Analysis of a Variational Framework for Exemplar Based Image Inpainting

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Linked publications

- P. Arias, V. Caselles, GF. Analysis of a Variational Framework for Exemplar-Based Image Inpainting. SIAM MMS 2012.
- P. Arias, GF, V. Caselles, G. Sapiro. A Variational Framework for Exemplar-Based Image Inpainting. IJCV 2011.
- P. Arias, V. Caselles, G. Sapiro. A variational framework for non-local image inpainting. EMMCVPR, 2009.
- GF, P. Arias, V. Caselles, G. Sapiro. Exemplar-based interpolation of sparsely sampled images. EMMCVPR, 2009.

Outline

1. **Introduction:** Exemplar-based inpainting
2. **Proposed model:** From non-local means to non-local inpainting
3. **Correspondences:** A limit case
4. **Results and conclusions**

Introduction

Inpainting: Problem statement

Definitions:

- image domain: Ω
- inpainting domain: O
- known data: $O^c = \Omega \setminus O$
- image: $\hat{u} : O^c \rightarrow \mathbb{R}$
- Patch of $\hat{u}$ centered at x : $p_{\hat{u}}(x)$
- Patch domain: Ω_p

Objective:

- Visually plausible completion u
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Local inpainting methods

- Continuation of level lines or gradients.
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Local inpainting methods

- Continuation of level lines or gradients.
- With Variational methods [Masnou'98] or PDE based [Bertalmío'00]
- Use only information at the boundary ∂O .
- Good results on **smooth** images, fail on **textured** images.

Exemplar-Based methods: texture synthesis

- Initiated by works on texture synthesis [Efros'99] [Wei'00] .

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- Also in ...
 - denoising **non-local means**, **UINTA** [Buades'05] [Awate'05].
 - superresolution [Protter'09] .
 - inspired a lot of work on variational denoising and restoration [Kinderman'06] [Gilboa'06] [Peyré'09] .
- Underlying assumption: image **self-similarity**.
A non-local notion of regularity.
- **Image patches** as basic units of information.

Self-similarity in inpainting

Idea: Enforce self-similarity

Correspondence map: $\varphi : \mathcal{O} \rightarrow \mathcal{O}^c$ [Demanet'03]

Image synthesis: $u(x) = \hat{u}(\varphi(x))$

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Hard to minimize:

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- Iteration of [Efros'99] scheme [Demanet'03] [Aujol'08]
- MRF solved via BP [Komodakis'07]
- Image and “map” **joint** minimization [Wexler'07] [Peyré'08] [Kawai'09]

Proposed model

Non-Local Means: Self-similarity for denoising

Given a noisy image $\hat{u}(x) = u^*(x) + n(x)$,

where $n(x)$ is a white Gaussian noise.

The NL-means formula is [Buades'05] [Awate'05]

$$u(x) = NLM(\hat{u}) := \int_{\Omega} w(x, y) \hat{u}(y) dy$$

Similarity weights:

$$w(x, y) \propto \exp \left(-\frac{1}{T} \|p_{\hat{u}}(x) - p_{\hat{u}}(y)\|^2 \right) \quad T > 0.$$

An Energy for *Non-Local Means*

$$E_w(u) = \int_{\Omega} \int_{\Omega} w(x, y) (u(x) - \hat{u}(y))^2 dy dx$$

Based on [Gilboa'06] 's non-local regularizer.

Weights can be estimated from the noisy patches.

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How to compute the weights for inpainting? (and other inverse problems)

Energies for Exemplar-Based Inpainting

Consider $w : \tilde{O} \times \tilde{O}^c \rightarrow \mathbb{R}^+$ as a **variable**,

$$\mathcal{E}_T(u, w) = \underbrace{\int_{\tilde{O}} \int_{\tilde{O}^c} w(x, y) \|p_u(x) - p_{\hat{u}}(y)\|^2 dy dx}_{\text{image energy}}$$

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subject to $\int_{\tilde{O}^c} w(x, y) dy = 1.$

Entropy of $w(x, \cdot)$: $H(w(x, \cdot)) = - \int_{\tilde{O}^c} w(x, y) \log w(x, y) dy$

$$\frac{\partial \mathcal{E}_T}{\partial w} = 0 \quad \Longleftrightarrow \quad w(x, y) = \frac{1}{q(x)} \exp \left(-\frac{T}{1} \|p_u(x) - p_{\hat{u}}(y)\|^2 \right)$$

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Minimized with alternating minimization, wrt w and u .

Energies for Exemplar-Based Inpainting

Impact of **patch-wise** energy on the image synthesis:

$$\mathcal{E}_T(u, w) = \int_{\tilde{O}} \int_{\tilde{O}^c} w(x, y) \|p_u(x) - p_{\hat{u}}(y)\|^2 dy dx - \dots$$

$$u(x) = \int_{\Omega_p} g(h) \int_{\tilde{O}^c} w(x - h, y) \hat{u}(y + h) dy dh$$

All patches overlapping x have something to say about $u(x)$.

Energies for Exemplar-Based Inpainting

$$\mathcal{E}_T(u, w) = \int_{\tilde{O}} \int_{\tilde{O}^c} w(x, y) U(x, y) dy dx - T \mathcal{H}(w)$$

- Patch NL-means : $U(x, y) = \|p_u(x) - p_{\hat{u}}(y)\|_2^2$

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- Patch NL-Poisson :

$$U(x, y) = \|p_{\nabla u}(x) - p_{\nabla \hat{u}}(y)\|_2^2$$

- Patch NL-Gradient Medians :

$$U(x, y) = \|p_{\nabla u}(x) - p_{\nabla \hat{u}}(y)\|_1$$

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- **Patch NL-means** : $U(x, y) = \|p_u(x) - p_{\hat{u}}(y)\|_2^2$
- **Patch NL-medians** : $U(x, y) = \|p_u(x) - p_{\hat{u}}(y)\|_1$
- **Patch NL-Poisson** : $\lambda \in [0, 1]$

$$U(x, y) = (1 - \lambda) \|p_{\nabla u}(x) - p_{\nabla \hat{u}}(y)\|_2^2 + \lambda \|p_u(x) - p_{\hat{u}}(y)\|_2^2$$

- **Patch NL-Gradient Medians** : $\lambda \in [0, 1]$

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Summary: four inpainting schemes

inpainting problem

initialization

Patch NL-means (L^2 comparison)

Patch NL-medians (L^1 comparison)

Patch NL-Poisson (L^2 grad comparison)

Patch NL-GM (L^1 grad comparison)

Summary: four inpainting schemes

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Means: blending

Medians: shocks

Poisson: smooth trans., average grad.

GM: smooth trans., median grad.

Existence of minima for Patch-NLM and NL-Poisson:

For Patch-NLM:

$$U(x, y) = \|p_u(x) - p_{\hat{u}}(y)\|^2 = g * (u(x + \cdot) - \hat{u}(y + \cdot))^2.$$

For Patch NL-Poisson:

$$U(x, y) = \|p_{\nabla u}(x) - p_{\nabla \hat{u}}(y)\|^2 = g * (\nabla u(x + \cdot) - \nabla \hat{u}(y + \cdot))^2.$$

Proposition

If $\nabla g \in L^1$ and $\hat{u} \in BV(O^c)$, there are minima (u, w) of the Patch-NLM energy.

For any minimum: $u \in W^{1,\infty}(O)$, $w \in W^{1,\infty}(\tilde{O} \times \tilde{O}^c)$.

For the NL-Poisson energy: If $\nabla g \in L^\infty$ and $\hat{u} \in W^{2,2}(O^c)$, then $u \in W^{1,p}(O)$ for any $p < \infty$, $w \in W^{1,\infty}(\tilde{O} \times \tilde{O}^c)$.

Correspondences

Limit when $T \rightarrow 0$: the correspondence case

For patch NL-means and -Poisson: As $T \rightarrow 0$, the energy $\mathcal{E}_T$ Gamma converges to

$$\mathcal{E}_0(u, \nu) := \int_{\tilde{O}} \int_{\tilde{O}^c} U(x, y) d\nu(x, y),$$

where $x \rightarrow \nu(x, \cdot)$ is a measurable probability-valued map.

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Intuitively: The weights converge to a delta function:

$$w(x, y) = \frac{1}{q(x)} \exp\left(-\frac{1}{T} U(x, y)\right) \xrightarrow{T \rightarrow 0} \delta(y - \varphi(x))$$

when $\varphi(x) = \arg \min_{y \in \tilde{O}^c} U(x, y)$ is unique.

Image update step with $T = 0$

For the case of patch NL-means:

$$u(x) = \int_{\Omega_p} g(h) \hat{u}(\varphi(x - h) + h) dh$$

Image update step with $T = 0$

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When φ is a translation, *i.e.* $\varphi(x) = x + t$:

$$u(x) = \hat{u}(\varphi(x))$$

$\Rightarrow$ rigid copy

Correspondence case: piece-wise translations

Experimental evidence for Piece-wise translation φ

$\varphi(x) = x + t(x)$, with t piece-wise constant.

Copy regions and transition band.

Regions R_1 and R_2 have a constant t . Data is rigidly translated (copied) from corresponding regions in O^c . Red band: transition between copy regions.

L^1 : Sharp transitions (between intensity/gradients)

L^2 : Smooth blending by averaging (of intensity/gradients)

Copy regions: maximal sets where φ is a translation.

Emergence of copy regions

Figure: Initialization

Evolution of iterates for $T = 0$. Red curves on the right are the boundaries between copy regions.

Emergence of copy regions

Figure: Iteration # 2

Evolution of iterates for $T = 0$. Red curves on the right are the boundaries between copy regions.

Emergence of copy regions

Figure: Iteration # 4

Evolution of iterates for $T = 0$. Red curves on the right are the boundaries between copy regions.

Emergence of copy regions

Figure: Iteration # 6

Evolution of iterates for $T = 0$. Red curves on the right are the boundaries between copy regions.

Emergence of copy regions

Figure: Iteration # 8

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Emergence of copy regions

Figure: Iteration # 10

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Emergence of copy regions

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Figure: Iteration # 32

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Figure: Iteration # 38

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Figure: Iteration # 44

Evolution of iterates for $T = 0$. Red curves on the right are the boundaries between copy regions.

Emergence of copy regions

Figure: Convergence

Evolution of iterates for $T = 0$. Red curves on the right are the boundaries between copy regions.

Emergence of copy regions

Obtained with a multiscale scheme.

$T = 0$: existence of mildly regular optimal correspondences

Proposition. There exist optima (u^*, ν^*) of $\mathcal{E}_0$ with $\nu^*(x, \cdot) = \delta_{\varphi(x)}$, where $\varphi : \tilde{O} \rightarrow \tilde{O}^c$ is a measurable map, such that

$$U(x, \varphi(x)) = \min_{y \in \tilde{O}^c} U(x, y).$$

Moreover, there exist such φ which are **uniform limit of BV maps with finitely many values**.

Results

Experiments

Experiments

Experiments

Experiments

Results with non-variational scheme

Departure from variational model:

1. Synthesize image using $\lambda_u \ll 1$.
2. Compute weights using λ_w . Parameter to be set for each image.

Greater flexibility, but it is not variational.

Summary

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- Variational model for exemplar based image inpainting
- Allows derivation of several inpainting schemes
- Model with $T = 0$ seems to favor piece-wise translations correspondences
- We know that solutions exist which are a uniform limit of BV maps
- Further analysis of the “copy regions”
 - Are there better regularity properties for φ ?
 - Should we enforce them? (as in [Demanet'03] [Aujol'08])

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Thank you.

More @ <http://gpi.upf.edu/static/vnli/>