{L_\ell}$, and so, b_ℓ is recursive, since the constituents
and $p_\ell^{L_\ell}$ are recursive.

§1. Limited Constituents

ℓ limits Px if and only if, for all $x \geq \ell$, we have

$$P\beta_\ell x \leftrightarrow Px$$

i.e. $P_{p_\ell^{L_\ell \cap \nu x}} \leftrightarrow Px$

p_i limits fx if and only if, for all $x \geq p_i$, we have

$$\beta_i fx = \beta_i f\beta_{p_i} x$$

i.e. $\beta_i fx = \beta_i f_{p_i^{L_{p_i} \cap \nu x}}$.

Clearly, $\beta_l x = \beta_l y$ if and only if (a) $\alpha_l x = \alpha_l y$

and (b) $Li \cap \nu x = Li \cap \nu y$.

Thus, π_l limits fx if and only if the following two conditions hold:

1. $\alpha_l fx = \alpha_l f p_{\pi_l}^{Li \cap \nu x}$ if $x \geq \pi_l$
2. $Li \cap \nu fx = Li \cap \nu f p_{\pi_l}^{Li \cap \nu x}$ if $x \geq \pi_l$.

Theorem π_l limits fx if fx is exponent free and $fx \geq i$,
for all $x \geq \pi_l$.

Proof Condition 1 holds, since $\alpha_l fx = i = \alpha_l f p_{\pi_l}^{Li \cap \nu x}$ if $x \geq \pi_l$

Condition 2 holds, since $Li \cap \nu fx = Li \cap \nu f p_{\pi_l}^{Li \cap \nu x}$, for $\nu fx = 1$,

which completes the proof.

The four corollaries below follow exactly as the corresponding corollaries in the preceding chapter, since we only need to show that the same conditions are satisfied.

Corollary 1. i limits $x!$.

Corollary 2. $i+1$ limits Lx .

Corollary 3 i limits p_x .

Corollary 4 $2i$ limits πx .

By construction, $a+1$ limits the predicate x is an a -th power.

Theorem i limits x^a , for all a .

Proof for all $a > 0$:

Condition 1 holds, since $\alpha_i x^a = i = \alpha_i p_i (Li \cap vx)^a$ if $x \geq i$.

Condition 2 holds, since $Li \cap v(x^a) = Li \cap v(p_i (Li \cap vx)^a)$

$$\text{i.e. } Li \cap avx = Li \cap (Li \cap vx)^a,$$

which completes the proof.

§2. Unlimited Constituents

Theorem $a|x$ is unlimited, for all $a > 1$.

Proof Suppose there is an $a > 1$, such that $a|x$ is limited.

Then, there exists $\ell > a$, such that ℓ limits $a|x$, and so for

all $x \geq \ell$, we have $a|p_\ell^{L\ell} \cap vx \leftrightarrow a|x$.

Then, either $a|p_\ell^{L\ell}$, giving $a|p_{\ell+1}^{L\ell}$, a contradiction,

or $a \nmid p_\ell^{L\ell}$, giving $a \nmid a\ell$, a contradiction.

Thus, $a|x$ cannot be limited, and so we have the theorem.

Theorem The predicate x is a power of a is unlimited, for
all $a > 1$.

Proof Suppose there is an $a > 1$, such that the predicate x is a power of a is limited. Then, there exists $\ell > a$, such that ℓ limits the predicate x is a power of a , and so for all $x \geq \ell$, we have

$$p_\ell^{L\ell} \cap vx \text{ is a power of } a \leftrightarrow x \text{ is a power of } a.$$

Then, either $p_\ell^{L\ell}$ is a power of a , giving $p_{\ell+1}^{L\ell}$ is a power of a ,
a contradiction,
or $p_\ell^{L\ell}$ is not a power of a , giving $a^{L\ell}$ is not a power of a ,
a contradiction.

Thus, " x is a power of a " cannot be limited, and so we have the theorem.

Theorem $\Pr(x)$ is unlimited.

Proof Suppose ℓ limits $\Pr(x)$. Then, for all $x \geq \ell$, we have

$$\Pr(p_\ell^{L\ell} \cap \nu x) \leftrightarrow \Pr(x)$$

and we obtain a contradiction by putting $x = 2^{L\ell+1}$.

Thus, $\Pr(x)$ cannot be limited, and so we have the theorem.

Theorem $x+1$ is unlimited.

Proof Suppose $x+1$ is limited. Then, there exists $p > 2$, such

that, for all $x \geq p$, we have $\nu(x+1) \equiv \nu(p_p^{Lp} \cap \nu x + 1) \pmod{L3}$

i.e. $\nu(x+1) \equiv \nu(p_p^{Lp} \cap \nu x + 1) \pmod{2}$,

since $L3 = 2$.

Putting $x = (Lp-1)(Lp+1)$, we obtain

$$p_p + 1 \equiv 0 \pmod{2}, \text{ since } x+1 = (Lp)^2$$

and so, there exists $y > 1$ such that $p_p + 1 = (y+1)^2$

i.e. $p_p = y(y+2)$, a contradiction.

Thus, $x + 1$ cannot be limited, and so we have the theorem.

Corollary $x + y$ is unlimited.

Theorem ax is unlimited, if a is not a square.

Proof Suppose there is some non-square a such that ax is limited.

Then, there exists $p > a$, such that for all $x \geq p$, we have

$$\nu(ax) \equiv \nu(a p_p^{Lp} \cap \nu x) \pmod{L3}$$

$$\text{i.e. } \nu(ax) \equiv \nu(a p_p^{Lp} \cap \nu x) \pmod{2}, \text{ since } L3 = 2$$

$$\text{i.e. } Sq(ax) \leftrightarrow Sq(a p_p^{Lp} \cap \nu x).$$

Putting $x = ap^2$, we obtain

$$Sq(a p_p), \text{ since } \nu x = 1, \text{ a contradiction.}$$

Thus, ax cannot be limited, and so we have the theorem.

Remark: It may be shown that the theorem holds in fact for all $a > 1$.

Corollary xy is unlimited.

Theorem a^x is unlimited, if a is not a square.

Proof Suppose there is some non-square a such that a^x is limited.

Then, there exists $p > a$ such that for all $x \geq p$, we have

$$\nu(a^x) \equiv \nu(a p_p^{Lp} \cap \nu x) \pmod{L3}$$

$$\text{i.e. } \nu(a^x) \equiv \nu(a p_p^{Lp} \cap \nu x) \pmod{2}, \text{ since } L3 = 2.$$

$$\text{i.e. } Sq(a^x) \leftrightarrow Sq(a p_p^{Lp} \cap \nu x)$$

and so, $2 \nmid x$.

Putting $x = 2p$, we obtain a contradiction.

Thus, a^x cannot be limited, and so we have the theorem.

Remark Again, it may be shown that the theorem holds in fact for all $a > 1$.

Corollary x^y is unlimited.

§3. A Decidable Class

Let $\mathcal{L}$ denote $\{x > a, x = a, x < a, x|a, x \text{ is an } a\text{-th power}, a, x, x^a, \min(a, x), \max(a, x), a \div x, x! Lx, p_x, \pi x : a \in \mathbb{N}\}$.

Clearly, $\mathcal{L}$ is an effectively limited class of recursive constituents. Thus, the class of sentences with constituents $\mathcal{L}$ is decidable.

Remark These constituents are the same as those of the preceding chapter, except for having "x is an a-th power" in place of "x is a power of a".

References

[1] R.L.GOODSTEIN and R.D.LEE,

A decidable class of equations in recursive arithmetic,
Zeitschr.f.math.Logik und Grundlagen d.Math. Bd.12,S.235-239(1966).

[2] G.ROUSSEAU,

A decidable class of number theoretic equations,
Journal London Math.Soc.,41(1966), 737-741.

Decidable Classes of Number-Theoretic Sentences: A Summary

The thesis is in two parts. In the first, I give a method for constructing decidable classes of number theoretic sentences, and in the second, I apply this method in the construction of particular decidable classes.

Let $\mathcal{B}_0, \mathcal{B}_1, \dots$ be an increasing sequence of finite Boolean algebras of subsets of the natural numbers, N , such that $\{l-1\} \in \mathcal{B}_l$, for all $l > 0$. We call the $\mathcal{B}_l$ limiters. We say that l limits a predicate P if the extension of each component of P belongs to $\mathcal{B}_l$. We say that a function $p(i)$ limits a function f if the inverse of each component of f maps $\mathcal{B}_i$ into $\mathcal{B}_{pi}$.

By a constituent, we mean either a predicate or a function. We call a set of constituents $\mathcal{C}$ effectively limited if there is an effective procedure for obtaining limits, which are recursive in the case of functions, for each constituent in $\mathcal{C}$. By a sentence with constituents $\mathcal{C}$ we mean a sentence generated from $\mathcal{C}$ and the equations $x = 0, x = 1, \dots$, by substitution, the propositional operations, bounded and unbounded quantification, and bounded and unbounded μ -operations. Our main result is that the class of sentences, with a given effectively limited set of recursive constituents, is decidable if b_l is recursive, where b_l denotes $\max\{\min B : B \in \mathcal{B}_l\}$.

In the second part, we consider several possible sequences of limiters. In each case, we show that b_l is recursive, and we find as large an effectively limited class $\mathcal{C}$ of recursive constituents as we can, so that the class of sentences with constituents $\mathcal{C}$ is decidable.

In the case of the simplest possible limiters, our work reveals the connection between the apparently unrelated methods, used by Goodstein and Lee and by Rousseau, to determine decidable classes of equations of the form $f = 0$, and provides a considerable extension of these classes.