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Two New Criteria for Normal Families

E. F. Clifford

Abstract

Let $a_0, \dots, a_{k-1}$ be analytic functions on a domain Ω . Let $\mathcal{F}$ be a family of meromorphic functions f on Ω such that $f \neq 0$ and $f^{(k)} + a_{k-1}f^{(k-1)} + \dots + a_0f \neq 0$ on Ω , for all $f \in \mathcal{F}$. Then $\{f'/f : f \in \mathcal{F}\}$ is a normal family. Furthermore, let $a_0, \dots, a_{k-1}$ be meromorphic functions on a domain Ω . Let $\mathcal{F}$ be a family of meromorphic functions f on Ω such that $f \neq 0$, $f' \neq 0$ and $f^{(k)} + a_{k-1}f^{(k-1)} + \dots + a_0f \neq 0$ on Ω , for all $f \in \mathcal{F}$. Then $\{f'/f : f \in \mathcal{F}\}$ is a normal family. These two new criteria for normal families extend a recent result of Bergweiler and Langley, [1, Corollary 1.1].¹

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1 Introduction

We say that a function is *nonvanishing* on a domain if it is without zeros there. Since at least as far back as Pólya [13], interest has been shown in determining meromorphic functions f such that f and $f^{(k)}$ are nonvanishing in $\mathbb{C}$ for $k \in \mathbb{N}$. In 1959, Hayman [7] conjectured that the following result would be true.

Theorem A *Let f be a meromorphic function in $\mathbb{C}$ and let $k \geq 2$ be an integer. Suppose f and $f^{(k)}$ are nonvanishing in $\mathbb{C}$. Then f has the form $f(z) = e^{az+b}$ or $f(z) = (az+b)^{-n}$ where $a, b \in \mathbb{C}$, $a \neq 0$ and $n \in \mathbb{N}$.*

Hayman [7] proved the $k = 2$ case for entire functions, and Clunie [3] proved the general case for entire functions. In 1976, Frank [4] proved the $k \geq 3$ case of Theorem A. In 1977, Frank, Hennekemper and Polloczek [6] considered the case where $k \geq 3$ and $ff^{(k)}$ has finitely many zeros. In 1993, Langley [10] proved the $k = 2$ case of Theorem A.

Interest has also been shown in determining meromorphic functions f such that f and $L(f)$ are nonvanishing on $\mathbb{C}$, where L is defined by

$$L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \dots + a_1f' + a_0f, \quad (1)$$

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for $k \in \mathbb{N}$. Steinmetz [16], following the work of Frank and Hellerstein [5], proved results for the case where the a_j are constants and $k \geq 3$. Brüggemann [2] proved results for the case where the a_j are polynomials, not all constant. Langley ([8], [9]) proved results for the case where the a_j are rational functions. In this paper, we prove results for the cases where the a_j are analytic functions and meromorphic functions.

To see how Theorem A provides a criterion for normal families, we first state the Bloch Principle, noting that this is a heuristic principle and counterexamples do exist, [14].

Bloch Principle *A family of meromorphic (analytic) functions which have a common property $\mathcal{P}$ on a domain Ω will in general be a normal family if $\mathcal{P}$ reduces a meromorphic (analytic) function in $\mathbb{C}$ to a constant.*

Next, we note that by Theorem A, if f is an entire function such that f and $f^{(k)}$ are nonvanishing on $\mathbb{C}$ for some $k \geq 2$, then we have that $f(z) = e^{az+b}$ and so $f'(z)/f(z)$ is constant. Then by the Bloch Principle, we see that Theorem A may provide a criterion for normal families. This is in fact the case. Schwick [15] proved this for families of analytic functions, and Bergweiler and Langley [1] proved it for families of meromorphic functions, their results being stated as follows:

Theorem B ([1]) *Let $k \geq 2$ and let $\mathcal{F}$ be a family of functions that are meromorphic in a domain Ω . Suppose that f and $f^{(k)}$ are nonvanishing in Ω , for all $f \in \mathcal{F}$. Then $\{f'/f : f \in \mathcal{F}\}$ is a normal family on Ω .*

We include the following example to show that Theorem B does not hold for $k = 1$.

Example Let $\mathcal{F} = \{f_n(z) = \frac{1}{e^{nz}-1} : n \in \mathbb{N}\}$. Then $f_n(z) \neq 0$ and $f'_n(z) = -\frac{ne^{nz}}{(e^{nz}-1)^2} \neq 0$ on $\mathbb{C}$, for all $n \in \mathbb{N}$. However, $\{f'_n(z)/f_n(z) = -ne^{nz}/(e^{nz}-1) : n \in \mathbb{N}\}$ is not a normal family on $\mathbb{C}$ since $f'_n(0)/f_n(0) = \infty$ whereas $f'_n(x)/f_n(x) \rightarrow 0$ for $x \in \mathbb{R}^-$, as $n \rightarrow \infty$.

In this paper, we first extend Theorem B to the following result.

Theorem 1.1 *Let $k \geq 2$ and let $\mathcal{F}$ be a family of meromorphic functions on a domain Ω . Let $a_0, \dots, a_{k-1}$ be analytic functions on Ω . For each $f \in \mathcal{F}$ define $L(f)$, as in (1), by*

$$L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \dots + a_1f' + a_0f.$$

Suppose that f and $L(f)$ are nonvanishing on Ω for each $f \in \mathcal{F}$. Then $\{f'/f : f \in \mathcal{F}\}$ is a normal family on Ω .

The following example shows that Theorem 1.1 cannot be extended to the case where $a_0, \dots, a_{k-1}$ are meromorphic functions, in the $k = 2$ case.

Example Let $f_n(z) = e^{nz}/z$ for $n \in \mathbb{N}$. Set $a_0(z) = 0$ and $a_1(z) = 2/z$. Then for $k = 2$, we have

$$L(f_n) = f_n^{(2)} + a_1(z)f'_n(z) + a_0(z)f_n(z) = \frac{n^2}{z}e^{nz}.$$

Thus for all $n \in \mathbb{N}$ we have that $f_n(z)$ and $L(f_n)$ are nonvanishing on $\mathbb{C}$. However $\{f'_n(z)/f_n(z) = (z - \frac{1}{n})/(z \cdot \frac{1}{n}) : n \in \mathbb{N}\}$ is not a normal family on $\mathbb{C}$, since $f'_n(0)/f_n(0) = \infty$, whereas $f'_n(\frac{1}{n})/f_n(\frac{1}{n}) = 0$.

Nevertheless, by including the extra condition that $f' \neq 0$ on Ω , for all $f \in \mathcal{F}$, we can extend Theorem 1.1 to the case where $a_0, \dots, a_{k-1}$ are meromorphic functions, as follows:

Theorem 1.2 *Let $k \geq 2$ and let $\mathcal{F}$ be a family of meromorphic functions on a domain Ω . Let $a_0, \dots, a_{k-1}$ be meromorphic functions on Ω . For each $f \in \mathcal{F}$ define $L(f)$, as in (1), by*

$$L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \dots + a_1f' + a_0f.$$

Suppose that f , f' and $L(f)$ are nonvanishing on Ω for each $f \in \mathcal{F}$. Then $\mathcal{G} = \{f'/f : f \in \mathcal{F}\}$ is a normal family on Ω .

2 Proof of Theorem 1.1

We follow a similar method of proof to that used by Bergweiler and Langley in their proof of Theorem B.

First, we define differential polynomials $\Psi_k(F)$ for $k \in \mathbb{N}$ by

$$\Psi_1(F) = F, \quad \Psi_{k+1}(F) = F\Psi_k(F) + (\Psi_k(F))'. \quad (2)$$

The link between these operators and nonvanishing derivatives is given by the following lemma from [1], which can be easily proved by induction.

Lemma 2.1 ([1]) *Let f be meromorphic on a domain Ω and let $F = f'/f$. Then for each $k \in \mathbb{N}$ we have $\Psi_k(F) = f^{(k)}/f$.*

Next, let $a_0, \dots, a_{k-1}$ be analytic functions on a domain Ω , and define differential polynomials $\Lambda_k(F)$ for $k \in \mathbb{N}$ by

$$\Lambda_k(F) = \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \dots + a_1\Psi_1(F) + a_0. \quad (3)$$

We then use the following theorem, which is an extension of [1, Theorem 1.3].

Theorem 2.2 *Let $k \geq 2$ and let $\mathcal{F}$ be a family of functions meromorphic in a domain Ω . Let $a_0, \dots, a_{k-1}$ be analytic functions on Ω . For $F \in \mathcal{F}$ define $\Lambda_k(F)$, as in (3), by*

$$\Lambda_k(F) = \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \dots + a_1\Psi_1(F) + a_0.$$

Suppose that there exists $\delta \in (0, 1]$ such that the following conditions hold for all $F \in \mathcal{F}$:

- (i) $\Lambda_k(F)$ has no zeros.
- (ii) if a is a simple pole of F then $|\text{Res}(F, a) - j| \geq \delta$ for $j \in \{0, 1, \dots, k-1\}$.
- (iii) if $c \in \Omega$ and $R > 0$ with $D(c, R) \subset \Omega$, if $D(c, \delta R)$ contains two poles of F , counting multiplicities, and if $D(c, R) \setminus D(c, \delta R)$ contains no poles of F , then

$$\left| \sum_{a \in D(c, \delta R)} \text{Res}(F, a) - (k-1) \right| \geq \delta.$$

Then $\mathcal{F}$ is a normal family.

We prove this theorem in § 4, and provide an example there to show that Theorem 2.2 cannot be extended to the case where $a_0, \dots, a_{k-1}$ are meromorphic functions.

Finally, we can deduce Theorem 1.1 from Theorem 2.2 as follows. First we note that conditions (ii) and (iii) of Theorem 2.2 are satisfied if we have that $\operatorname{Re}(\operatorname{Res}(F, a)) \leq -\delta$ for all poles a of F . This is the case, in particular, if $F = f'/f$ for some nonvanishing meromorphic function f . Then by (3) and since $\Psi_k(F) = f^{(k)}/f$ for $k \in \mathbb{N}$ by Lemma 2.1, we have that Theorem 1.1 follows.

3 Proof of Theorem 1.2

Proof Let $\{\alpha_j : j \in J\}$ be the set of poles of $a_0, \dots, a_{k-1}$ in Ω . By Theorem 1.1, the result is true in the case where $a_0, \dots, a_{k-1}$ are analytic functions, and so it is sufficient to prove that $\mathcal{G}$ is normal at the α_j , for $j \in J$.

Suppose there exists α_j such that $\mathcal{G}$ is not normal at α_j . Choose $\delta > 0$ such that the punctured disc $\Omega_j = \{z : 0 < |z - \alpha_j| < \delta\}$ is contained in Ω , and such that Ω_j does not contain any poles of $a_0, \dots, a_{k-1}$. Then we have that there exists a sequence (F_n) in $\mathcal{G}$, where $F_n = f'_n/f_n$ for some $f_n \in \mathcal{F}$, such that (f'_n/f_n) has no subsequence that converges locally uniformly on $\Omega_j \cup \{\alpha_j\}$. However, since $a_0, \dots, a_{k-1}$ are analytic in Ω_j , we have by Theorem 1.1 that $\mathcal{G}$ is normal on Ω_j . Then there exists a subsequence of (f'_n/f_n) , denoted (f'_n/f_n) without loss of generality, which converges uniformly on compact sub-regions of Ω_j , either to a meromorphic limit ϕ , or identically to ∞ . Then there are two cases.

Case 1. (f'_n/f_n) converges uniformly to a meromorphic limit ϕ on compact sub-regions of Ω_j .

We note first that for $n \in \mathbb{N}$, since $f_n \neq 0$ on Ω , the poles of f'_n/f_n can only arise at poles of f'_n , and therefore only at poles of f_n . We note that for the remainder of this proof, we refer to the poles of f'_n/f_n only as the poles of f_n . We note also that f'_n/f_n has no zeros on Ω since $f'_n \neq 0$ there.

Let Γ be a circular contour in Ω_j which goes once anti-clockwise around α_j and which does not pass through any poles of ϕ . Since ϕ is a meromorphic limit, Γ lies in the interior of a closed annulus $\mathcal{A}$ on which ϕ has no poles. Further, since (f'_n/f_n) converges uniformly to ϕ on compact sub-regions of Ω_j , we have that there exists $n_0 \in \mathbb{N}$ such that f_n has no poles on $\mathcal{A}$ for $n \geq n_0$.

Let Ω_Γ be the domain enclosed by Γ . By the Argument principle, and since each f_n has no zeros in Ω , we have that for each $n \in \mathbb{N}$,

$$\int_{\Gamma} \left(\frac{f'_n}{f_n} \right) (z) dz = -2\pi i q_n \quad (4)$$

where q_n is the number of poles of f_n in Ω_Γ , counting multiplicities. Then since,

$$\lim_{n \rightarrow \infty} \int_{\Gamma} \left(\frac{f'_n}{f_n} \right) (z) dz = \int_{\Gamma} \phi(z) dz = \lambda$$

for some $\lambda \in \mathbb{C}$, we have that there exists $n_1 \in \mathbb{N}$ such that $n_1 \geq n_0$ and such that $\int_{\Gamma} (f'_n/f_n)(z) dz$ is constant for $n \geq n_1$. Then by (4), we must have that for $n \geq n_1$, the f_n have the same number of poles, say q , in Ω_Γ .

We list the poles by $\gamma_{n,1}, \dots, \gamma_{n,q}$ repeating according to multiplicity. Then we can write for $n \geq n_1$,

$$\left(\frac{f'_n}{f_n}\right)(z) = \sum_{l=1}^q \left(-\frac{1}{z - \gamma_{n,l}}\right) + \psi_n(z), \quad (5)$$

where ψ_n is an analytic function on $\Omega_\Gamma \cup \mathcal{A}$.

Next, we show that the ψ_n are uniformly bounded on Ω_Γ . We note first that $\gamma_{n,1}, \dots, \gamma_{n,q}$ are not in the closed annulus $\mathcal{A}$, and so $|\phi(z)| \leq C_1$ and $|z - \gamma_{n,l}| \geq c_1$ for some positive constants C_1 and c_1 , for $z \in \Gamma$. Since (f'_n/f_n) converges uniformly to ϕ on Γ , there exists $n_2 \in \mathbb{N}$ such that $n_2 \geq n_1$ and such that, on Γ , $|(f'_n/f_n)(z)| \leq C_1 + 1$ for $n \geq n_2$. Since $|z - \gamma_{n,l}| \geq c_1$, we have that $\left|\sum_{l=1}^q \left(-\frac{1}{z - \gamma_{n,l}}\right)\right| \leq q/c_1$. Then we have that, on Γ ,

$$|\psi_n(z)| \leq \left|\left(\frac{f'_n}{f_n}\right)(z)\right| + \left|\sum_{l=1}^q \left(-\frac{1}{z - \gamma_{n,l}}\right)\right| \leq C_1 + 1 + q/c_1,$$

for $n \geq n_2$. Therefore the ψ_n are uniformly bounded on Γ . By the maximum principle, the ψ_n are uniformly bounded on Ω_Γ .

Choose a subsequence of $(\gamma_{n,l})$, denoted $(\gamma_{n,l})$ without loss of generality, such that $\gamma_{n,l} \rightarrow \gamma_l$ as $n \rightarrow \infty$ for $l = 1, \dots, q$. Then there are two subcases, depending on whether some of $\gamma_1, \dots, \gamma_q$ are equal to α_j .

Case 1.1. *Some of $\gamma_1, \dots, \gamma_q$ are equal to α_j .*

Rearrange $\gamma_1, \dots, \gamma_q$ so that $\gamma_1, \dots, \gamma_p$ are equal to α_j and $\gamma_{p+1}, \dots, \gamma_q$ are not equal to α_j , for some $p \in \{1, \dots, q\}$. Then by (5), we have for $n \geq n_2$,

$$\left(\frac{f'_n}{f_n}\right)(z) = \sum_{l=1}^p \left(-\frac{1}{z - \gamma_{n,l}}\right) + \sum_{l=p+1}^q \left(-\frac{1}{z - \gamma_{n,l}}\right) + \psi_n(z). \quad (6)$$

Next, since $\gamma_{n,p+1}, \dots, \gamma_{n,q}$ tend to $\gamma_{p+1}, \dots, \gamma_q$ as $n \rightarrow \infty$, and $\gamma_{p+1}, \dots, \gamma_q$ are not equal to α_j , we can choose $\delta_1 > 0$ such that $B(\alpha_j, 3\delta_1)$ is contained in $\Omega_\Gamma \setminus \mathcal{A}$ and does not contain $\gamma_{p+1}, \dots, \gamma_q$. Then there exists $n_3 \in \mathbb{N}$ such that $n_3 \geq n_2$ and such that, for $n \geq n_3$ and $z \in B(\alpha_j, \delta_1)$, we have $|z - \gamma_{n,l}| \geq \delta_1$ for $l = p+1, \dots, q$. Then for each $n \geq n_3$, we have $\left|\sum_{l=p+1}^q \left(-\frac{1}{z - \gamma_{n,l}}\right)\right| \leq (q-p)/\delta_1$ and so $\sum_{l=p+1}^q \left(-\frac{1}{z - \gamma_{n,l}}\right)$ is uniformly bounded and analytic in $B(\alpha_j, \delta_1)$. Then since the ψ_n are uniformly bounded on Ω_Γ , there exists a large positive constant M such that for $n \geq n_3$,

$$\left|\sum_{l=p+1}^q \left(-\frac{1}{z - \gamma_{n,l}}\right) + \psi_n(z)\right| \leq M \quad (7)$$

on $B(\alpha_j, \delta_1)$.

Now choose $\delta_2 > 0$ such that δ_2/δ_1 is small, and consider the circle $S(\alpha_j, \delta_2) = \{z : |z - \alpha_j| = \delta_2\}$. We have that $\gamma_{n,1}, \dots, \gamma_{n,p}$ each tend to α_j as $n \rightarrow \infty$, and so each $\left|-\frac{1}{z - \gamma_{n,l}}\right|$ is large on $S(\alpha_j, \delta_2)$ for $l = 1, \dots, p$. Then, in particular, we have that

$$\left|\sum_{l=1}^p -\frac{1}{z - \gamma_{n,l}}\right| \rightarrow \left|\sum_{l=1}^p -\frac{1}{z - \gamma_l}\right| = \left|-\frac{p}{z - \alpha_j}\right| = \frac{p}{\delta_2} > 2M,$$

as $n \rightarrow \infty$, for a suitable choice of δ_2 .

Then by (6) and (7), there exists $n_4 \in \mathbb{N}$ such that $n_4 \geq n_3$ and such that f'_n/f_n is large on $S(\alpha_j, \delta_2)$ for $n \geq n_4$, and thus f_n/f'_n is small on $S(\alpha_j, \delta_2)$ for $n \geq n_4$. Next, we know that each f_n/f'_n is analytic in Ω since $f'_n \neq 0$ on Ω . Then for $n \geq n_4$, by the maximum principle and since f_n/f'_n is small on $S(\alpha_j, \delta_2)$, each f_n/f'_n is small on $B(\alpha_j, \delta_2)$. Then (f_n/f'_n) is a uniformly bounded sequence of analytic functions on $B(\alpha_j, \delta_2)$, and by the Montel-Vitali theorem, we have that (f_n/f'_n) is normal on $B(\alpha_j, \delta_2)$. Therefore (f'_n/f_n) is normal on $B(\alpha_j, \delta_2)$, and thus, in particular, (f'_n/f_n) is normal at α_j . This is a contradiction.

Case 1.2. $\alpha_j \neq \gamma_l$ for all $l = 1, \dots, q$.

Then we can choose $\delta_3 > 0$ such that $B(\alpha_j, 3\delta_3)$ is contained in $\Omega_\Gamma \setminus \mathcal{A}$, and does not contain $\gamma_1, \dots, \gamma_q$. Then there exists $n_5 \in \mathbb{N}$ such that $n_5 \geq n_3$ and such that, for $n \geq n_5$ and $z \in B(\alpha_j, \delta_3)$, we have $|z - \gamma_{n,l}| \geq \delta_3$ for $l = 1, \dots, q$. Then $\left| \sum_{l=1}^q -\frac{1}{z - \gamma_{n,l}} \right| \leq \frac{q}{\delta_3}$, and since $\sum_{l=1}^q \left(-\frac{1}{z - \gamma_{n,l}} \right)$ is uniformly bounded and analytic in $B(\alpha_j, \delta_3)$ and since the ψ_n are uniformly bounded on Ω_Γ , we have that (7) holds with $p = 0$, on $B(\alpha_j, \delta_3)$.

Therefore, (f'_n/f_n) is a uniformly bounded sequence of analytic functions on $B(\alpha_j, \delta_3)$. Then by the Montel-Vitali theorem, we have that (f'_n/f_n) is normal on $B(\alpha_j, \delta_3)$, and thus, in particular, (f'_n/f_n) is normal at α_j . This is a contradiction.

Case 2. (f'_n/f_n) converges identically to ∞ on Ω_j .

Then we have that (f_n/f'_n) converges identically to 0 on Ω_j . We note that for each n , we have that f_n/f'_n is analytic in $\Omega_j \cup \{\alpha_j\}$ since $f'_n \neq 0$ on Ω . Then by the maximum principle, we have that $((f_n/f'_n)(\alpha_j))$ converges to 0, and so (f'_n/f_n) converges identically to ∞ on $\Omega_j \cup \{\alpha_j\}$. Therefore (f'_n/f_n) is normal on $\Omega_j \cup \{\alpha_j\}$. This is a contradiction. Therefore $\mathcal{G}$ is a normal family.

4 Proof of Theorem 2.2

We need several lemmas for the proof of Theorem 2.2. The first assertion in the following lemma is proved in [1], and the second is an extension which follows immediately.

Lemma 4.1 ([1]) *Let $k \geq 2$ be an integer. Let y be meromorphic on a domain Ω , such that if a is a simple pole of y then $\text{Res}(y, a) \notin \{1, \dots, k-1\}$. Let $n \in \mathbb{N}$ be such that $n \leq k$. If y has a pole at a of multiplicity m then $\Psi_n(y)$ has a pole at a of multiplicity nm , and $\Lambda_n(y)$ has a pole at a of multiplicity nm , where $\Lambda_n(y)$ is defined as in (3) by*

$$\Lambda_n(y) = \Psi_n(y) + a_{n-1}\Psi_{n-1}(y) + \dots + a_1\Psi_1(y) + a_0$$

where $a_0, \dots, a_{n-1}$ are analytic functions on Ω .

We need the following theorems of Bergweiler and Langley.

Theorem 4.2 ([1]) *Let $k \geq 3$ be an integer, and let F be meromorphic and nonconstant in the plane and satisfy both of the following conditions:*

- (i) $\Psi_k(F)$ has no zeros.

(ii) if a is a simple pole of F then $\text{Res}(F, a) \notin \{1, \dots, k-1\}$.

Then F has the form

$$F(z) = \frac{(k-1)z + \alpha}{z^2 + \beta z + \gamma}, \quad (8)$$

or

$$F(z) = \frac{1}{\alpha z + \beta}. \quad (9)$$

Here $\alpha, \beta, \gamma \in \mathbb{C}$ with $\alpha \neq 0$ in (9).

Conversely, if F has the form (8) or (9), and if (ii) holds, then $\Psi_k(F)$ has no zeros. If F has the form (8) or (9), but (ii) does not hold, then $\Psi_k(F) \equiv 0$.

Theorem 4.3 ([1]) Let F be meromorphic and nonconstant in the plane, such that

(i) $\Psi_2(F) = F' + F^2$ has no zeros.

(ii) if a is a simple pole of F , then $\text{Res}(F, a) \neq 1$.

(iii) there exists $\delta > 0$ such that if a is a simple pole of F then $|\text{Res}(F, a)| \geq \delta$.

Then either F has the form (8) with $k = 2$, or the form (9).

We also need the following lemma of Pang and Zalcman, see [1], [11] and [12].

Lemma 4.4 ([12]) Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc $D(0, 1)$. Suppose that there exists $\delta > 0$ such that if $f \in \mathcal{F}$ has a simple pole a , then $|\text{Res}(f, a)| \geq \delta$. Then if $\mathcal{F}$ is not normal, there exist a number $r \in (0, 1)$, points $z_n \in D(0, r)$, functions $F_n \in \mathcal{F}$ and positive numbers ρ_n tending to zero such that

$$\rho_n F_n(z_n + \rho_n z) \rightarrow F(z)$$

locally uniformly in $\mathbb{C}$, where F is a nonconstant meromorphic function on $\mathbb{C}$ such that $F^\sharp(z) \leq F^\sharp(0) = 1 + 1/\delta$ for all $z \in \mathbb{C}$, in which $F^\sharp$ denotes the spherical derivative.

The proof of Theorem 2.2 will involve rescaling, as in Lemma 4.4, and so we need the following results which can be easily proved.

Lemma 4.5 Let F and g be functions such that

$$g(z) = \rho F(a + \rho z)$$

where $\rho > 0$, $a \in \mathbb{C}$. Then the following statements are true:

(a) If g has a pole at b , then F has a pole at $a + \rho b$, and $\text{Res}(g, b) = \text{Res}(F, a + \rho b)$.

(b) For each $j \in \mathbb{N}$, we have

$$\Psi_j(g)(z) = \rho^j \Psi_j(F)(a + \rho z)$$

where Ψ_j is defined as in (2).

(c) If $a_0, \dots, a_{k-1}$ are analytic functions and

$$\Lambda_k(F) = \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \dots + a_0$$

as defined by (3), then

$$\Lambda_k(F)(a + \rho z) = \rho^{-k} \left[\Psi_k(g)(z) + \sum_{j=1}^{k-1} a_j(a + \rho z) \rho^{k-j} \Psi_j(g)(z) + \rho^k a_0(a + \rho z) \right]. \quad (10)$$

We proceed to the proof of Theorem 2.2. We note that it uses essentially the same methods as [1, Theorem 1.3].

Proof Since normality is a local property we can assume, without loss of generality, that Ω is a disc. We can assume also that the a_j are bounded on Ω since they are analytic functions. Using a linear change of variables $g(z) = \rho F(a + \rho z)$, for suitable choice of $\rho > 0$ and $a \in \mathbb{C}$, we may assume that Ω is the open unit disc $D(0, 1)$ since Lemma 4.5 (a) shows that the residues of $g(z)$ are unaltered, and Lemma 4.5 (c) shows that if $\Lambda_k(F)$ is nonvanishing then $\tilde{\Lambda}_k(g)$ is nonvanishing, where the coefficients of $\tilde{\Lambda}_k$ are given by (10).

Suppose now that $\mathcal{F}$ is not normal. Then by condition (ii) of Theorem 2.2, with $j = 0$, we can apply Lemma 4.4. Let r, z_n, F_n, ρ_n and F be as in Lemma 4.4, so that,

$$g_n(z) = \rho_n F_n(z_n + \rho_n z) \rightarrow F(z)$$

locally uniformly in $\mathbb{C}$ as $n \rightarrow \infty$.

Let a be a simple pole of F . Then, by Hurwitz' Theorem, if n is sufficiently large, g_n has a simple pole at a_n with $a_n \rightarrow a$ as $n \rightarrow \infty$. By Lemma 4.5 (a), $z_n + \rho_n a_n$ is a simple pole of F_n with $\text{Res}(F_n, z_n + \rho_n a_n) = \text{Res}(g_n, a_n)$. Hence, with $\delta \in (0, 1]$, we deduce from condition (ii) of Theorem 2.2 that $|\text{Res}(g_n, a_n) - j| \geq \delta$ for $j \in \{0, 1, \dots, k-1\}$. Then we have that $|\text{Res}(F, a) - j| \geq \delta$ for $j \in \{0, 1, \dots, k-1\}$. And so, by Lemma 4.1, every pole of F is a pole of $\Psi_k(F)$.

Next, by Lemma 4.5 (c), we have

$$\Lambda_k(F_n)(z_n + \rho_n z) = \rho_n^{-k} \left[\Psi_k(g_n)(z) + \sum_{j=1}^{k-1} a_j(z_n + \rho_n z) \rho_n^{k-j} \Psi_j(g_n)(z) + \rho_n^k a_0(z_n + \rho_n z) \right].$$

By condition (i) of Theorem 2.2, this has no zeros. Hence,

$$\tilde{\Lambda}_k(g_n)(z) = \Psi_k(g_n(z)) + \sum_{j=1}^{k-1} a_j(z_n + \rho_n z) \rho_n^{k-j} \Psi_j(g_n(z)) + \rho_n^k a_0(z_n + \rho_n z)$$

has no zeros, since ρ_n is a sequence of positive numbers. We know by (2) that $\Psi_j(g_n)$ is a linear combination of products of g_n and its derivatives. Let $\mathbb{E}$ be the set of poles of F . Then, by the Weierstrass Theorem, we have that

$$\tilde{\Lambda}_k(g_n)(z) \rightarrow \Psi_k(F)(z)$$

as $n \rightarrow \infty$, locally uniformly on $\mathbb{C} \setminus \mathbb{E}$, since $\rho_n \rightarrow 0$ and the a_j are bounded. By Hurwitz' Theorem, either $\Psi_k(F) \equiv 0$ or $\Psi_k(F)$ has no zeros on $\mathbb{C} \setminus \mathbb{E}$. In the latter case, we deduce that $\Psi_k(F)$ has no zeros at all since every pole of F is a pole of $\Psi_k(F)$.

Case 1. $\Psi_k(F) \equiv 0$.

Since $|\operatorname{Res}(F, a) - j| \geq \delta$ for $j \in \{0, 1, \dots, k-1\}$, if a is a simple pole of F , we deduce from Lemma 4.1 that F has no poles. Thus F is entire and so is the function f defined by setting $f(z) = \exp(\int_0^z F(t)dt)$. Then $F = f'/f$ and thus $f^{(k)}/f = \Psi_k(F) \equiv 0$ by Lemma 2.1. Hence f is a polynomial. This implies that f is constant. Hence $F \equiv 0$, which is a contradiction, since F is nonconstant.

Case 2. $\Psi_k(F)$ has no zeros.

It follows from Theorem 4.2 for $k \geq 3$ and from Theorem 4.3 for $k = 2$ that F has the form (8) or (9).

Suppose first that F has the form (9). Then $1/|\alpha| = |\operatorname{Res}(F, -\beta/\alpha)| \geq \delta$ so that $|\alpha| \leq 1/\delta$. On the other hand, $|\alpha| \geq |\alpha|/(1 + |\beta|^2) = F^\sharp(0) = 1 + 1/\delta$. This is a contradiction.

Suppose second that F has the form (8) but is not of the form (9). Then F has two poles, counting multiplicity. We also observe that if F is of the form (8), then

$$\sum_{a \in F^{-1}(\{\infty\})} \operatorname{Res}(F, a) = k - 1 \quad (11)$$

by the Residue Theorem. Next, choose $R > 0$ such that these poles are contained in $D(0, \delta R)$. Since F has no other poles we deduce from Hurwitz' theorem that for n sufficiently large, g_n has two poles in $D(0, \delta R)$, but no poles in $D(0, R) \setminus D(0, \delta R)$. Thus F_n has two poles in $D(z_n, \delta \rho_n R)$, but no poles in $D(z_n, \rho_n R) \setminus D(z_n, \delta \rho_n R)$. By Lemma 4.5 (a) and condition (iii) of Theorem 2.2, we deduce that

$$\left| \sum_{a \in D(0, \delta R)} \operatorname{Res}(g_n, a) - (k - 1) \right| = \left| \sum_{a \in D(z_n, \delta \rho_n R)} \operatorname{Res}(F_n, a) - (k - 1) \right| \geq \delta.$$

But this gives

$$\left| \sum_{a \in D(0, \delta R)} \operatorname{Res}(F, a) - (k - 1) \right| \geq \delta,$$

which contradicts (11). This completes the proof of Theorem 2.2.

Finally, we use an example to show that Theorem 2.2 cannot be extended to the case where $a_0, \dots, a_{k-1}$ are meromorphic functions.

Example Let $k = 2$ and let $\mathcal{F}$ be the family of meromorphic functions $\{F_n(z) = \frac{1}{nz^3} : n \in \mathbb{N}\}$. Let $a_0(z) = 0$ and $a_1(z) = 3/z$. Then a_0 and a_1 are meromorphic functions and for $F_n \in \mathcal{F}$ we have

$$\begin{aligned} \Lambda_2(F_n)(z) &= \Psi_2(F_n)(z) + a_1(z)\Psi_1(F_n)(z) + a_0(z) \\ &= \frac{1}{n^2 z^6} - \frac{3}{nz^4} + \frac{3}{z} \frac{1}{nz^3} = \frac{1}{n^2 z^6}. \end{aligned}$$

This has no zeros, and so condition (i) is satisfied.

Also, each $F \in \mathcal{F}$ has a triple pole at $z = 0$, and no other poles. Thus conditions (ii) and (iii) are trivially satisfied. Thus all the conditions of Theorem 2.2 are satisfied except for the analyticity of the a_j . However $\mathcal{F}$ is not a normal family since $F_n(0) = \infty$ while $F_n(z) = \frac{1}{nz^3} \rightarrow 0$ as $n \rightarrow \infty$ for $z \in \mathbb{R}^+$. Therefore Theorem 2.2 cannot be extended to the case where $a_0, \dots, a_{k-1}$ are meromorphic functions, in the $k = 2$ case. Similar counterexamples can be constructed in the general case.

It is interesting to note here, with regard to Theorem 1.2, that although the F_n in this example have no zeros, they cannot be written in the form $F_n = f'_n/f_n$ where f_n is a nonvanishing meromorphic function.

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