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STEADY MULTIPOLAR PLANAR VORTICES WITH NONLINEAR CRITICAL LAYERS

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Abstract

This article considers a family of steady multipolar planar vortices which are the superposition of an axisymmetric mean flow, and an azimuthal disturbance in the context of inviscid, incompressible flow. This configuration leads to strongly nonlinear critical layers when the angular velocities of the mean flow and the disturbance are comparable. The poles located on the same critical radius possess the same uniform vorticity, whose weak amplitude is of the same order as the azimuthal disturbance. This problem is examined through a perturbation expansion in which relevant nonlinear terms are retained in the critical layer equations, while viscosity is neglected. In particular, the associated singularity at the meeting point of the separatrices is treated by employing appropriate re-scaled variables. Matched asymptotic expansions are then used to obtain a complete analytical description of these vortices.

1 Introduction

Laboratory experiments and numerical observations have only recently shown the occurrence of multipolar vortices in two-dimensional turbulent flows. In general, these vortices are coherent structures with an ensemble of vorticity extrema, while the total circulation is arbitrary. Here, we consider multipolar vortices composed of an axisymmetric core and an azimuthal disturbance with $N-1$ poles, thus making an N -polar structure. Multipolar vortices are generated due to strongly nonlinear processes. Indeed, it can be demonstrated that weak asymmetrical disturbances inside a circular vortex are rapidly damped except for the azimuthal mode one perturbation responsible for the appearance of a dipolar vortex (Bernoff and Lingeitch (1994)). The most commonly-found compound coherent structure is thus the dipole and a large number of articles have been devoted to it. On the other hand, the relatively more complicated tripolar vortex is not so common, and hence is a puzzling feature. A single tripole has, for instance, been observed in the ocean, in the Bay of Biscay (Pingree and Le Cann (1992)). Special conditions are probably necessary for its formation, as we will demonstrate here.

Tripoles first appeared in numerical simulations as the outcome of modon collisions (Larichev and Reznik (1983)), in forced two-dimensional turbulent flow (Legras et al. (1988)) and in von Karman streets (Carton et al. (1989)). Polvani and Carton (1990) numerically found stable tripoles in a two-dimensional flow from the interaction of three patches of constant vorticity (contour surgery method), or nearly constant vorticity (pseudo-spectral method). In experiments, tripoles are observed to emerge from the instability of barotropic cyclones in a rotating homogeneous fluid (Kloosterziel and van Heijst (1991)). This process was numerically verified by Carton et al. (1989) and Orlandi & van Heijst (1992). A tripolar structure can be generated from an initial vortex composed of a circular disk surrounded by an annulus of oppositely-signed vorticity. This configuration leads to an intense barotropic instability of azimuthal mode 2, which sometimes breaks the vortex. Mode 2 seems to be the most unstable mode for a large family of vorticity profiles (Carton and Legras (1994)). Tripoles can also appear in a stratified fluid, again due to the instability of a monopole (Flór et al. (1993)). The steepness of the axisymmetric vorticity profile appears to be a key feature for the stability of the tripole. If the profile is steep, the tripolar vortex reaches a quasi-stationary state resulting from nonlinear saturation. However, if it is too steep, saturation does not occur, and instead the core vortex is split into two, and two dipoles are created from both of the satellites; a stable triangular vortex can then appear if a satellite happens to break (Flór (1994)). Morel and Carton (1994) analysed the formation and stability of multipolar structures from the instability of circular step-like vortices. Their study found that tripoles and quadrupoles are stable features, but more complex multipoles usually decay into dipoles or tripoles, but seldom into quadrupoles. In the same way, Kloosterziel and Carnevale (1999) numerically investigated the nonlinear stability of compound vortices created from the instability of a family of axisymmetric continuous profiles in an infinite domain. They found that tripoles were the most robust of the multipolar vortices. Triangular structures can emerge, but with more difficulty, and square vortices are unstable to infinitesimal perturbations. The robustness of the tripole was also observed by Rossi et al. (1997) in a numerical simulation of the evolution of the strong destabilization of a Lamb vortex in a viscous flow; they also pointed out the absence of a true steady-state, and the difficulty to describe the long-time structure when enstrophy is gradually damped. We thus infer that multipolar vortices can sometimes characterize quasi-steady states.

Until now, the various studies concerning stationary coherent structures in Euler's equation have often assumed that a linear relationship existed between the vorticity and the streamfunction (the inverse Laplacian of the vorticity), for instance, for monopoles (Leith (1984)), dipoles (Chaplin (1902); Lamb (1945)), or multipolar vortical structures (van de Fliert (1993)). A common approach is to use a variational principle. Vortices would attain equilibrium states defined by extrema of a constant of motion. Thus Leith's monopoles were obtained by minimizing the production of enstrophy inside the vortex (Leith (1984)), while Turkington's dipoles are minimal energy structures (Turkington (1983)). It is also possible to obtain vortical patterns by searching for a state maximizing an entropy based upon a statistical distribution of vorticity in the flow (Chavanis and Sommeria (1996)). In this context, the literature on tripoles is sparse; an analytical solution can be derived if the structure is enclosed in a bounded domain, or is embedded in a sheared velocity field (Vranješ et al. (2000)), but we are not aware of analogous results for tripoles in an infinite domain. Tripoles can be obtained as critical points of the energy (van de Fliert (1993)) but the solutions are obtained numerically. Studies of asymmetrical structures become rapidly quite complex. Most theoretical models of multipolar vortices are based on the representation of vortices as point-vortices. For instance, a tripole can be modelled by three point-vortices (Carton and Legras (1994)). Recently, Crowdy exhibited a class of multipolar structures obtained by a distribution of point vortices and a patch of uniform vorticity (Crowdy (1999)).

It can be shown that a stationary tripole cannot be described in a two-dimensional inviscid flow by a linear relationship between vorticity ω and streamfunction ψ ((2000)). Indeed, a more complex relationship is observed in numerical simulations, or in experiments (Legras et al. (1988); Flór (1994)); this is related through statistical mechanics concepts to the statistical distribution of vorticity in the flow. Assuming a linear relationship leads one to suppose Gaussian statistics, and therefore the absence of true developed turbulence, characterized by an intermittent energy cascade. Indeed, observations of two-dimensional turbulence do not show these features of intermittency (Dubos et al. (2001)). In this paper, we will consider a basic vortex which has a linear relationship between the vorticity and streamfunction in the vortex core, but a constant vorticity in the outer flow. The corresponding azimuthal velocity field may then have a set of zeros, leading to the introduction of one or more critical levels when a free multipolar perturbation is imposed onto this basic flow. In this work, we resolve the consequent singularities through the introduction of nonlinear critical layers. These layers we shall describe in detail and show that their structure is essential in determining the structure of the whole multipolar vortex. The most relevant and interesting case is when we have a tripole with just one critical layer located at the boundary of the vortex core, and we shall give most attention to this case.

Before proceeding, it is pertinent to mention briefly some works from the large literature on the evolution of disturbances in nonlinear critical layers. In general, critical layers occur when a stream-wise oscillation perturbs a shear flow with a phase speed in the range of the shear flow velocity. Then, the equation governing the development of a linear, inviscid, steady disturbance is singular at the critical level. Neglected terms involving nonlinearity, transience or viscosity must be reintroduced within the critical layer in order to smooth out the singularity and correctly model the dynamics. The theory of this layer was significantly advanced by Benney and Bergeron (1969) and Davis (1969), who considered purely inviscid steady, nonlinear dynamics. Indeed, we use this theory in this paper. However the Benney-Bergeron theory requires corrections at the higher-order terms in the asymptotic expansion, and in particular needs correcting for the separatrices bounding the open and closed streamlines. Here, we treat these difficulties by introducing a radius-like streamwise coordinate which uniquely parametrizes each streamline (cf. Davis (1969)). We find that the inner flow nevertheless still diverges at the stagnation points where the separatrices intersect, and so we will introduce a third region with its own scaling located around these points. The singularity is then cancelled, and the velocity correctly vanishes there at each order in the asymptotic expansion. This correction is accomplished without taking viscosity into account.

Warn and Warn (1978) studied a forced Rossby-wave critical layer keeping both inertial and nonlinear terms. Numerical integration of the critical layer equations revealed an absence of any steady state. On the contrary, a strong temporal dependency within the cat's eyes structure was observed. With a long-wave assumption, Stewartson (1978) found an analytical solution of this problem. Later, Killworth and MacIntyre (1985) demonstrated that this solution was linearly unstable. The nonlinear saturation of the instabilities finally leads to a complete rearrangement of the vorticity distribution, with an unsteady and disordered motion within the critical layer (Haynes (1989)). Brown and Stewartson (1978) showed that a steady state could be attained after a long time by adding viscosity.

Our study here is quite distinct from these, due on the one hand to our consideration of a free mode, that is, our multipolar vortices are unforced solutions, and on the hand, to the different limiting processes involved; that is, we assume that $t \rightarrow \infty$ and then $\nu \rightarrow 0$, whereas the preceding works set $\nu \rightarrow 0$ and then $t \rightarrow \infty$. Recently, Balmforth, Llewellyn and Young (2001) analysed the evolution of an asymmetric perturbation to a class of compact vortices approximating a Gaussian profile of vorticity. The use of

compact vortices allows for the existence of Kelvin modes. A weakly nonlinear approach was undertaken by introducing a small distribution of $O(\epsilon)$ of vorticity outside the base vortices in order to obtain a critical layer of width $O(\epsilon)$. These disturbances of $O(\epsilon^2)$ then become quasi-modes which are resonantly excited by a forcing of $O(\epsilon^3)$. The critical layer dynamics comes down to two coupled nonlinear time-dependent equations which are integrated numerically. As in the previously quoted works, the vorticity develops on an increasing cross-stream fine-scale structure as time proceeds. According to the amplitude of the forcing, the quasi-mode may eventually decay inviscidly, or nonlinearity may lead to the formation of a multipolar structure.

2 Formulation

We consider an inviscid, two-dimensional steady flow, for which the vorticity ω is related to the streamfunction ψ by

$$\omega = \Delta\psi = F(\psi). \quad (1)$$

Here $F(\psi)$ is an arbitrary function of ψ , which, importantly from our perspective, may be multi-valued. Indeed, we shall describe a steady multipolar vortex which has a weakly nonlinear relationship between streamfunction and vorticity in the *core* of the vortex. In the poles of the vortex, we invoke the Prandtl-Batchelor theorem (Batchelor (1956)) that, due to mixing and diffusion over a long time, the vorticity becomes uniform in each pole, with the same value by symmetry. In general, the use of a single-valued functional form must then be relaxed since there does not necessarily exist a one-to-one correspondence between streamlines, and the value of the streamfunction; one value of ψ may correspond to two different streamlines.

This difficulty is overcome here by introducing a new coordinate η which is constant along a streamline (Davis (1969)):

$$\frac{d\psi}{d\eta} = \frac{1}{\eta} \int_0^\eta \omega(s) s ds = V(\eta). \quad (2)$$

The representation, in polar coordinates, of a streamline (cf. figure 2) is given by

$$r = \eta + h(\eta, \theta), \quad (3)$$

where h is the deviation from axisymmetry. It is assumed to be of order ϵ with ϵ a very small dimensionless parameter. $V(r)$ is the leading order axisymmetric component to the azimuthal velocity of the vortex. The full velocity field is described by

$$u = \frac{1}{r} \frac{V(\eta) \partial_\theta h}{1 + \partial_\eta h}, \quad v = \frac{V(\eta)}{1 + \partial_\eta h}, \quad (4)$$

whereas the vorticity is now defined, in place of (1), by

$$\omega = \Delta\psi = \frac{1}{\eta} \frac{d(\eta V(\eta))}{d\eta}. \quad (5)$$

On using (3), this becomes

$$\begin{aligned} (1 + \frac{1}{r^2} \partial_\theta h^2) \partial_\eta^2 h + (1 + \partial_\eta h)^2 \frac{1}{r^2} \partial_\theta^2 h + \frac{1}{r\eta} (h + r \partial_\eta h) (1 + \partial_\eta h)^2 \\ - \frac{2}{r^2} \partial_\theta h \partial_{\eta\theta}^2 h (1 + \partial_\eta h) = \frac{V'}{V} (1 + \partial_\eta h) \left(1 + \frac{1}{r^2} \partial_\theta h^2 - (1 + \partial_\eta h)^2\right). \end{aligned} \quad (6)$$

We shall choose $V(\eta)$ so that V is $O(\eta)$ as $\eta \rightarrow 0$ and, relative to a rigid body rotation, V is $O(1/\eta)$ as $\eta \rightarrow \infty$. The boundary conditions on h are then that h is $O(\eta)$ as $\eta \rightarrow 0$ and h is $o(1/\eta)$ as $\eta \rightarrow \infty$. These boundary conditions ensure that the perturbation to the axisymmetric vortex does not change the structure at the origin or at infinity.

We will study a $n + 1$ -pole vortex with a constant angular velocity $-\Omega/2$ in the exterior region, so that when the motion is considered in a steady frame, the exterior flow has a constant vorticity Ω . Its inner boundary is $r = R_b = 1 + \epsilon \eta_b \cos n\theta + \dots$. The deviation h from axisymmetry is then expanded as follows:

$$h = \epsilon h_1 + \epsilon^2 h_2 + \dots \quad (7)$$

Substitution of (7) into (6) yields the following linear equation for h_1 , with analogous equations for h_2 etc.,

$$\partial^2 h_1 + \frac{1}{r^2} \partial_\theta^2 h_1 + \frac{1}{r} \partial_\eta h_1 + \frac{h_1}{r} - 2 \partial_\eta h_1 \frac{V'}{V} \quad (8)$$

Let us impose the multipolar condition that $h_1(1, \theta) = \eta_b \cos n\theta$. Then we look for a solution in the form, $h_1 = H_{1,n}(\eta) \cos n\theta$. Let us denote η_c as the critical level where $V(\eta_c) = 0$ and then let $z = \eta - \eta_c$. This point is a singularity of (8) and prevents the expansion (7) being globally valid. Indeed, as we will show below, an inner expansion is needed around such critical levels. In the meantime, we seek an outer solution from (8) in the form of a Frobenius series, valid in a neighbourhood of η_c . Let

$$V(\eta) = V'_c z + \frac{1}{2} V''_c z^2 + \frac{1}{6} V'''_c z^3 + \dots$$

Then a regular Frobenius solution is:

$$\phi_1(z) = 1 + \frac{n^2 - 1}{6\eta_c^2} \left(z^2 - \frac{1}{6} \left(\frac{7}{\eta_c} + \frac{V''_c}{V'_c} \right) z^3 + O(z^4) \right), \quad (9)$$

provided that $V'_c \neq 0$, and a singular Frobenius solution is

$$\phi_2(z) = \left(\frac{V''_c}{V'_c} + \frac{1}{\eta_c} \right) \phi_1(z) \ln z + \frac{1}{z} \left[1 + \left(\frac{n^2 - 3}{2\eta_c^2} - \frac{3}{4} \left(\frac{V''_c}{V'_c} \right)^2 - \frac{V''_c}{\eta_c V'_c} + \frac{V'''_c}{3V'_c} \right) z^2 + O(z^3) \right]. \quad (10)$$

The general solution for $H_{1,n}$ is then a linear combination of these solutions:

$$H_{1,n}(\eta) = a \phi_1(z) + b \phi_2(z) \quad (11)$$

where a, b are constants to be determined. In the particular case where $n = 1$, ϕ_1 is reduced to a constant and $H_{1,1}$ can be determined by a mere quadrature,

$$H_{1,1}(\eta) = a \int \frac{d\eta}{\eta V(\eta)^2} + b.$$

After using the boundary conditions, the only possible solution is $H_{1,1} = 0$. According to previous studies on the stability of an axisymmetric flow, a sufficient condition for the nonexistence of unstable modes is a decreasing profile of vorticity in the whole space (Rayleigh, 1880). If we assume that $\omega(\eta)$ satisfies such a condition, then the existence of a mode with a critical level implies that the derivative of vorticity at the critical level must vanish (Briggs et al. (1970)); that is: $V''_c + [V'_c/\eta_c] = 0$. The terms multiplying the logarithm in equation (10) are then cancelled and only the singularity in $1/z$ remains. The Rayleigh equation (that is, the analogue of (8) for the streamfunction) is regular at the critical level. The general solution is therefore a regular neutral mode.

3 Analytical solution

3.1 The basic flow

To make further progress, we must now make a specific choice for the basic flow. We first observe that $H_{1,n}V$ obeys the differential equation:

$$\partial_\eta^2(H_{1,n}V) + \frac{1}{\eta} \partial_\eta(H_{1,n}V) - \frac{n^2}{\eta^2} H_{1,n}V = -\lambda^2 H_{1,n}V, \quad (12)$$

provided that V is solution of

$$\partial_\eta^2 V + \frac{1}{\eta} \partial_\eta V - \frac{1}{\eta^2} V = -\lambda^2 V. \quad (13)$$

Hence, when $\lambda \neq 0$, $V \equiv J_1(\lambda\eta)$ and $H_{1,n}V \equiv J_n(\lambda\eta) + \kappa Y_n(\lambda\eta)$ where J_p and Y_p (is singular at the origin) are respectively Bessel functions of the first and second kind, and of order p , while κ is a constant. When $\lambda = 0$, $V \equiv \alpha\eta + \beta\eta^{-1}$ ($\omega \equiv 2\alpha$) and $H_{1,n}V \equiv \eta^{-n}$, where α and β are constants.

Without loss of generality, we choose to set the vorticity amplitude at the centre of the vortex equal to unity, so the vorticity field is represented by

$$\begin{aligned} \omega(\eta) &= J_0(\lambda\eta), & \eta \leq 1, \\ \omega(\eta) &= \Omega, & \eta \geq 1, \end{aligned} \quad (14)$$

with $\Omega = J_0(\lambda)$, yielding a basic azimuthal velocity

$$\begin{aligned} V(\eta) &= \frac{J_1(\lambda\eta)}{\lambda}, & \eta \leq 1, \\ V(\eta) &= \frac{\Omega}{(n^2 - 1)} + \frac{J_1(\lambda)}{\lambda}, & \eta \geq 1 \end{aligned} \quad (15)$$

Such vortices are not isolated. Indeed, their velocity in the Earth's frame decreases as $v \simeq 1/r$ very far from the core, with a circulation

$$C = -\pi\Omega \left(1 - \frac{2J_1(\lambda)}{\lambda J_0(\lambda)}\right).$$

Note that in the vortical region $\eta < 1$, there is a linear relationship between ω and ψ , namely $\omega = -\lambda^2\psi$, and consequently the governing equation (1) is the linear Helmholtz equation in this region. Consequently, there are no intrinsic critical layers in this region, although as will be seen below, we can always impose a nonlinear critical layer there if needed. Similarly, in the outside region $\eta > 1$, since $\omega = \Omega$ is a constant, (1) is again a linear equation. Hence, any intrinsic nonlinearity in this model resides around $\eta = 1$.

3.2 The perturbed flow

3.2.1 $O(\epsilon)$

The equation for $H_{1,n}$ has a critical level at $\eta = \eta_c$ where $V(\eta_c) = 0$, as discussed above in Section 2 for the general case. Here, with the choice (15), we see directly from (13) that the condition $V_c'' + [V_c'/\eta_c] = 0$ holds here, so that there is no logarithmic term in the singular solution near $\eta = \eta_c$. Consequently, this is a regular neutral mode. Further, as discussed above, since the governing equation (1) is linear for $\eta \neq 1$, any intrinsic nonlinearity occurs around $\eta = 1$. Hence, most interest attaches to the case when $\eta_c = 1$. In fact, we shall ignore the possibility that $\eta_c > 1$, but we will allow for the possibility that $\eta_c \leq 1$, and is not necessarily exactly $\eta_c = 1$.

Consider then the critical level closest to $\eta = 1$, with $\eta_c \leq 1$. Then from (15), we see that $\eta_c = \mu_i/\lambda$ where μ_i is the i^{th} root of J_1 . The leading order perturbation on either side of the critical level is then given by

$$h_1 = a_1^- \lambda \frac{J_n(\lambda\eta) + \kappa^- Y_n(\lambda\eta)}{J_1(\lambda\eta)} \cos n\theta, \quad \eta < \eta_c, \quad (16)$$

$$h_1 = a_1^+ \lambda \frac{J_n(\lambda\eta) + \kappa^+ Y_n(\lambda\eta)}{J_1(\lambda\eta)} \cos n\theta, \quad \eta_c < \eta \leq 1, \quad (17)$$

$$h_1 = \frac{2b_1\Omega\eta^{1-n}\cos n\theta}{\Omega(\eta^2 - 1) + 2J_1(\lambda)/\lambda}, \quad \eta \geq 1. \quad (18)$$

The superscripts $-$, $+$ denote the flows closer to the core and to the boundary respectively. Here we have imposed the boundary condition that h_1 is $o(1/\eta)$ as $\eta \rightarrow \infty$, while the boundary condition as $\eta \rightarrow 0$ gives $\kappa^- = 0$ if η_c is the only critical level, and an analogous condition otherwise. At the vortex boundary, we impose the conditions that the streamfunction and velocity are continuous, that is, h_1 and $\partial_\eta h_1$ are continuous at $\eta = 1$, which gives

$$b_1\Omega = a_1^+ (J_n(\lambda) + \kappa^+ Y_n(\lambda)) = \eta_b \frac{J_1(\lambda)}{\lambda},$$

$$J_{n-1}(\lambda) + \kappa^+ Y_{n-1}(\lambda) = 0. \quad (19)$$

This last equation has used one of the recurrence relations between Bessel functions (Abramowitz and Stegun (1972)). Once κ^+ has been determined, (19) is the dispersion relation determining λ . First, let us consider the case of most interest, when $\eta_c = 1$, so that $\lambda = \mu_i$, $i = 1, 2, \dots$. If $i = 1$ and so $\lambda = \mu_1$, then there are no other critical levels in $\eta < 1$, and so $\kappa_+ = \kappa_- = 0$. In this case, the boundary deformation blows up ($\eta_b \rightarrow \infty$), and the above relations collapse to give $J_{n-1}(\lambda) = 0$. This can hold if and only if $n = 2$. The tripole is thus the only vortex which can have a single critical level placed at its boundary.

Next, let $\lambda = \mu_i$ with $i \geq 2$ and $\eta_c = 1$ still. The boundary deformation again blows up, but in this case there are $(i - 1)$ other critical levels at $\eta_c = \mu_j/\mu_i < 1$, $j = 1, \dots, i - 1$. A detailed analysis of the nonlinear critical layers, given in section 6, shows that $\kappa^+ = \kappa^- = 0$ across each critical level, thus eliminating all Y_n functions. It follows then that $J_{n-1}(\lambda) = 0$; thus, we must have $n = 2$ and the case $n > 2$ cannot occur if $\eta_c = 1$. Thus, in fact we have the stronger statement that the tripole ($n = 2$) is the only case for which there can be a critical level at the vortex boundary, $\eta_c = 1$.

We next discuss the general case when $\eta = 1$ is not a critical level, and $\eta_c < 1$ is the closest critical level to $\eta = 1$. Since $\eta_c = \mu_i/\lambda < 1$, for some $i = 1, 2, \dots$, λ must be larger than μ_i . In general, there may be an infinite number of allowed values, lying in the bands, $\mu_i < \lambda < \mu_{i+1}$, where some of these bands may be empty. Further for $\mu_1 < \lambda < \mu_2$, ($i = 1$) there is just one critical level possible in $\eta < 1$, while for $\mu_2 < \lambda < \mu_3$, ($i = 2$), there are two critical levels possible, and so on. At the boundaries of these bands, the critical level coincides with the boundary, and the boundary deformation blows up.

$\mathbf{n}$	n_c	λ	η_c	$\eta_b/(\lambda a_1^+)$	Ω
2	1	3.8317	1	∞	-0.4028
	2	7.0156	0.5462 1	∞	0.3001
3	1	5.1356	0.7461	-1	-0.1323
	2	8.4172	0.4552 0.8335	-1	0.0645
4	1	6.3802	0.6006	-1.5951	0.2397
	2	9.7610	0.3926 0.7187	-2.4401	-0.2285

Table 1: First modes and their critical levels

Figure 1: Profiles of vorticity (solid line) and azimuthal velocity (dashed line) in the mean flow in the vortex frame. (a) $n = 2$: $C/\pi = -\Omega = 0.4028$; (b) $n = 4$: $\lambda = 9.761$, $\eta_c = 0.3926$, $\eta_c = 0.7187$, $\chi = -2.4401$, $\Omega = -0.2285$ and $C/\pi = 0.2494$.

6 that a detailed analysis of the nonlinear critical layers reveals that $\kappa^+ = \kappa^- = 0$ across each critical level. It follows that $J_{n-1}(\lambda) = 0$, thus determining the allowed values of λ . Of course, we must then have $n \geq 3$, so that $\eta = 1$ is not a critical level. Thus we see that the interaction of an asymmetric mode n with the J_0 -vorticity profile can yield one or more critical levels inside the vortex. For n_c critical levels, the critical vorticities, from the core to the boundary, are $\Omega_c = J_0(\mu_i)$, $i = 1, n_c$. For the first three first roots of J_1 , the corresponding vorticities are: $\Omega = -0.4028$, 0.3001 and -0.2497 . Table 1 displays the first two critical levels for the first three modes.

In the absence of viscosity, the concomitant singularity is removed by keeping nonlinear terms in the leading order equation in a zone around η_c , called the nonlinear critical layer. We note that when $n = 2$ and $\lambda = \mu_1$, so that there is just a single critical level at $\eta_c = 1$, the chosen profile (14) is decreasing in all space and that $\omega'(\eta_c) = 0$ (see figure 1 (a)). This mean flow is therefore neutrally stable to this particular mode-2 perturbation (Briggs et al. (1970)). But, otherwise, and for all $n \geq 3$ (see figure 1 (b)), the mean flow possesses at least one inflexion point, and no inference can be made about its linear stability.

3.2.2 $O(\epsilon^2)$

The equation satisfied by h_2V is

$$\begin{aligned} \partial_\eta^2(h_2V) + \frac{1}{\eta}\partial_\eta(h_2V) + \frac{1}{\eta^2}\partial_\theta^2(h_2V) + \lambda^2h_2V = V' \left(\frac{1}{\eta^2}(\partial_\theta h_1)^2 - 3(\partial_\eta h_1)^2 \right) \\ + 2V \left(\frac{1}{\eta^3}h_1\partial_\theta^2h_1 - \frac{1}{\eta^2}\partial_\eta h_1\partial_\theta^2h_1 + \frac{h_1^2}{2\eta^3} - \frac{1}{\eta^2}h_1\partial_\eta h_1 - \frac{1}{\eta}(\partial_\eta h_1)^2 + \frac{1}{\eta^2}\partial_\theta h_1\partial_\eta^2h_1 \right). \end{aligned} \quad (20)$$

The solution of this equation is separated into the modes 0, n and $2n$; $h_2 = H_{2,0} + H_{2,n}\cos n\theta + H_{2,2n}\cos 2n\theta$. Then, $H_{2,p}$ may be obtained by two quadratures, as follows. First, we note that the homogeneous equation for h_2V obtained by equating the right-hand side to zero has independent solutions $J_p(\eta)$, $Y_p(\eta)$ for $\eta < 1$ and η^{-p} , η^p ($p > 1$), $1, \ln(\eta)$ ($p = 0$) for $\eta > 1$. Of these we select the first one, as

N_p . It is made continuous at $\eta = 1$. Then multiplying (20) by ηN_p and integrating once we obtain

$$\eta(N_p G'_p - G_p N'_p) = INH_p(\eta) = \int_{0,\infty}^{\eta} NH_p(t)N_p(t)t dt. \quad (21)$$

Here G_p denotes $H_{2,p}V$, a prime denotes the derivative with respect to η and NH_p is the inhomogeneous term on the right-hand side of (20). In obtaining (21), we have applied boundary conditions at $\eta = 0, \infty$ respectively. Then, one further integration gives

$$H_{2,p}(\eta) = \frac{N_p(\eta)}{V(\eta)} \int_{\eta_I}^{\eta} \frac{INH_p(r)}{rN_p(r)^2} dr. \quad (22)$$

Here η_I are chosen to lie in $\eta < 1 (> 1)$ respectively, and represent an arbitrary constant of integration in each of $\eta < 1 (> 1)$. They are eventually determined by applying the boundary conditions at $\eta \rightarrow 0, (\infty)$ respectively.

3.3 The perturbed flow inside the critical layer

Henceforth, we exclusively use the terms *inner* and *outer* to define the critical layer and the regions outside the critical layer respectively. It is necessary to rescale both h and η to obtain the relevant inner expansion.

$$h = \epsilon^{\frac{1}{2}} \hat{h}_1 + \epsilon \hat{h}_2 + \epsilon^{\frac{3}{2}} \hat{h}_3 + O(\epsilon^2), \quad (23)$$

$$\hat{\eta} = \frac{\eta - \eta_c}{\epsilon^{\frac{1}{2}}}. \quad (24)$$

The mean vorticity and azimuthal velocity are also expanded in the critical layer

$$\omega(\hat{\eta}) = \Omega_c \left(1 - \epsilon \frac{1}{2} \lambda^{\pm 2} \hat{\eta}^2 + \epsilon^{\frac{3}{2}} \frac{1}{6} \lambda^{\pm 2} \frac{\hat{\eta}^3}{\eta_c} \right) + O(\epsilon^2), \quad (25)$$

$$V^{\pm}(\hat{\eta}) = \Omega_c \left(\epsilon^{\frac{1}{2}} \hat{\eta} - \epsilon \frac{\hat{\eta}^2}{2\eta_c} + \epsilon^{\frac{3}{2}} \left(\frac{1}{2} - \frac{1}{6} \lambda^{\pm 2} \eta_c^2 \right) \frac{\hat{\eta}^3}{\eta_c^2} \right) + O(\epsilon^2), \quad (26)$$

with $\Omega_c = J_0(\lambda\eta_c)$. Here, we use the notation that $\lambda^- = \lambda$ inside the vortex and $\lambda^+ = 0$ outside.

4 The tripolar vortex

The main case of interest is $n = 2$ when there is a single critical level at the vortex boundary ($\eta_c = 1$). We shall give all the details for this case, for which $J_1(\lambda) = 0$, with $\lambda = \mu_1$, and V given by (15). In subsection 4.1, we present leading-order solutions of the outer flow and their asymptotic expansions near $\eta = \eta_c$ in terms of the inner variable. Then, in subsection 4.2, we give the solution of the flow within the critical layer and match this flow with the preceding expansions at the edges of the critical layer and with the flow within the poles (subsection 4.3). This latter flow has a uniform vorticity as a consequence of the Prandtl-Batchelor theorem. We will find that the matching on the dividing streamlines is possible up to $O(\epsilon)$ for the vorticity and $O(\epsilon^{\frac{3}{2}})$ for the azimuthal velocity. In subsection 4.4, we refine our model by inserting a fifth layer in the flow which surrounds the stagnation points and find a new variable which enables us to suppress the singularities which appear in subsection 4.2. The values of the pole vorticity and velocity distortions are unchanged in this refinement.

4.1 The outer flow

For this case, equations (16,17,18) collapse to

$$h_1^- = a_1 \lambda \frac{J_2(\lambda\eta)}{J_1(\lambda\eta)} \cos 2\theta, \quad \eta < 1, \quad (27)$$

$$h_1^+ = \frac{2b_1 \cos 2\theta}{\eta(\eta^2 - 1)}, \quad \eta > 1. \quad (28)$$

At the next order, from (22), $H_{2,0}$ can readily be found, using the formulae linking Bessel functions and their derivatives,

$$H_{2,0}^- = a_1^2 \lambda^2 \left(\frac{\lambda}{2} \frac{J_2(\lambda\eta)}{J_1(\lambda\eta)} - \frac{5}{4\eta} \frac{J_2(\lambda\eta)^2}{J_1(\lambda\eta)^2} + \frac{\lambda}{4} \frac{J_2(\lambda\eta)^3}{J_1(\lambda\eta)^3} \right) + \xi_{2,0} a_1^2 \lambda \frac{J_0(\lambda\eta)}{J_1(\lambda\eta)}, \quad \eta \leq 1,$$

$$H_{2,0}^+ = b_1^2 \left(\frac{3 - 5\eta^2}{\eta^2} - \frac{2\xi_{2,0}\eta}{\eta^2} \right), \quad \eta > 1$$

Here $\xi_{2,0}$ and $\zeta_{2,0}$ are the afore-mentioned integration constants. However, on applying the boundary conditions as $\eta \rightarrow \infty, 0$, we readily find that $\zeta_{2,0} = 0, \xi_{2,0} = 0$ respectively. Indeed, in the latter case, on analysing the flow around $\eta = 0$ with a stretched variable, we find a non-zero azimuthal velocity in the core proportional to $\xi_{2,0}$. This is clearly unacceptable, and so $\xi_{2,0} = 0$.

Next we find that $H_{2,2}^{\pm} \cos 2\theta$ are multiples of $h_1^{\pm}$ (27,28),

$$H_{2,2}^{-} = 4a_2\lambda \frac{J_2(\lambda\eta)}{J_1(\lambda\eta)} a_1^2, \quad H_{2,2}^{+} = \frac{8b_2b_1^2}{\eta(\eta^2-1)}, \quad (29)$$

with integration constants a_2, b_2 analogous to a_1, b_1 respectively. For the present, we retain these terms, but note that we could set $a_2 = b_2 = 0$ by absorbing the expressions (29) into the first-order terms (27, 28) with a consequent redefinition of the parameter ϵ .

The calculations are complex for the case $p = 4$, but again using formulae linking Bessel functions and their derivatives we can show that,

$$\begin{aligned} H_{2,4}^{-} &= a_1^2 \lambda^2 \left(\frac{\lambda}{2} \frac{J_2(\lambda\eta)}{J_1(\lambda\eta)} - \frac{5}{4\eta} \frac{J_2(\lambda\eta)^2}{J_1(\lambda\eta)^2} + \frac{\lambda}{4} \frac{J_2(\lambda\eta)^3}{J_1(\lambda\eta)^3} \right) + a_1^2 \lambda^3 \xi_{2,4} \frac{J_4(\lambda\eta)}{J_1(\lambda\eta)} \\ \text{while for } \eta > 1, \quad H_{2,4}^{+} &= \frac{b_1^2}{\eta^3} \left(\frac{2\zeta_{2,4}}{\eta^2-1} + \frac{3-5\eta^2}{(\eta^2-1)^3} \right). \end{aligned}$$

The nonhomogeneous parts of $H_{2,0}^{-}$ and $H_{2,4}^{-}$ are identical, this is a result, apparently after (20), only valid for the mode 2. Here $\xi_{2,4}, \zeta_{2,4}$ are integration constants, determined later after matching across the critical layer. We can easily check that all these solutions satisfy the required boundary conditions as $\eta \rightarrow 0, \infty$. Note that the singularity at $\eta = 1$ is $O((\eta-1)^{-3})$.

These outer terms will need to be matched, as $\eta \rightarrow 1$ with the inner expansion. This is achieved by introducing the inner variable (24), that is $\hat{\eta} = (\eta-1)/\epsilon^{\frac{1}{2}}$ here, and then expanding with respect to $\epsilon^{\frac{1}{2}}$. The outcome for h_1 is,

$$\begin{aligned} \epsilon h_1^{-} &= -\epsilon^{\frac{1}{2}} \frac{a_1}{\hat{\eta}} \left(1 - \epsilon^{\frac{1}{2}} \frac{3}{2} \hat{\eta} + \epsilon \left(\frac{7}{4} - \frac{1}{3} \lambda^2 \right) \hat{\eta}^2 + O(\epsilon^{\frac{3}{2}}) \right) \cos 2\theta, \quad \hat{\eta} < 0 \\ \epsilon h_1^{+} &= \epsilon^{\frac{1}{2}} \frac{b_1}{\hat{\eta}} \left(1 - \epsilon^{\frac{1}{2}} \frac{3}{2} \hat{\eta} + \epsilon \frac{7}{4} \hat{\eta}^2 + O(\epsilon^{\frac{3}{2}}) \right) \cos 2\theta, \quad \hat{\eta} > 0. \end{aligned} \quad (30)$$

Similarly, we find that the expansions of $H_{2,0}$ and $H_{2,4}$ in terms of the inner variable, yield

$$\begin{aligned} \epsilon^2 H_{2,0}^{-} &= -\epsilon^{\frac{1}{2}} \frac{a_1^2}{4\hat{\eta}^3} \left(1 + \epsilon^{\frac{1}{2}} \frac{1}{2} \hat{\eta} + \epsilon (\lambda^2 - 8 - 4\xi_{2,0}) \hat{\eta}^2 + O(\epsilon^{\frac{3}{2}}) \right), \quad \hat{\eta} < 0, \\ \epsilon^2 H_{2,0}^{+} &= -\epsilon^{\frac{1}{2}} \frac{b_1^2}{4\hat{\eta}^3} \left(1 + \epsilon^{\frac{1}{2}} \frac{1}{2} \hat{\eta} + \epsilon 4(\zeta_{1,0} - 2) \hat{\eta}^2 + O(\epsilon^{\frac{3}{2}}) \right), \quad \hat{\eta} > 0. \end{aligned} \quad (31)$$

$$\begin{aligned} \epsilon^2 H_{2,4}^{-} &= -\epsilon^{\frac{1}{2}} \frac{a_1^2}{4\hat{\eta}^3} \left(1 + \epsilon^{\frac{1}{2}} \frac{1}{2} \hat{\eta} - \epsilon 4(\lambda^2 - 24) \xi_{2,4} \hat{\eta}^2 + \epsilon \hat{\eta}^2 (\lambda^2 - 8) + O(\epsilon^{\frac{3}{2}}) \right), \quad \hat{\eta} < 0, \\ \epsilon^2 H_{2,4}^{+} &= -\epsilon^{\frac{1}{2}} \frac{b_1^2}{4\hat{\eta}^3} \left(1 + \epsilon^{\frac{1}{2}} \frac{1}{2} \hat{\eta} - \epsilon 4(2 + \zeta_{2,4}) \hat{\eta}^2 + O(\epsilon^{\frac{3}{2}}) \right), \quad \hat{\eta} > 0. \end{aligned} \quad (32)$$

4.2 The inner flow

4.2.1 $O(\epsilon^{\frac{1}{2}})$

Substituting (23), (24) and (26) into equation (6) and collecting the highest order terms lead to the equation governing $\hat{h}_1$:

$$\partial_{\hat{\eta}}^2 \hat{h}_1 = \frac{1}{\hat{\eta}} \left((1 + \partial_{\hat{\eta}} \hat{h}_1) - (1 + \partial_{\hat{\eta}} \hat{h}_1)^3 \right). \quad (33)$$

The general solution of this equation is

$$\hat{h}_1^s = s \sqrt{\hat{\eta}^2 + f_1^s(\theta)} - \hat{\eta} + g_1(\theta), \text{ where } s = \pm. \quad (34)$$

To determine the functions $f_1(\theta)$ and $g_1(\theta)$, we need to match this solution as $\hat{\eta} \rightarrow \pm\infty$ with the outer flow as $\eta \rightarrow 1$, that is, with the expansions (30) above. The matching of these outer and inner expansions then gives that $s = \text{sgn}[\hat{\eta}]$, $f_1^{-}(\theta) = -2a_1 \cos 2\theta$, $f_1^{+}(\theta) = 2b_1 \cos 2\theta$ and $g_1(\theta) = 0$. The solution (34) is

This boundary of the domain of validity is characterized by a separatrix surrounding a family of closed streamlines and will be denoted by a zero subscript, so that $\hat{\eta}_0^{-2} = -2a_1$ and $\hat{\eta}_0^{+2} = 2b_1$. At this stage, the ratio a_1/b_1 is not known, but later we will show that in fact $a_1/b_1 = -1$, so that, from (27,28) we see that then the outer solution h_1 is such that $h_1 V$ is continuous at $\eta = 1$. On the inside separatrix, denoted by ψ_0^- , the vorticity is $\Omega(1 - \epsilon \frac{1}{2} \lambda^2 (\hat{\eta}_0^-)^2)$ and on the outside separatrix ψ_0^+ , it is Ω . From the Prandtl-Batchelor theorem, the vorticity inside the closed streamlines becomes constant after a long time for a high Reynolds number flow due to viscous diffusion of vorticity (Batchelor (1956)). A distortion of vorticity thus exists through the separatrix, of order ϵ , weaker than the usual one of $\epsilon^{\frac{1}{2}}$ when $\omega'_c \neq 0$. This jump is distributed across a thin boundary layer along ψ_0 in the limit of infinitesimal viscosity. Nevertheless, the width of the critical layer is still of order $\epsilon^{\frac{1}{2}}$, which will enable us to obtain vortices with the same spatial extent of the poles as would be expected for a singular neutral mode. We can quantify the departure from axisymmetry by the eccentricity of the ellipse bounding the tripole, given by $\sigma = \sqrt{b_1 \epsilon}$. It vanishes as $\epsilon \rightarrow 0$, but only as $\sqrt{\epsilon}$. The base vortex is thus quite strongly deformed by the poles as the plots show (cf. figure 4). Comparisons with eccentricities computed from the works quoted in the Introduction show a good agreement: σ varying between 0.17 and 0.33. Nevertheless, our perturbative approach requires an $O(\epsilon)$ weakness of the vorticity amplitude within the poles. This circumstance is not due to the use of the nonlinear critical layer theory but instead is due to the linear relationship between the streamfunction and the vorticity elsewhere which has enabled us to obtain an exact solution of our equations. If instead, we had adopted a nonlinear relationship, we suspect we would have obtained an $O(\sqrt{\epsilon})$ vorticity amplitude within the poles. This remark is in agreement with the nonlinear character of the functional $\omega = F(\psi)$ found in the works quoted in the Introduction. Here, we have focused on providing a possible explanation of the formation of multipolar vortices from the interaction of an axisymmetric flow and an azimuthal disturbance.

4.2.2 $O(\epsilon)$

At this order we get the equation,

$$\partial_{\hat{\eta}}^2 \hat{h}_2 + \frac{1}{\hat{\eta}} \partial_{\hat{\eta}} \hat{h}_2 \left(3(D\hat{h}_1)^2 - 1 \right) = -\frac{1}{2} D\hat{h}_1 (\partial_{\hat{\eta}} \hat{h}_1)^2, \quad \text{where} \quad D\hat{h}_1 = 1 + \partial_{\hat{\eta}} \hat{h}_1. \quad (35)$$

The general solution is

$$\hat{h}_2^s = \frac{\hat{\eta}^2}{6} - \frac{1}{6} \frac{s\hat{\eta}^3}{\sqrt{\hat{\eta}^2 + f_1^s(\theta)}} - \frac{f_2(\theta)}{\sqrt{\hat{\eta}^2 + f_1^s(\theta)}} + g_2^s(\theta). \quad (36)$$

Matching with the second terms of (30) then yields $f_2(\theta) = 0$, $g_2^-(\theta) = 5/3a_1 \cos 2\theta$ and $g_2^+(\theta) = -5/3b_1 \cos 2\theta$.

4.2.3 $O(\epsilon^{\frac{3}{2}})$

The relevant equation at this order is

$$\begin{aligned} \partial_{\hat{\eta}}^2 \hat{h}_3 + \frac{1}{\hat{\eta}} \partial_{\hat{\eta}} \hat{h}_3 \left(3D\hat{h}_1^2 - 1 \right) &= -(\partial_{\theta} \hat{h}_1)^2 \partial_{\hat{\eta}}^2 \hat{h}_1 - \frac{1}{2} \partial_{\hat{\eta}} \hat{h}_1^2 \partial_{\hat{\eta}} \hat{h}_2 \\ &+ D\hat{h}_1 \left(\lambda^{s2} \frac{\hat{\eta}}{3} (D\hat{h}_1^2 - 1) - \frac{3}{\hat{\eta}} (\partial_{\hat{\eta}} \hat{h}_2)^2 - \partial_{\hat{\eta}} \hat{h}_1 \partial_{\hat{\eta}} \hat{h}_2 + \frac{1}{4} \hat{\eta} (D\hat{h}_1^2 - 4D\hat{h}_1 + 3) \right. \\ &\quad \left. + 2\partial_{\theta} \hat{h}_1 \partial_{\hat{\eta}}^2 \hat{h}_1 - D\hat{h}_1 (\hat{h}_1 + \partial_{\theta}^2 \hat{h}_1) + \frac{1}{\hat{\eta}} (\partial_{\theta} \hat{h}_1)^2 \right) \end{aligned}$$

and its general solution is

$$\begin{aligned} \hat{h}_3^s &= -\frac{\hat{\eta}^3}{18} - \frac{s\hat{\eta}^6}{72\sqrt{\hat{\eta}^2 + \hat{\eta}_0^{2s} \cos 2\theta}} + \frac{5s}{72} \sqrt{\hat{\eta}^2 + \hat{\eta}_0^{2s} \cos 2\theta}^3 - s \frac{\hat{\eta}_0^{2s}}{6} (\lambda^{s2} - \frac{9}{2}) \sqrt{\hat{\eta}^2 + \hat{\eta}_0^{2s} \cos 2\theta} \cos 2\theta \\ &+ \left(f_3^s(\theta) + s\hat{\eta}_0^{4s} \left(1 - \frac{\lambda^{2s}}{6} \right) \cos^2 2\theta \right) \frac{1}{\sqrt{\hat{\eta}^2 + \hat{\eta}_0^{2s} \cos 2\theta}} + g_3(\theta). \end{aligned} \quad (37)$$

The matching of $\hat{h}_3$ with (30-31-32) gives $g_3 = 0$ and the following expression for f_3^s :

$$\begin{aligned} f_3^-(\theta) &= a_1^2 \left(\frac{3}{4} - \frac{\lambda^2}{4} + 4a_2 \cos 2\theta - \left[\frac{\lambda^2}{4} - \frac{3}{4} + \xi_{2,4}(\lambda^2 - 24) \right] \cos 4\theta \right), \\ f_3^+(\theta) &= -b_1^2 \left(\frac{3}{4} - 4b_2 \cos 2\theta - \left[\zeta_{2,4} - \frac{3}{4} \right] \cos 4\theta \right). \end{aligned}$$

The integration constants remaining in these expressions are eventually determined by matching across

4.2.4 Reformulation in the radial coordinate

At any stage, we may use the transformations (2, 3) to express the solution in terms of the radial variable up to the order considered. We display here only the outer expression for the vorticity in $r < 1$,

$$\omega(r, \theta) = J_0(\lambda r) + \epsilon a_1(1 + 4\epsilon a_1 a_2) \lambda^2 J_2(\lambda r) \cos 2\theta + \epsilon^2 a_1^2 \lambda^2 \xi_{2,4} J_4(\lambda r) \cos 4\theta + O(\epsilon^3).$$

Thus, there is no singularity in $r < 1$, and all the singular terms in h are in fact cancelled in this reformulation. This is to be expected, since, as noted earlier, for this outer solution, ω is a linear function of ψ ($\omega = -\lambda^2 \psi$), and so the governing equation for ω is just the linear Helmholtz equation, with no intrinsic, or imposed, singularity. A similar situation holds in $r > 1$, where $\omega = \Omega$, a constant, and so the governing equation for ψ is again a linear equation, with no singularities.

At this point, it might be thought that it would have been simpler to use the radial coordinate r from the outset. However, as noted earlier in Sections 2 there would then have been difficulties in determining the nonlinear critical layer correctly. Hence, our choice to work with the implicit variable η , which defines streamlines directly.

4.3 Expansion inside the poles

We here define the motion within the poles and try to match it with the inner flow. We recall that the critical layer contains two domains, each bounded by the separatrices $\hat{\eta}^2 = -2a_1$ and $\hat{\eta}^2 = 2b_1$ on the inside and outside respectively. These domains are the satellites of our tripole, have closed streamlines, and constant vorticity, denoted by ω_0 . Within these domains, a general solution for the streamfunction is,

$$\psi(r) = \frac{\omega_0}{4} \left(r^2 + (A_0 - 2) \ln r + 2C + \sum_{p=1}^{\infty} A_p (r^{2p} + r^{-2p}) \cos 2p\theta + 2B_p r^{-2p} \cos 2p\theta \right). \quad (38)$$

The presence of the coefficients A_0 and B_p leads to an asymmetry of the critical layer with respect to $r = 1$ ($A_0 = B_p = 0 \rightarrow v = 0$ on the unity radius). Next r is expanded around the critical level, $r = 1 + \epsilon^{\frac{1}{2}} \hat{\eta} + \epsilon^{\frac{1}{2}} \hat{h}^c$ and in the same way A_p , B_p and C are also expanded (for instance, $A_p = \epsilon A_{p,1} + \epsilon^{\frac{3}{2}} A_{p,2} + O(\epsilon^2)$). The superscript c will denote the variables inside the closed streamlines of the poles. The expression of ψ can also be rewritten from (2) in terms of the variable $\hat{\eta}$ as follows.

$$\psi(\hat{\eta}) = \frac{\omega_0}{4} (1 + \epsilon 2\hat{\eta}^2 - \epsilon^{\frac{3}{2}} \frac{2}{3} \hat{\eta}^3 + \epsilon^2 \frac{1}{2} \hat{\eta}^4 + O(\epsilon^{\frac{5}{2}})). \quad (39)$$

Equating (38) and (39) at the same order, we obtain $A_{0,1} = C_1 = 0$ and

$$(\hat{h}_1^c + \hat{\eta})^2 = \hat{\eta}^2 - \sum_{p=1}^{\infty} (A_{p,1} + B_{p,1}) \cos 2p\theta. \quad (40)$$

Matching h at the separatrix, on using (34), leads to the determination of the constants. That is, $A_{1,1} + B_{1,1} = -\hat{\eta}_0^{2s}$, $A_{p,1} + B_{p,1} = 0$ for $p > 1$ and finally we obtain the relationship between a_1 and b_1 ; $\hat{\eta}_0^+ = \hat{\eta}_0^- = \hat{\eta}_0$ or $b_1 = -a_1$. The matching of the velocity on ψ_0 yields, on using (4),

$$v_1(\psi_0) = s \sqrt{2} \Omega \hat{\eta}_0 |\cos \theta| = s \sqrt{2} \omega_0 \hat{\eta}_0 |\cos \theta|. \quad (41)$$

This is in fact the equality of ω_0 and Ω at the leading order, the discrepancy between them appearing only at order ϵ . We denote from now on: $\omega_0 = \Omega(1 + \epsilon \delta \Omega)$ whereas the vorticity at the edge of the separatrix within the vortex is written $\omega_0^- = \Omega(1 + \epsilon \Delta \Omega)$, $\Delta \Omega = -\frac{1}{2} \lambda^2 \hat{\eta}_0^2$. We note incidentally that the scaling with the variable $\hat{\eta}$ and function $\hat{h}$ is not strictly valid in the core of the critical layer where $\hat{\eta} \approx 0$ in Eq. (40), as here the streamlines vary strongly with respect to θ . But here, of course, (38) can be used.

At the next order, we have

$$\begin{aligned} \hat{h}_2^c = & -\frac{1}{4} A_{0,2} + \frac{1}{6} (\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta) - \frac{s}{2\sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta}} \left(\sum_{p=1}^{\infty} (A_{p,2} + B_{p,2}) \cos 2p\theta + C_2 \right) \\ & - \frac{1}{6} \frac{s \hat{\eta}^3}{\sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta}} + \sum_{p=1}^{\infty} B_{p,1} \cos 2p\theta, \end{aligned}$$

and with $C_2 = 0$, $A_{0,2} = 0$, $A_{1,1} = 0$, $B_{1,1} = -\hat{\eta}_0^2$, $A_{p,1} = B_{p,1} = 0$, $p > 1$ and $A_{p,2} + B_{p,2} = 0$ according to Eq. (36). Asymmetry thus appears at the second order as was observed at Eq. (36). The second order orthoradial velocity on ψ_0 is then

$$v_2(\psi_0) = -\frac{1}{2} \Omega \hat{\eta}_0^2 (1 + \cos 2\theta) - \frac{1}{2} \frac{\Omega \hat{\eta}_0^2}{\sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta}}$$

At the order $\epsilon^{\frac{3}{2}}$,

$$\begin{aligned} \hat{h}_3^c = & -\frac{1}{4}A_{0,3} - \frac{\hat{\eta}^3}{18} + \frac{1}{18} \frac{s\hat{\eta}^6}{\sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta}^3} + \frac{13}{36} s\hat{\eta}_0^6 \frac{3 \cos 2\theta + \cos 6\theta}{\sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta}^3} + \frac{23}{24} \frac{s\hat{\eta}_0^2 \hat{\eta}^4 \cos 2\theta}{\sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta}^3} \\ & + \frac{7}{6} \frac{s\hat{\eta}_0^4 \hat{\eta}^2 (1 + \cos 4\theta)}{\sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta}^3} - \frac{s}{2\sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta}} \left(\sum_{p=1}^{\infty} (A_{p,3} + B_{p,3}) \cos 2p\theta + C_3 \right) + \sum_{p=1}^{\infty} p B_{p,2} \cos 2p\theta \end{aligned}$$

The Fourier's coefficients determined after matching on ψ_0 with (37) are given in the Appendix A. The series in (38) converges very slowly. Indeed, a large number of harmonics are present inside the cat's eyes, as has been previously reported (Benney and Bergeron (1969)). This issue is associated with the breakdown of this scaling when $\hat{\eta} \approx 0$.

The order $\epsilon^{\frac{3}{2}}$ velocity given by the inner expansion on ψ_0 is

$$\begin{aligned} \frac{v_3^s(\psi_0)}{\Omega} = & s \frac{\hat{\eta}_0^3}{9} - \frac{\lambda^{2s}}{6} \frac{s\hat{\eta}_0^3}{\sqrt{1 + \cos 2\theta}} \\ & + \frac{s\hat{\eta}_0^3 (70 + 21 \cos 2\theta - 42 \cos 4\theta + 11 \cos 6\theta)}{288\sqrt{1 + \cos 2\theta}^3} + \frac{f_3^s(\theta)}{\hat{\eta}_0 \sqrt{1 + \cos 2\theta}} \end{aligned} \quad (42)$$

and inside the separatrices by

$$\begin{aligned} \frac{v_3^{cs}(\psi_0)}{\Omega} = & s\Omega\hat{\eta}_0\sqrt{1 + \cos 2\theta} + s \frac{\hat{\eta}_0^3}{9} + \frac{1}{18} \frac{s\hat{\eta}_0^3}{\sqrt{1 + \cos 2\theta}^3} - \frac{5}{24} \frac{s\hat{\eta}_0^3 \cos 2\theta}{\sqrt{1 + \cos 2\theta}^3} - \frac{1}{3} \frac{s\hat{\eta}_0^3 \cos 4\theta}{\sqrt{1 + \cos 2\theta}^3} \\ & - \frac{1}{18} \frac{s\hat{\eta}_0^3 \cos 6\theta}{\sqrt{1 + \cos 2\theta}^3} - \frac{s}{2\hat{\eta}_0\sqrt{1 + \cos 2\theta}} \left(\sum_{p=1}^{\infty} (A_{p,3} + B_{p,3}) \cos 2p\theta + C_3 \right). \end{aligned} \quad (43)$$

The matching of $v_3^{cs}(\psi_0)$ and $v_3^s(\psi_0)$ cannot be obtained (see Appendix A), thus leaving a velocity jump across the separatrix boundary.

4.3.1 Determination of the vorticity inside the poles

We now focus on the thin boundary layer which is located along the separatrix ψ_0 . First, we note that the two-dimensional steady state viscous momentum equation is,

$$\omega \times \mathbf{u} + \nabla \left(\frac{\mathbf{p}}{\rho} + \frac{|\mathbf{u}|^2}{2} \right) = -\nu \nabla \times \omega, \quad (44)$$

where $\mathbf{u} = \mathbf{k} \times \nabla \psi$, $\omega = \omega \mathbf{k}$ and we recall that $\omega = \Delta \psi$ (see 1); ν is the kinematic viscosity and ρ is the (constant) density. Elimination of the pressure then yields the viscous vorticity equation,

$$\mathbf{u} \cdot \nabla \omega = \nu \Delta \omega. \quad (45)$$

Next, we use the Von Mises transformation to the curvilinear coordinates l, ψ , where l follows the streamlines and ψ is perpendicular to them (Curle (1962), Davis (1969)). The vorticity equation (45) then becomes, after also making a boundary layer approximation,

$$\frac{\partial \omega}{\partial l} = \nu \frac{\partial}{\partial \psi} \left(U \frac{\partial \omega}{\partial \psi} \right), \quad (46)$$

where U is the curvilinear velocity along a streamline. We choose a characteristic length L (radius of the vortex, for example), and a characteristic velocity U_0 so that the Reynolds number is $R = U_0 L / \nu$. Since the velocity in the critical layer is $O(\epsilon^{\frac{1}{2}})$, we set

$$U \rightarrow \epsilon^{\frac{1}{2}} U_0 \Omega_c \hat{\eta}_0 U, \quad \psi \rightarrow \epsilon^{\frac{1}{2}} U_0 \Omega_c \hat{\eta}_0 \delta \psi,$$

where δ is the boundary layer thickness. This is now determined by equating the nonlinear and viscous terms, so that

$$\delta = \epsilon^{-\frac{1}{4}} R^{-\frac{1}{2}} (\Omega_c \hat{\eta}_0)^{-\frac{1}{2}} L.$$

This analysis is valid provided that δ is then smaller than the width of the critical layer, of order $O(\epsilon^{\frac{1}{2}} 2\sqrt{2}\hat{\eta}_0)$. Hence, we must have a Reynolds number such that

$$R \gg (\epsilon \hat{\eta}_0^2)^{-\frac{3}{2}}$$

The critical Reynolds number thus depends on $(\epsilon a_1)^{-\frac{3}{2}}$. If the amplitude ϵa_1 is too weak, then for a fixed Reynolds number, the above condition is not realized and the azimuthal mode is presumably damped, and the vortex evolves back to the J_0 profile. This result also holds for a flow with several critical layers, when Ω_c must then be the minimum critical vorticity.

Next, following the argument of Batchelor (1956), we take the line integral of the momentum equation (44) around a closed streamline to obtain

$$\oint \nabla \times \omega \, d\mathbf{l} = \mathbf{0}.$$

This holds for all $\nu > 0$, and hence is valid in the limit $\nu \rightarrow 0$, that is, as $R \rightarrow \infty$. Evaluating this for large R , and in the curvilinear coordinate system yields

$$\oint U \frac{\partial \omega}{\partial \psi} dl = 0. \quad (47)$$

But the velocity U is independent of ψ to leading order in the boundary layer approximation. That is, we can expand U as $\mathcal{U}(l) + O(\epsilon^{-1/4} R^{-1/2})$, where $\mathcal{U}$ is the inviscid velocity outside the boundary layer. Hence, (47) can be integrated once with respect to ψ to yield,

$$\oint \mathcal{U} \omega \, dl = \text{constant}. \quad (48)$$

This expression is then evaluated around the separatrices S_1 and S_2 , as shown in figures 2 and 3. Thus,

$$\oint_{S_1} \mathcal{U} \omega \, dl = \oint_{S_2} \mathcal{U} \omega \, dl,$$

where S_1 is a closed streamline inside the pole, and S_2 is another closed streamline inside the inner flow. We then take the limit $S_1 \rightarrow S_2 \rightarrow \psi_0$.

By using the previous relations for continuity of the velocity, we thus get

$$\delta\Omega \int_{\psi_0^+} v_1^+ d\theta = \int_{\psi_0^+} (v_3^+ - v_3^{c+}) d\theta + \frac{1}{2} \Delta\Omega \int_{\psi_0^+} v_1^+ d\theta - \frac{1}{2} \Omega \int_{\psi_0^+} \frac{f_3^+(\theta) + f_3^-(\theta)}{\hat{\eta}_0 \sqrt{1 + \cos 2\theta}} d\theta - \frac{\lambda^2}{12} \Omega \hat{\eta}_0^3 \int_{\psi_0^+} \frac{d\theta}{\sqrt{1 + \cos 2\theta}}$$

Evaluating each integrand, we then obtain

$$\delta\Omega \int_{\psi_0^+} v_1^+ d\theta = -\delta\Omega \int_{\psi_0^+} v_1^+ d\theta + \frac{1}{2} \Delta\Omega \int_{\psi_0^+} v_1^+ d\theta + \frac{\lambda^2}{12} \Omega \hat{\eta}_0^3 \int_{\psi_0^+} \frac{(\cos 2\theta + \cos 4\theta)}{\sqrt{1 + \cos 2\theta}} d\theta$$

The singularities have now vanished, and finally we get the distortion in vorticity given by

$$\delta\Omega = \frac{\Delta\Omega}{4} + \frac{\hat{\eta}_0^2 \lambda^2}{24} \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos 2\theta + \cos 4\theta}{\sqrt{1 + \cos 2\theta}} d\theta}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + \cos 2\theta} d\theta} \quad (49)$$

$$\text{or} \quad \delta\Omega = -\frac{5}{36} \lambda^2 \hat{\eta}_0^2, \quad (50)$$

on evaluating the integrals and substituting for $\Delta\Omega$. The distortion is proportional to the amplitude of the asymmetric mode and is obviously less than $\Delta\Omega$ in absolute value. The vorticity jump is caused by the change of the eigenvalue λ from inside to outside the vortex (recall that $\lambda = \lambda^+ = 0$ outside, and that $\lambda = \lambda^- = \mu_1$ inside the vortex).

4.4 Removing the singularity at the intersection of the separatrices

A singularity appears at order ϵ in the inner expansion at the saddle point where the separatrices meet, that is at $\theta = \pm\pi/2$. In order to remove this, a further rescaling is needed (see figure 3 for a schematic sketch of all asymptotic regions used in this analysis of the nonlinear critical layer). We introduce a new coordinate which is of order one in the neighbourhood of the saddle points but is large at remote locations,

$$\xi = \frac{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta}{\epsilon^{\frac{1}{2}}}. \quad (51)$$

Figure 2: The tripole and a few streamlines denoted by the value of η in the case where $\epsilon = 0.1$, $\hat{\eta}_0 = 1$. The streamlines S_1 and S_2 are used to perform the vorticity distortions (cf. subsection 4.3.1). The streamlines are characterized by an increasing value of η all the more so that they are located far from the core. The separatrices possess the same value of the streamfunction; however, they have two distinct values of η

Figure 3: The tripole and its various asymptotic layers (bounded by dashed lines): COF the core outer flow, CIF the core inner flow, SPF the flow around the stagnation point, EIF the external inner flow and EOF the external outer flow. The dark area represents the zone within the separatrices (solid lines) where the critical layer scalings used for our expansions fail to model the flow, due to a strong θ -dependence.

The expansion of the inner solution in this new coordinate is

$$h = -\epsilon^{\frac{1}{2}}\hat{\eta} + \epsilon^{\frac{3}{4}}s\sqrt{\xi} - \epsilon^{\frac{3}{4}}\frac{s\hat{\eta}^3}{6\sqrt{\xi}} - \epsilon^{\frac{3}{4}}\frac{s\hat{\eta}^6}{72\sqrt{\xi}^3} + \epsilon g_2(\theta) - \epsilon\frac{1}{6}\hat{\eta}_0^2 \cos 2\theta \\ + \epsilon^{\frac{5}{4}}\frac{f_3(\theta)}{\sqrt{\xi}} + \epsilon^{\frac{5}{4}}s(1 - \frac{1}{6}\lambda^2)\frac{\hat{\eta}_0^4 \cos^2 2\theta}{\sqrt{\xi}} + \epsilon^{\frac{3}{2}}\frac{1}{6}(\xi - \frac{\hat{\eta}^3}{3}) + \epsilon^{\frac{7}{4}}s((\frac{1}{4} + \frac{\lambda^{2s}}{6})\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta)\sqrt{\xi} + O(\epsilon^2). \quad (52)$$

The new expansion of h is thus

$$\bar{h} = \epsilon^{\frac{1}{2}}\bar{h}_1(\xi, \hat{\eta}, \theta) + \epsilon^{\frac{3}{4}}\bar{h}_2(\xi, \hat{\eta}, \theta) + \epsilon\bar{h}_3(\xi, \hat{\eta}, \theta) + \epsilon^{\frac{5}{4}}\bar{h}_4(\xi, \hat{\eta}, \theta) + \epsilon^{\frac{3}{2}}\bar{h}_5(\xi, \hat{\eta}, \theta) \dots \quad (53)$$

The results obtained from this expansion (53) for the first few terms, are successively, after integrating the related differential equations,

$$\begin{aligned} \partial_\xi \bar{h}_1 &= 0, & \bar{h}_1 &= \bar{h}_1(\hat{\eta}, \theta), \\ \partial_{\xi^2} \bar{h}_2 + 2(\partial_\xi \bar{h}_2)^3 &= 0, & \bar{h}_2 &= s\sqrt{\xi + F_1(\hat{\eta}, \theta)} + G_1(\hat{\eta}, \theta), \\ \partial_{\xi^2} \bar{h}_3 + 6(\partial_\xi \bar{h}_2)^2 \partial_\xi \bar{h}_3 &= 0, & \bar{h}_3 &= \frac{F_2(\hat{\eta}, \theta)}{\sqrt{\xi + F_1(\hat{\eta}, \theta)}} + G_2(\hat{\eta}, \theta). \end{aligned}$$

The matching of $\bar{h}$ with (52) enables us to determine the functions $F_{1,2}$ and $G_{1,2}$. We find that $\bar{h}_1 = -\hat{\eta}$, $F_1(\hat{\eta}, \theta) = -1/3\hat{\eta}^3$, $G_1(\hat{\eta}, \theta) = 0$, $F_2(\hat{\eta}, \theta) = 0$ and $G_2(\hat{\eta}, \theta) = g_2(\theta) - \hat{\eta}_0^2/6 \cos 2\theta = -\hat{\eta}_0^2 \cos 2\theta$. The higher-order terms are then

$$\begin{aligned} \partial_{\xi^2} \bar{h}_4 + \frac{3}{2} \frac{\partial_\xi \bar{h}_4}{\xi - \frac{\hat{\eta}^3}{3}} &= 0, & \bar{h}_4 &= \frac{F_3^s(\hat{\eta}, \theta)}{\sqrt{\xi - \frac{\hat{\eta}^3}{3}}}, \\ \partial_{\xi^2} \bar{h}_5 + \frac{3}{2} \frac{\partial_\xi \bar{h}_5}{\xi - \frac{\hat{\eta}^3}{3}} &= \frac{1}{4(\xi - \frac{\hat{\eta}^3}{3})}, & \bar{h}_5 &= \frac{1}{6}(\xi - \frac{\hat{\eta}^3}{3}) + \frac{F_4(\hat{\eta}, \theta)}{\sqrt{\xi - \frac{\hat{\eta}^3}{3}}} + G_4(\hat{\eta}, \theta). \end{aligned}$$

The leading term of order $\epsilon^{\frac{7}{4}}$ ($\xi \rightarrow \infty$) is

$$\bar{h}_6^s = s\sqrt{\xi - \frac{\hat{\eta}^3}{3}} \left(\left(\frac{1}{4} + \frac{\lambda^{2s}}{6} \right) \hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta \right) + O(1/\sqrt{\xi}).$$

In the same way,

$$\begin{aligned} F_3^s(\hat{\eta}, \theta) &= f_3^s(\theta) + s(1 - \frac{1}{6}\lambda^{2s})\hat{\eta}_0^4 \cos^2 2\theta, \\ F_4(\hat{\eta}, \theta) &= 0, \\ G_4(\hat{\eta}, \theta) &= 0. \end{aligned}$$

We note that ξ is always associated with $-\hat{\eta}^3/3$, and so the true coordinate is $\xi - \hat{\eta}^3/3$ which vanishes at $\theta = \pm\pi/2$ on ψ_0 at the saddle points.

To make $F_3^s(\hat{\eta}, \theta)/\sqrt{\xi - \hat{\eta}^3/3}$ regular at these points gives two conditions on $f_3^s(\theta)$ and thus determining a_2 and b_2 :

$$a_2 = -\frac{5}{8} + \frac{\lambda^2}{24} - \frac{\xi_{2,4}}{4}(\lambda^2 - 24), \quad (54)$$

$$b_2 = \frac{5}{8} + \frac{1}{4}\zeta_{2,4}. \quad (55)$$

At this point we recall that we can put $a_2 = b_2 = 0$ by redefining ϵ . In this case (54,55) determine the integration constants $\xi_{2,4}, \zeta_{2,4}$, thus providing a closure to the expansion at the second order.

We can now check that $\bar{h}_4(\psi_0) = O(\theta \mp \pi/2)$ in the vicinity of $\pm\pi/2$. The presence of mode 2 at the $O(\epsilon^2)$ is a feedback of the critical layer flow onto the outer flow. A uniformly valid expansion describing the streamlines in the critical layer outside the two satellites is obtained by superposing the plain and improved inner expansions while removing the intermediary zones between the layers EIF and SPF or CIF and SPF (cf. figure 3)

$$h = \epsilon^{\frac{1}{2}}\bar{h}_1 + \epsilon^{\frac{3}{4}}\bar{h}_2 + \epsilon\bar{h}_3 + \epsilon^{\frac{5}{4}}\bar{h}_4 + \epsilon^{\frac{3}{2}}\bar{h}_5 + \epsilon^{\frac{3}{2}}s((\frac{1}{4} + \frac{\lambda^{2s}}{6})\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta)\sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}}\frac{\hat{\eta}^3}{3}} \\ - \epsilon^{\frac{3}{2}}\frac{s}{2}(\lambda^{2s} + \frac{13}{12})\sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}}\frac{\hat{\eta}^3}{3}} + o(\epsilon^{\frac{3}{2}}). \quad (56)$$

Indeed, $h \rightarrow \hat{h}$ for $\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta$ fixed and $\epsilon \rightarrow 0$, that is $\xi \rightarrow \infty$, while $h \rightarrow \bar{h}$ for ξ fixed and $\epsilon \rightarrow 0$, that is $\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta \rightarrow 0$. As the separatrices are not defined any longer by $\hat{\eta}^2 = \hat{\eta}_0^2$ but by $\eta_0 = 1 + s\epsilon^{\frac{1}{2}}\hat{\eta}_0 + 1/6\epsilon\hat{\eta}_0^2 + 5/72s\epsilon^{\frac{3}{2}}\hat{\eta}_0^3 + \dots$, it is necessary to introduce small corrections in the above square roots. The saddle point is located at $r = 1 + \epsilon\hat{\eta}_0^2$.

The azimuthal velocity is expanded as $\bar{v} = \epsilon^{\frac{3}{4}}\bar{v}_1 + \epsilon\bar{v}_2 + \epsilon^{\frac{5}{4}}\bar{v}_3 + \dots$, giving

$$\begin{aligned}\bar{v}_1 &= s\Omega \sqrt{\xi - \frac{\hat{\eta}^3}{3}}, \quad \bar{v}_2 = 0, \\ \bar{v}_3 &= \frac{\Omega F_3^s(\theta)}{\sqrt{\xi - \frac{\hat{\eta}^3}{3}}}, \quad \bar{v}_4 = -\frac{\Omega}{3}(\xi - \frac{\hat{\eta}^3}{3}) \\ \text{and} \quad \bar{v}_5 &= s\Omega \sqrt{\xi - \frac{\hat{\eta}^3}{3}} \left(\frac{1}{2} - \frac{\lambda^2}{6} \right) \hat{\eta}^2 + s \frac{\Omega F_3^{s2}(\theta)}{\sqrt{\xi - \frac{\hat{\eta}^3}{3}}}.\end{aligned}$$

We can easily check that the velocity is zero at the saddle points. We can do the same remark for the radial velocity. In the same way, a uniformly valid expansion of the azimuthal velocity is

$$\begin{aligned}v &= \epsilon^{\frac{3}{4}}\bar{v}_1 + \epsilon^{\frac{5}{4}}\bar{v}_3 + \epsilon^{\frac{3}{2}}\bar{v}_4 \\ &+ \epsilon^{\frac{3}{2}} \frac{s\Omega}{72} \left((29 - 12\lambda^2) \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3}} + 6\hat{\eta}_0^2(4\lambda^2 - 15) \cos 2\theta \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3}} \right). \quad (57)\end{aligned}$$

The expansion for the streamlines inside the separatrices is not changed, but the matching must now be done on the new separatrices. The relationships concerning the Fourier coefficients obtained by the condition of continuity of h are identical to these previously obtained by the previous inner expansion. However, the improved inner expansion now gives for the velocity on ψ_0 :

$$\begin{aligned}\frac{v^s(\psi_0)}{\Omega} &= \epsilon^{\frac{1}{2}} s \sqrt{2} \hat{\eta}_0 |\cos \theta| - \epsilon \frac{2}{3} \hat{\eta}_0^2 \cos^2 \theta + \epsilon^{\frac{3}{2}} \frac{F_3^s(\theta)}{\hat{\eta}_0 \sqrt{1 + \cos 2\theta}} \\ &+ \epsilon^{\frac{3}{2}} \frac{s\hat{\eta}_0^3}{72} \left((29 - 12\lambda^2) \sqrt{1 + \cos 2\theta}^3 + 6(4\lambda^2 - 15) \cos 2\theta \sqrt{1 + \cos 2\theta} \right). \quad (58)\end{aligned}$$

The two first orders can be matched by the expansion inside the separatrices, but at order $\epsilon^{\frac{3}{2}}$, we have

$$\begin{aligned}\frac{v_3^{cs}(\psi_0)}{\Omega} &= s\Delta\Omega \hat{\eta}_0 \sqrt{1 + \cos 2\theta} \\ &+ \frac{s\hat{\eta}_0^3}{\sqrt{1 + \cos 2\theta}} \left(\frac{7}{24} - \frac{C_3}{2\hat{\eta}_0^4} - \frac{4}{9} \cos 2\theta - \frac{1}{9} \cos 4\theta - \frac{1}{2} \sum_{p=1}^{\infty} \frac{A_{p,3} + B_{p,3}}{\hat{\eta}_0^4} \cos 2p\theta \right) \quad (59)\end{aligned}$$

which cannot be matched with Eq. (58). Although both of these velocities are regular at the saddle points, they lead to the same distortion relationships as in (A1, A2). The singularities present in (42) and (43) are thus identical and cancel in the distortion, which enables us to determine the vorticity jump (50) by using the original inner expansion.

4.4.1 New determination of the vorticity ω_0

Now, we have a vorticity distortion scaling with ϵ which can be evaluated from the equation (48),

$$\begin{aligned}\Delta\Omega \int_{\psi_0^+} v_1^+ d\theta &= \int_{\psi_0^+} (v_3^+ - v_3^{c+}) d\theta + \frac{1}{2} \Delta\Omega \int_{\psi_0^+} v_1^+ d\theta - \frac{\lambda^2}{12} \hat{\eta}_0^3 \int_{\psi_0^+} \sqrt{1 + \cos 2\theta} (1 - \cos 2\theta) d\theta \\ &- \frac{\hat{\eta}_0^3 \lambda^2}{12} \Omega \int_{\psi_0^+} \frac{\cos^2 2\theta}{\sqrt{1 + \cos 2\theta}} d\theta - \frac{1}{2} \Omega \int_{\psi_0^+} \frac{f_3^+(\theta) + f_3^-(\theta)}{\hat{\eta}_0 \sqrt{1 + \cos 2\theta}} d\theta.\end{aligned}$$

There is much simplification, and we have

$$\begin{aligned}\Delta\Omega &= \frac{\Delta\Omega}{4} + \frac{\lambda^2 \hat{\eta}_0^2}{24} \frac{\int_{-\pi/2}^{\pi/2} \sqrt{1 + \cos 2\theta} (2 \cos 2\theta - 1) d\theta}{\int_{-\pi/2}^{\pi/2} \sqrt{1 + \cos 2\theta} d\theta} \\ \text{or} \quad \Delta\Omega &= -\frac{5}{36} \lambda^2 \hat{\eta}_0^2.\end{aligned} \quad (60)$$

This distortion is the same as the previous one in (50). Thus the use of the new coordinate (51) has allowed us to suppress the singularities in our expansions, but has had no effect on the velocity and

4.4.2 Uniformly valid asymptotic expansion

An expansion uniformly valid everywhere except within the satellites is obtained, in the core of the vortex, by adding the outer and inner expansions and subtracting the intermediary region CIF/COF

$$\begin{aligned}
h = & \epsilon h_1^- + \epsilon^2 h_2^- + \epsilon^{\frac{1}{2}} \bar{h}_1^- + \epsilon^{\frac{3}{4}} \bar{h}_2^- + \epsilon \bar{h}_3^- + \epsilon^{\frac{5}{4}} \bar{h}_4^- + \epsilon^{\frac{3}{2}} \bar{h}_5^- \\
& - \epsilon^{\frac{3}{2}} \left(\left(\frac{1}{4} + \frac{\lambda^2}{6} \right) \hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta \right) \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3} + \epsilon^{\frac{3}{2}} \frac{1}{6} \left(\lambda^2 + \frac{13}{12} \right) \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3}}}^3 \\
& + \epsilon^{\frac{1}{2}} \frac{a_1}{\hat{\eta}} \left(1 - \epsilon^{\frac{1}{2}} \frac{3}{2} \hat{\eta} + \epsilon \left(\frac{7}{4} - \frac{1}{3} \lambda^2 \right) \hat{\eta}^2 \right) \cos 2\theta + \epsilon^{\frac{1}{2}} \frac{a_1^2}{4\hat{\eta}^3} \left(1 + \epsilon^{\frac{1}{2}} \frac{1}{2} \hat{\eta} + \epsilon (\lambda^2 - 8) \hat{\eta}^2 \right) \\
& + \epsilon^{\frac{1}{2}} \frac{a_1^2}{4\hat{\eta}^3} \left(1 + \epsilon^{\frac{1}{2}} \frac{1}{2} \hat{\eta} - \epsilon 4(\lambda^2 - 24) \xi_{2,4} \hat{\eta}^2 + \epsilon (\lambda^2 - 8) \hat{\eta}^2 \right) \cos 4\theta + \epsilon^{\frac{3}{2}} a_2 \frac{\hat{\eta}_0^4}{\hat{\eta}} \cos 2\theta + o(\epsilon^{\frac{3}{2}}).
\end{aligned}$$

In the same way, outside the vortex, we have the final expansion

$$\begin{aligned}
h = & \epsilon h_1^+ + \epsilon^2 h_2^+ + \epsilon^{\frac{1}{2}} \bar{h}_1^+ + \epsilon^{\frac{3}{4}} \bar{h}_2^+ + \epsilon \bar{h}_3^+ + \epsilon^{\frac{5}{4}} \bar{h}_4^+ + \epsilon^{\frac{3}{2}} \bar{h}_5^+ \\
& + \epsilon^{\frac{3}{2}} \left(\frac{1}{4} \hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta \right) \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3} - \epsilon^{\frac{3}{2}} \frac{13}{72} \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3}}}^3 \\
& + \epsilon^{\frac{1}{2}} \frac{a_1}{\hat{\eta}} \left(1 - \epsilon^{\frac{1}{2}} \frac{3}{2} \hat{\eta} + \epsilon \frac{7}{4} \hat{\eta}^2 \right) \cos 2\theta + \epsilon^{\frac{1}{2}} \frac{a_1^2}{4\hat{\eta}^3} \left(1 + \epsilon^{\frac{1}{2}} \frac{1}{2} \hat{\eta} - \epsilon 8 \hat{\eta}^2 \right) \\
& + \epsilon^{\frac{1}{2}} \frac{a_1^2}{4\hat{\eta}^3} \left(1 + \epsilon^{\frac{1}{2}} \frac{1}{2} \hat{\eta} \epsilon^2 - \epsilon 4(2 + \zeta_{2,4}) \hat{\eta}^2 \right) \cos 4\theta - \epsilon^{\frac{3}{2}} b_2 \frac{\hat{\eta}_0^4}{\hat{\eta}} \cos 2\theta + o(\epsilon^{\frac{3}{2}}).
\end{aligned}$$

5 Mean flow adjustment

When we consider the generation of the steady solution described in the previous sections, we expect the vorticity distortion may be cancelled by the generation of an $O(\epsilon^{1/2})$ critical-layer induced disturbance (e.g. Haberman 1972). Here, however, as $\omega'_c = 0$, this induced flow has only an order- ϵ vorticity distortion. Hence, we now eliminate the leading-order vorticity distortion by adjusting the mean flow, as follows. We let,

$$V(\eta) = V_0(\eta) + \Omega(\epsilon^{\frac{1}{2}} V_1(\eta) + \epsilon V_2(\eta) + \dots), \quad (61)$$

where $V_0(\eta)$ is the mean flow given by (15), with of course here $J_1(\lambda) = 0$. On expanding around $\eta_c = 1$, with $z = \eta - \eta_c$ as before,

$$V_1^s(\eta) = V_{1c}' z + \frac{1}{2} V_{1c}'' z^2 + \dots$$

with a similar expression for $V_2(\eta)$, while the corresponding vorticity is given by

$$\omega_1^s(\eta) = \Omega(V_{1c}'^s + \omega_{1c}'^s z + \dots),$$

where $\omega_{1c}' = V_{1c}' + V_{1c}''$ and the superscript “ $s = \pm$ ” as before. We anticipate that we shall need to set $\omega_{1c}^{'+} = 0$ so that $V_{1c}''^{+} = -V_{1c}'^{+}$. Continuity of the vorticity then implies that

$$V_{1c}^{'+} = V_{1c}'^{-} \quad \text{at} \quad O(\epsilon^{\frac{1}{2}}),$$

$$V_{2c}^{'+} - V_{2c}'^{-} = \Delta\Omega - \omega_{1c}'^{-} \hat{\eta}_0, \quad \text{at} \quad O(\epsilon).$$

Consequently, the vorticity within the poles becomes equal, at the leading order, to the vorticity outside the vortex which is: $\Omega(1 + \epsilon^{\frac{1}{2}} V_{1c}'^{+} + \epsilon V_{2c}'^{+})$. The outer flow perturbative expansion now becomes:

$$h = \epsilon h_1 + \epsilon^{\frac{3}{2}} h_{\frac{3}{2}} + \epsilon^2 h_2 + \epsilon^{\frac{5}{2}} h_{\frac{5}{2}} + \dots$$

The Frobenius expansion of the $O(\epsilon^{\frac{3}{2}})$ new term is

$$h_{\frac{3}{2}} = \hat{\eta}_0^2 \left(\frac{1}{2} \omega_{1c}' \ln |z| + a_{\frac{3}{2},1} + a_{\frac{3}{2},2} z \ln |z| + a_{\frac{3}{2},3} z + \dots \right) \cos 2\theta,$$

while for h_2 we get for the $H_{2,2} \cos 2\theta$ -component

$$H_{2,2} = \hat{\eta}_0^2 \left(\frac{1}{2} \omega_{1c}' - V_{1c}'^{+} \right) \ln |z| + a_{\frac{3}{2},1} + a_{\frac{3}{2},2} z \ln |z| + a_{\frac{3}{2},3} z + \dots \cos 2\theta$$

The previous inner expansion is now modified at $O(\epsilon^{\frac{3}{2}})$,

$$h = \epsilon^{\frac{1}{2}} \hat{h}_1 + \epsilon \hat{h}_2 + \epsilon^{\frac{3}{2}} \ln \epsilon \frac{1}{4} \hat{\eta}_0^2 \omega'_{1c} \cos 2\theta + \epsilon^{\frac{3}{2}} \hat{h}_3 + \dots$$

with

$$\hat{h}_3 = \frac{1}{2} \hat{\eta}_0^2 \cos 2\theta \left(\ln \left[\frac{|\hat{\eta}| + \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta}}{2\hat{\eta}_0} \right] - \frac{s\hat{\eta}}{\sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta}} \right) \omega'_{1c} + g_3(\theta).$$

Matching $\hat{h}_3$ with $h_{\frac{3}{2}}$ and $h_{\frac{5}{2}}$ as $\hat{\eta} \rightarrow \infty$, we get $a_{\frac{3}{2},1} = -\frac{1}{2} \ln \hat{\eta}_0 \omega'_{1c}$, $g_3(\theta) = \frac{1}{2} \hat{\eta}_0^2 \omega'_{1c} \cos 2\theta$. The improved inner expansion is modified to contain,

$$\bar{h}_5 = \frac{1}{2} \hat{\eta}_0^2 \cos 2\theta \omega'_{1c} \ln \frac{|\hat{\eta}|}{\hat{\eta}_0} + g_3(\theta),$$

$F_3^s(\theta)$ is now given by

$$F_3^s(\theta) = f_3^s(\theta) + s(1 - \frac{\mu^2}{6}) \frac{\hat{\eta}_0^4}{\hat{\eta}_c^2} \cos^2 2\theta + s \frac{\hat{\eta}_0^4}{2\hat{\eta}} \cos^2 2\theta \omega'_{1c},$$

and the regularity of $\bar{h}_5$ yields the amplitude of the homogeneous part of $H_{2,2}$:

$$a_2 = -\frac{5}{8} + \frac{\lambda^2}{24} - \frac{\xi_{2,4}}{4} (\lambda^2 - 24) + \frac{\omega'_{1c}}{2\hat{\eta}_0}.$$

The uniformly valid expansion in the critical layer is now,

$$\begin{aligned} h = & \epsilon^{\frac{1}{2}} \bar{h}_1 + \epsilon^{\frac{3}{4}} \bar{h}_2 + \epsilon \bar{h}_3 + \epsilon^{\frac{5}{4}} \bar{h}_4 + \epsilon^{\frac{3}{2}} \ln \epsilon \frac{1}{4} \hat{\eta}_0^2 \omega'_{1c} \cos 2\theta + \epsilon^{\frac{3}{2}} \bar{h}_5 \\ & + \epsilon^{\frac{3}{2}} s \left(\left(\frac{1}{4} + \frac{\mu_1^2}{6} \right) \hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta \right) \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3}} - \epsilon^{\frac{3}{2}} \frac{s}{6\hat{\eta}_c^2} (\mu_1^2 + \frac{13}{12}) \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3}} \\ & + \epsilon^{\frac{3}{2}} \frac{1}{2} \left(\ln \left[\frac{|\hat{\eta}| + \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3}}}{2\hat{\eta}_0} \right] - \ln \frac{|\hat{\eta}|}{\hat{\eta}_0} - \frac{s}{\hat{\eta}} \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3}} \right) \hat{\eta}_0^2 \cos 2\theta \omega'_{1c} + o(\epsilon^{\frac{3}{2}}), \end{aligned} \quad (62)$$

while the velocity is given by,

$$\begin{aligned} v = & \epsilon^{\frac{3}{4}} \bar{v}_1 + \epsilon^{\frac{5}{4}} \bar{v}_3 + \epsilon^{\frac{3}{2}} \bar{v}_4 \\ & + \epsilon^{\frac{3}{2}} \frac{s\Omega}{72} (29 - 12\mu_1^2) \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3}} + \epsilon^{\frac{3}{2}} s\Omega \hat{\eta}_0^2 \left(\frac{\mu_1^2}{3} - \frac{5}{4} \right) \cos 2\theta \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3}} \\ & - \epsilon^{\frac{3}{2}} \frac{1}{3} \Omega \hat{\eta}_0^2 V'_{1c} \cos 2\theta \frac{\sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3}}}{\sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta} + |\hat{\eta}|} + \epsilon^{\frac{3}{2}} \frac{s\Omega}{2\hat{\eta}} \omega'_{1c} (\hat{\eta}^2 - \hat{\eta}_0^2 \cos 2\theta) \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3}} \\ & - \epsilon^{\frac{3}{2}} \frac{1}{3} s\hat{\eta} \Omega V'_{1c} \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3}} + \epsilon^{\frac{3}{2}} s\Omega V'_{2c} \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3}} \end{aligned} \quad (63)$$

with $\bar{v}_3$ containing $s\Omega V'_{1c} \sqrt{\xi - \frac{\hat{\eta}^3}{3}}$. The velocity on the separatrix ψ_0 is modified to become

$$\begin{aligned} v_2^c &= s\hat{\eta}_0 \Omega V'_{1c} \sqrt{1 + \cos 2\theta}, \\ v_3^c &= -\frac{\hat{\eta}_0^2}{3} \Omega V'_{1c} (1 + \cos 2\theta) + s\Omega \delta\Omega \hat{\eta}_0 \sqrt{1 + \cos 2\theta}, \end{aligned}$$

here $\delta\Omega = V'_{2c} + \Delta\Omega - \omega'_{1c} \hat{\eta}_0 + V'_{2c}$.

In order to eliminate the leading order vorticity distortion, we must again use the expression (48) which now leads to

$$\begin{aligned} \int_{\psi_0^+} (v_3^+ - v_3^{c+}) d\theta &= \frac{\lambda^2}{12} \hat{\eta}_0^3 \Omega \int_{\psi_0^+} \sqrt{1 + \cos 2\theta} (1 - \cos 2\theta) d\theta + \left(\frac{\hat{\eta}_0^3 \lambda^2}{12} + \frac{\hat{\eta}_0^2 \omega'_{1c}}{4} \right) \Omega \int_{\psi_0^+} \frac{\cos^2 2\theta}{\sqrt{1 + \cos 2\theta}} d\theta \\ &+ \frac{1}{2} \Omega \int_{\psi_0^+} \frac{f_3^+(\theta) + f_3^-(\theta)}{\hat{\eta}_0 \sqrt{1 + \cos 2\theta}} d\theta + \frac{1}{2} (V'_{2c} + - V'_{2c}) \int_{\psi_0^+} v_1^+ d\theta \\ &- \frac{1}{2} \Omega \hat{\eta}_0^2 V'_{1c} \int_{\psi_0^+} 1 + \cos 2\theta d\theta + \frac{1}{2} \Omega \hat{\eta}_0^2 \omega'_{1c} \int_{\psi_0^+} (1 - \cos 2\theta) \sqrt{1 + \cos 2\theta} d\theta \end{aligned} \quad (64)$$

Figure 4: The tripole: $\hat{\eta}_0 = 1$, $\Omega = -0.4028$, $\epsilon = 0.1$ and $R_c \simeq 10$: (a) the streamlines; (b) vorticity $\omega(r)$ at $\theta = 0$, $\theta = \pi/2$ shown by the — curves. The $-\cdot-\cdot-$ curves describe the inner flow obtained by adding a distorted mean flow, $V'_{1,c} = -0.2$, $\omega'_{1,c} = -9.479$, $\omega'_2 = V'_{2,c} = -2.138$, $V'_{2,c} = 0$. The curve $\theta = \pi/2$ reaches the vortex boundary at $r = 0.973$. If the curve $\theta = 0$ is slightly modified, then on the contrary the curve $\theta = \pi/2$ shows a strong reduction of the vortex area. The curve $\cdots \omega(r) = J_0(\lambda r)$.

Finally, evaluating all terms, we get the desired expression for the vorticity gradient ω'_{1c}

$$\omega'_{1c} = -\frac{8}{9} \frac{\sqrt{2}\lambda^2\hat{\eta}_0}{2\sqrt{2} - \ln[1 + \sqrt{2}]} \quad (65)$$

Since ω'_{1c} is not zero, $V_1(\eta)$ cannot be proportional to $J_1(\lambda\eta)$, and so now the relationship between the vorticity and the streamfunction is no longer linear, and instead is weakly nonlinear. Indeed, now

$$\begin{aligned} \omega = -\lambda^2\psi + \epsilon^{\frac{1}{2}} \Big(\Omega V'_{1c} - \text{sgn}[\Omega]\omega'_{1c} \sqrt{2\Omega(\psi + \Omega/\lambda^2)} + v(\psi + \Omega/\lambda^2) + \dots \Big) \\ + \epsilon \Big(\text{sgn}[\Omega] \left(\frac{1}{2} V'_{1c} \omega'_{1c} - \omega'_2 \right) \sqrt{2\Omega(\psi + \Omega/\lambda^2)} + \dots \Big) \end{aligned} \quad (66)$$

with $v = \omega'_{1c}/3 + V''_{1c} + (\lambda^2 - 2)V'_{1c} + V'''_{1c}$.

6 Multipolar Vortices

6.1 The plain inner expansion

We now consider the general case, when $n \geq 2$ and there may be more than one critical level in $\eta \leq 1$. As noted earlier, such vortices have at least one inflexion point and hence may be unstable. Hence, we shall give only an outline of their construction, which follows the same procedure described in detail above for the case $n = 2$. Let us then consider the general case where there are n_c critical levels, $\eta_c = \eta_{c,i}$, $i = 1, \dots, n_c$, in the flow inside the vortex. Let D_i , $i = 1, \dots, n_c + 1$, be the domain $\eta_{c,i-1} < \eta < \eta_{c,i}$, where by convention, $\eta_{c,0} = 0$ and $\eta_{c,n_c+1} = 1$. We retain the possibility that $\eta_c = 1$ may also be a critical level.

The first term of the inner expansion is identical to (34) but with

$$f_1^s(\theta) = 2a_{1,j} \{ (J_n(\mu_i) + \kappa_j Y_n(\mu_i)) / J_0(\mu_i) \} \cos n\theta$$

with the notation, $j = i$ ($s = -1$) or $j = i + 1$ ($s = 1$). The expansions of the outer solution with the inner variable $\hat{\eta}$ are given in Appendix B. Assuming that the critical points are located at $\theta = \pi/n[2\pi/n]$, we then have

$$\hat{\eta}_0^s = s_1 \sqrt{2a_{1,j} \frac{J_n(\mu_i) + \kappa_j Y_n(\mu_i)}{J_0(\mu_i)}} \quad (67)$$

The vorticity on each separatrix $\eta_0 = \eta_c + \epsilon^{\frac{1}{2}} s \hat{\eta}_0 + \epsilon^{\frac{1}{6}} \hat{\eta}_0^2 / \eta_c + \epsilon^{\frac{3}{2}} s 5 / 72 \hat{\eta}_0^3 / \eta_c^2$ is

$$\omega_0 = \Omega_c (1 - \epsilon \lambda^2 \frac{\hat{\eta}_0^2}{2} + \epsilon^2 \lambda^4 (\frac{1}{3} - \frac{1}{\mu_i^2}) \hat{\eta}_0^4 + O(\epsilon^{\frac{5}{2}})).$$

Note that there is no $O(\epsilon^{\frac{3}{2}})$ term here. If $\eta_c \neq 1$, there is no vorticity jump through the critical layer up to $O(\epsilon^{\frac{5}{2}})$. The second order solution is

$$\hat{h}_2 = \frac{\hat{\eta}^2}{6\eta_c} - \frac{s\hat{\eta}^3}{6\eta_c \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos n\theta}} + g_2(\theta) \quad (68)$$

with $g_2^s(\theta) = (\rho_{n-1}^s + 1/3 - n) (\hat{\eta}_0^2 / 2\eta_c) \cos n\theta$ and $\rho_k^s = \mu_i (J_k(\mu_i) + \kappa_j Y_k(\mu_i)) / J_n(\mu_i)$. The third order solution is given by

$$\begin{aligned} \hat{h}_3 = & -\frac{\hat{\eta}^3}{18\eta_c^2} - \frac{s\hat{\eta}^6}{72\eta_c^2 \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos n\theta}^3} + \left(f_3^s(\theta) + s \frac{\hat{\eta}_0^4}{\eta_c^2} \left(\frac{n^2}{4} - \frac{\mu_i^2}{6} \right) \cos^2 n\theta \right) \frac{1}{\sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos n\theta}} \\ & + \epsilon^{\frac{3}{2}} s \left(\left(\frac{1}{4} + \frac{\mu_i^2}{6} \right) \frac{\hat{\eta}^2}{\eta_c^2} + \frac{n^2}{4} \frac{\hat{\eta}_0^2}{\eta_c^2} \cos n\theta \right) \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos n\theta} - \epsilon^{\frac{3}{2}} \frac{s}{6\eta_c^2} \left(\mu_i^2 + \frac{13}{12} \right) \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos n\theta}^3. \end{aligned} \quad (69)$$

Here $f_3^s(\theta)$ is given in Appendix B.

6.2 Improved inner expansion

The new coordinate is now

$$\xi = \frac{\hat{\eta}^2 + \hat{\eta}_0^2 \cos n\theta}{\epsilon^{\frac{1}{2}}}. \quad (70)$$

The expansion of the perturbation around the critical point is identical to (53)

$$\bar{h} = -\epsilon^{\frac{1}{2}} \hat{\eta} + \epsilon^{\frac{3}{4}} s \sqrt{\xi - \frac{\hat{\eta}^3}{3\eta_c}} + \epsilon G_2(\theta) + \epsilon^{\frac{5}{4}} s \frac{F_3^s(\theta)}{\sqrt{\xi - \frac{\hat{\eta}^3}{3\eta_c}}} + \frac{\epsilon^{\frac{3}{2}}}{6\eta_c} \left(\xi - \frac{\hat{\eta}^3}{3\eta_c} \right) + o(\epsilon^{\frac{3}{2}}), \quad (71)$$

the unknown functions being determined by matching with 34, 68 and 69 for $\xi \rightarrow \infty$:

$$\begin{aligned} G_2^s(\theta) &= g_2^s(\theta) - \frac{\hat{\eta}_0^2}{6\eta_c} \cos n\theta, \\ F_3^s(\theta) &= f_3^s(\theta) + s \left(\frac{n^2}{4} - \frac{\mu_i^2}{6} \right) \frac{\hat{\eta}_0^4}{\eta_c^2} \cos^2 n\theta. \end{aligned}$$

The velocity is in this zone

$$\bar{v} = \epsilon^{\frac{3}{4}} s \Omega_c \sqrt{\xi - \frac{\hat{\eta}^3}{3\eta_c}} + \epsilon^{\frac{5}{4}} \Omega_c \frac{F_3^s(\theta)}{\sqrt{\xi - \frac{\hat{\eta}^3}{3\eta_c}}} - \epsilon^{\frac{3}{2}} \frac{\Omega_c}{3\eta_c} \left(\xi - \frac{\hat{\eta}^3}{3\eta_c} \right). \quad (72)$$

Uniformly valid expansions describing the open streamlines in the critical layer are,

$$\begin{aligned} h &= \epsilon^{\frac{1}{2}} \bar{h}_1 + \epsilon^{\frac{3}{4}} \bar{h}_2 + \epsilon \bar{h}_3 + \epsilon^{\frac{5}{4}} \bar{h}_4 + \epsilon^{\frac{3}{2}} \bar{h}_5 + \epsilon^{\frac{3}{2}} s \left(\left(\frac{1}{4} + \frac{\mu_i^2}{6} \right) \frac{\hat{\eta}^2}{\eta_c^2} + \frac{n^2}{4} \frac{\hat{\eta}_0^2}{\eta_c^2} \cos n\theta \right) \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos n\theta} - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3\eta_c} \\ &\quad - \epsilon^{\frac{3}{2}} \frac{s}{6\eta_c^2} \left(\mu_i^2 + \frac{13}{12} \right) \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos n\theta} - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3\eta_c} + o(\epsilon^{\frac{3}{2}}), \\ v &= \epsilon^{\frac{3}{4}} \bar{v}_1 + \epsilon^{\frac{5}{4}} \bar{v}_3 + \epsilon^{\frac{3}{2}} \bar{v}_4 + \epsilon^{\frac{3}{2}} \frac{s \Omega_c}{72\eta_c^2} (29 - 12\mu_i^2) \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos n\theta} - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3\eta_c} \\ &\quad + \epsilon^{\frac{3}{2}} s \Omega_c \frac{\hat{\eta}_0^2}{\eta_c^2} \left(\frac{\mu_i^2}{3} - \frac{1+n^2}{4} \right) \cos n\theta \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos n\theta} - \epsilon^{\frac{1}{2}} \frac{\hat{\eta}^3}{3\eta_c} + o(\epsilon^{\frac{3}{2}}). \end{aligned} \quad (74)$$

6.3 Expansion inside the poles

The streamfunction of a π/n -periodic flow possessing a constant vorticity ω_0 is given by

$$\psi(r) = \frac{\omega_0}{4} \left(r^2 - 2\eta_c^2 \ln \frac{r}{\eta_c} + 2C + \sum_{p=1}^{\infty} A_p \left(\left(\frac{r}{\eta_c} \right)^{np} + \left(\frac{r}{\eta_c} \right)^{-np} \right) \cos np\theta + 2B_p \left(\frac{r}{\eta_c} \right)^{-np} \cos np\theta \right). \quad (75)$$

But ψ is also given around $\eta = \eta_c$ by

$$\psi(r) = \frac{\omega_0}{4} \left(\eta_c^2 + 2\epsilon \hat{\eta}^2 - \epsilon^{\frac{3}{2}} \frac{2}{3} \frac{\hat{\eta}^3}{\eta_c} + \epsilon^2 \frac{1}{2} \frac{\hat{\eta}^4}{\eta_c^2} + O(\epsilon^{\frac{5}{2}}) \right). \quad (76)$$

Equating (75) and (76) with the same expansion as in subsection (??) yields the equality of $\hat{h}_1^c$ (34) on ψ_0 , with analogous conclusions to those of subsection ???. Next, equality of $\hat{h}_2^c$ and (68) on ψ_0 holds provided that $g_2^-(\theta) = g_2^+(\theta)$. In turn, this then finally implies that $\kappa^+ = \kappa^-$. Since $\kappa_1 = 0$, it then follows that $\kappa_i = 0$, $i = 2, n_c + 1$, and all the functions Y_n are finally discarded. The dispersion relation becomes: $J_{n-1}(\lambda) = 0$ and so only the tripole can have a critical level at $\eta = 1$.

We also have $B_{1,1} = ((\rho_{n-1}/n) - 1)\hat{\eta}_0^2$, $A_{1,1} = -(\rho_{n-1}/n)\hat{\eta}_0^2$, $A_{p,1} = B_{p,1} = 0$, $p > 1$. At the next order, $\hat{h}_3^c$ is given by

$$\begin{aligned} \hat{h}_3^c = & -\frac{\hat{\eta}^3}{18\eta_c^2} - \frac{s\hat{\eta}^6}{72\eta_c^2 \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos n\theta}} + \frac{5s}{72\eta_c^2} \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos 2\theta} \\ & + s \frac{\hat{\eta}_0^2}{4\eta_c^2} (n^2 - 1) \cos n\theta \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos n\theta} + \frac{s\hat{\eta}_0^4 \cos^2 n\theta}{8\eta_c^2 \sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos n\theta}} (1 + (n - \rho_{n-1})^2) \\ & - \frac{s}{2\sqrt{\hat{\eta}^2 + \hat{\eta}_0^2 \cos n\theta}} \left(\sum_{p=1}^{\infty} (A_{p,3} + B_{p,3}) \cos np\theta + C_3 \right) + \sum_{p=1}^{\infty} np \frac{B_{p,2}}{2\eta_c} \cos np\theta. \end{aligned}$$

After matching with (73), the coefficients are found and given in the Appendix B. Matching of the velocity is then satisfied on ψ_0 up to $O(\epsilon)$.

However, a velocity distortion appears at $O(\epsilon^{\frac{3}{2}})$, where the velocity on the separatrix is

$$\begin{aligned} \frac{\bar{v}_3^{cs}}{\Omega_c} = & s\Delta\Omega\hat{\eta}_0\sqrt{1+\cos n\theta} - \frac{s}{2\hat{\eta}_0\sqrt{1+\cos n\theta}} \left(C_3 + \sum_{p=1}^{\infty} (A_{p,3} + B_{p,3}) \cos np\theta \right) \\ & + \frac{s\hat{\eta}_0^3}{72\eta_c^2 \sqrt{1+\cos n\theta}} \left(39 - 9n^2 + \frac{9}{2}(n - \rho_{n-1})^2 + 2(20 - 9n^2) \cos n\theta + \frac{3}{2}(10 - 9n^2 + \frac{9}{2}(n - \rho_{n-1})^2) \cos 2n\theta \right), \end{aligned} \quad (77)$$

while the related velocity outside the closed streamlines is

$$\frac{v_3^s}{\Omega_c} = \frac{F_3^s(\theta)}{\hat{\eta}_0\sqrt{1+\cos n\theta}} + \frac{s\hat{\eta}_0^3}{72\eta_c^2} (29 - 12\mu_i^2) \sqrt{1+\cos n\theta} + s \frac{\hat{\eta}_0^3}{\eta_c^2} \left(\frac{\mu_i^2}{3} - \frac{1+n^2}{4} \right) \cos n\theta \sqrt{1+\cos n\theta}. \quad (78)$$

As for the previous case considered in subsection 4.3, a complete matching of the velocity cannot be obtained.

6.4 Vorticity in the poles

Now, the relation (48) comes down to a simpler expression which yields the leading order vorticity distortion

$$\Delta\Omega \int_{\psi_0^+} v_1^+ d\theta = \int_{\psi_0^+} (v_3^+ - v_3^{c+}) d\theta + \Delta\Omega \int_{\psi_0^+} v_1^+ d\theta - \frac{1}{2}\Omega_c \int_{\psi_0^+} \frac{f_3^+(\theta) + f_3^-(\theta)}{\hat{\eta}_0\sqrt{1+\cos n\theta}} d\theta,$$

and using the relationships (77) and (78) we get,

$$\begin{aligned} \Delta\Omega = & \frac{\Delta\Omega}{2} + \frac{\hat{\eta}_0^2 \lambda^2}{12} \frac{\int_{-\pi/n}^{\pi/n} \frac{\cos n\theta + \cos 2n\theta}{\sqrt{1+\cos n\theta}} d\theta}{\int_{-\pi/n}^{\pi/n} \sqrt{1+\cos n\theta} d\theta}, \\ \text{or} \quad \Delta\Omega = & -\frac{5}{18} \lambda^2 \hat{\eta}_0^2. \end{aligned}$$

$\Delta\Omega$ has increased in absolute value because the distortion $\Delta\Omega - \Delta\Omega$ must be weaker since there does not exist a jump of vorticity through the critical layer due to the jump of λ^s as in the case where $\eta_c = 1$. When $n = 2$, $\hat{\eta}_0$ is identical for all critical levels; as a result, $\Delta\Omega$ is the same except for $\eta_c = 1$ where $\Delta\Omega$ is

6.5 Outside the vortex

In this subsection, we shall exhibit the constraints which arise due to the critical layer which is the closest to the boundary with the flow outside the vortex core. Because h may be singular for some values of n , it is more suitable here to use the usual variable and coordinate ψ and r . But this will then create a difficulty here with matching the flow inside the vortex. $H_{2,n}$ is given in D_i by

$$H_{2,n}(\eta) = a_{2,i} \frac{\hat{\eta}_0^4 J_n(\lambda\eta) + \sigma_i Y_n(\lambda\eta)}{\eta_c^2 V(\eta)},$$

with $\sigma_1 = 0$. The deformation of the boundary is,

$$R_b = 1 + \epsilon \eta_b \cos n\theta + \epsilon^2 H_{2,0}(1) + \epsilon^2 H_{2,n}(1) \cos n\theta + \epsilon^2 H_{2,2n}(1) \cos 2n\theta + \dots$$

Note that here $\eta_b = \chi \lambda a_{1,n_c+1}$ where $\chi = \lambda J_n(\lambda)/J_1(\lambda)$.

The streamfunction outside the vortex is given by

$$\psi = \frac{1}{4} \Omega r^2 + \alpha \ln r + \beta + \sum_{p=1}^{\infty} \gamma_p r^{-np} \cos np\theta.$$

This streamfunction and the azimuthal velocities must be matched at the boundary R_b with $\psi = -J_0(\lambda\eta)/\lambda^2$ and $v = J_1(\lambda)/\lambda$. Hence, α , β and γ are expanded as $\alpha = \alpha_0 + \epsilon \alpha_1 + \epsilon^2 \alpha_2$, yielding,

$$O(0): \quad \alpha_0 = \frac{J_1(\lambda)}{\lambda} - \frac{1}{2} \Omega, \quad \beta_0 = -\Omega \left(\frac{1}{\lambda^2} + \frac{1}{4} \right), \quad \gamma_{p,0} = 0,$$

$$O(\epsilon \cos n\theta): \quad \alpha_1 = 0, \quad \beta_1 = 0, \quad \gamma_{1,1} = -\chi J_1(\lambda) a_{1,n_c+1}, \quad H'_{1,n}(1) = -\chi \lambda (n-1 + \lambda \frac{J_0(\lambda)}{J_1(\lambda)}) a_{1,n_c+1}.$$

This last relation is equivalent to the dispersion relation (19) and so is automatically satisfied.

$$O(\epsilon^2 \cos n\theta): \quad \gamma_{1,2} = -\frac{J_1(\lambda)}{\lambda} H_{2,n}(1), \quad H'_{2,n}(1) = -(n-1 + \lambda \frac{J_0(\lambda)}{J_1(\lambda)}) H_{2,n}(1).$$

We now get from the last relation that $J_{n-1}(\lambda) + \sigma_{n_c+1} Y_{n-1}(\lambda) = 0$ and so,

$$\sigma_{n_c+1} = \kappa_{n_c+1} = 0. \quad (79)$$

$$\begin{aligned} O(\epsilon^2): \quad \alpha_2 &= \frac{1}{2} \chi^2 \lambda^2 (n(n+1) \frac{J_1(\lambda)}{\lambda} + \frac{1}{2} \Omega) a_{1,n_c+1}^2 - \frac{J_1(\lambda)}{\lambda} H'_{2,0}(1) + (\frac{J_1(\lambda)}{\lambda} - \Omega) H_{2,0}(1), \\ \beta_2 &= \frac{1}{2} \chi^2 \lambda^2 ((\frac{1}{2} - n) \frac{J_1(\lambda)}{\lambda} - \frac{1}{2} \Omega) a_{1,n_c+1}^2 - \frac{J_1(\lambda)}{\lambda} H_{2,0}(1), \\ O(\epsilon^2 \cos 2n\theta): \quad \gamma_{2,2} &= \frac{1}{2} \chi^2 \lambda^2 ((\frac{1}{2} - n) \frac{J_1(\lambda)}{\lambda} - \frac{1}{2} \Omega) a_{1,n_c+1}^2 - \frac{J_1(\lambda)}{\lambda} H_{2,2n}(1), \\ H'_{2,2n}(1) &= \frac{1}{2} \chi^2 \lambda^2 (2n - n^2 - \lambda(n - \frac{1}{2}) \frac{J_0(\lambda)}{J_1(\lambda)}) a_{1,n_c+1}^2 + (1 - 2n - \lambda \frac{J_0(\lambda)}{J_1(\lambda)}) H_{2,2n}(1). \end{aligned} \quad (80)$$

This last equation leads to a relationship between the integration constants $\xi_{2,2n}$ and $\xi'_{2,2n}$ (see Appendix B). There do not exist any similar constraint for the 0-mode. Inside the vortex, the corresponding integration constants are determined by the cancellation of $F_3^s(\theta)$ at the meeting point of the separatrices. But, as we still have more unknowns than equations, an analysis of the inner flow at the $O(\epsilon^2)$ is required to determine these. The equality of the half-widths of the cat's eyes: $\hat{\eta}_0^- = -\hat{\eta}_0^+$ (see (67)) allows the determination of the amplitudes $a_{1,i}$. At the second order, only the phase $\sigma_{n_c+1} = 0$ is known and σ_i , $i = 2, n_c$ are determined at the order ϵ^2 of the inner expansion by evaluating the jumps $\sigma_{i+1} - \sigma_i$. We shall not, however, give further details here.

As for the case $n = 2$, we may now calculate the additional mean flow needed to remove the vorticity distortion. The calculations are similar to those in subsection 5, leading eventually to the counterpart of (65). For $\eta_c \neq 1$ this is

$$V_{2c}^{' +} - V_{2c}^{' -} = -(\omega_{1c}^{' +} + \omega_{1c}^{' -}) \hat{\eta}_0, \quad (81)$$

$$\omega_{1c}^{' +} - \omega_{1c}^{' -} = \frac{16\sqrt{2}\lambda^2 \hat{\eta}_0}{9(2\sqrt{2} - |\eta[1 + \sqrt{2}]|)}. \quad (82)$$

7 Discussion

We have described the explicit analytic construction by a perturbative expansion of a planar steady state flow representing a multipolar vortex, localized in space and embedded in an infinite domain. Although our construction is for the two-dimensional Euler equations, the existence of such multipolar vortices is highly relevant to geophysical flows. For instance, as well as the examples noted in the Introduction, the results obtained here may be relevant to the dynamics of the polar vortex, which when perturbed by external forces can give rise to the birth to a tripole, or more complex multipolar patterns.

We have considered a basic vortex which has a linear relationship between the vorticity and streamfunction in the vortex core, and constant vorticity in the outer flow. The corresponding azimuthal velocity field may then be singular, leading to the introduction of one or more nonlinear critical layers when a multipolar perturbation is imposed onto this basic flow. These layers we have described in detail. The most relevant and interesting case is when we have a tripole with just one critical layer located at the boundary of the vortex core (see figure 4). However, our detailed analysis of the general case shows that a tripole is the only possibility when there is a critical level at the vortex core boundary, and all higher-order multipolar vortices must have any critical levels located inside the vortex core.

In a similar study to ours, Le Dizès (2000) considered the properties of asymmetric eigenmodes of the Rayleigh equation for an axisymmetric mean flow interacting with a normal mode azimuthal disturbance, in a two-dimensional viscous flow. These eigenmodes are forced by a rotating multipolar strain, modelling the neighbouring vortices. Le Dizès demonstrated that compound vortices can survive without the external strain field when the angular velocity of the pattern takes a certain specific value. This is similar to the situation we have discussed here, with the presence of a nonlinear critical layer, since here the rotation Ω is related to the wavenumber λ . Le Dizès's result is valid for $n = 2$ and $n = 3$, and for several mean flows; when $n > 3$, the vorticity profile has a strong steepness. In the presence of viscosity, a mode amplitude threshold exists which depends on the Reynolds number. Below this threshold, and without any forcing, the eigenmode is eroded.

However, the inclusion of viscous stresses is not essential for a mathematically satisfactory description of the dynamics in the critical layer. As shown originally by Davis (1969); Benney and Bergeron (1969), nonlinear terms in the perturbative expansion may cancel the singularity at the critical level and lead to a zero change of phase across the critical layer, in contrast to the viscous critical layer which implies a jump in the phase of π . Further, in the nonlinear critical layer considered here, the nonlinear terms are able to cancel the singularity at the meeting point of the separatrices, and the velocity is there zero. The jump of vorticity through the separatrix is here proportional to ϵ , (rather than $O(\epsilon^{\frac{1}{2}})$ as it would have been if $\omega'_c \neq 0$). Further, it may be cancelled by introducing a distortion (of order ϵ) to the mean flow. The velocity distortion is of order $\epsilon^{\frac{3}{2}}$. Consequently, the vorticity-streamfunction relationship becomes weakly nonlinear with a dependence on the square root of the streamfunction.

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A The tripole

The order ϵ^2 coefficients related to the expansion of the subsection ?? are

$$\begin{aligned} C_3 &= (\zeta_{1,0} - \xi_{2,0} + \frac{5}{12}\lambda^2)\frac{\hat{\eta}_0^4}{4}, \\ A_{2,3} + B_{2,3} &= (a_2 - b_2 + \frac{\lambda^2}{6})\hat{\eta}_0^4, \\ A_{4,3} + B_{4,3} &= ((24 - \lambda^2)\xi_{2,4} - \zeta_{2,4} + \frac{5\lambda^2}{12})\frac{\hat{\eta}_0^4}{4}, \\ A_{2p,3} + B_{2p,3} &= 0, \quad p > 2. \end{aligned}$$

The coefficients $B_{p,2}$ are determined by the following linear relationships:

$$\begin{aligned} \sum_{p=1}^{\infty} p I_{p,0} B_{2p,2} &= (\frac{5}{12}\lambda^2 - \xi_{2,0} - \zeta_{1,0})\frac{\hat{\eta}_0^3}{8}, \\ \sum_{p=1}^{\infty} p I_{2p,2} B_{2p,2} &= (a_2 + b_2 + \frac{\lambda^2}{6})\frac{\hat{\eta}_0^3}{4}, \\ \sum_{p=1}^{\infty} p I_{2p,4} B_{2p,2} &= \left((24 - \lambda^2)\xi_{2,4} + \zeta_{4,2} + \frac{5\lambda^2}{12}\right)\frac{\hat{\eta}_0^3}{16}, \\ \sum_{p=1}^{\infty} p I_{2p,2q} B_{2p,2} &= 0, \quad q > 2, \end{aligned}$$

with $I_{2p,2q} = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 + \cos 2\theta} \cos 2p\theta \cos 2q\theta d\theta$.

$p I_{2p,2q}$ has an asymptotic behaviour as $1/p$. So, the convergence of the system is very slow.

The $O(\epsilon^{\frac{3}{2}})$ velocity distortion is clearly apparent in these relations:

$$\begin{aligned} \frac{v_3^c}{\Omega} &= \frac{v_3^-}{\Omega} - \frac{\hat{\eta}_0^3}{48} \left(\frac{5}{2}\lambda^2 + 48\frac{\delta\Omega}{\hat{\eta}_0^2} - 6\xi_{2,0} - 6\zeta_{1,0} + 4(6a_2 + 6b_2 + 12\frac{\delta\Omega}{\hat{\eta}_0^2} - \lambda^2) \cos 2\theta \right. \\ &\quad \left. + (6\zeta_{2,4} - \frac{11}{2}\lambda^2 + 6(24 - \lambda^2)\xi_{2,4}) \cos 4\theta \right) / \sqrt{1 + \cos 2\theta}, \end{aligned} \quad (A1)$$

$$\begin{aligned} \frac{v_3^c}{\Omega} &= \frac{v_3^+}{\Omega} - \frac{\hat{\eta}_0^3}{48} \left(\frac{5}{2}\lambda^2 - 48\frac{\delta\Omega}{\hat{\eta}_0^2} - 6\xi_{2,0} - 6\zeta_{1,0} + 4(6a_2 + 6b_2 - 12\frac{\delta\Omega}{\hat{\eta}_0^2} + \lambda^2) \cos 2\theta \right. \\ &\quad \left. + (6\zeta_{2,4} + \frac{5}{2}\lambda^2 + 6(24 - \lambda^2)\xi_{2,4}) \cos 4\theta \right) / \sqrt{1 + \cos 2\theta}. \end{aligned} \quad (A2)$$

The matching of v_3^c and v_3 cannot be obtained for it would imply that

$$\delta\Omega = 0, \quad \lambda = 0,$$

which is, of course, not the case.

B Structures with $\eta_c \neq 1$

B1 The outer flow

The expansion of (17) with the inner variable $\hat{\eta}$ is at either side of the i^{th} critical level

$$h_1 = \epsilon^{-\frac{1}{2}} \frac{\hat{\eta}_0^2}{2\hat{\eta}} \left(1 + \epsilon^{\frac{1}{2}} (2\rho_{n-1}^j + 1 - 2n) \frac{\hat{\eta}}{2\eta_c} - \epsilon(3 + 4\mu_i^2 - 6n^2) \frac{\hat{\eta}^2}{12\eta_c^2} \right) \cos n\theta, \quad (\text{B3})$$

where $\rho_k = \mu_i \frac{J_k(\mu_i)}{J_n(\mu_i)}$. At the order ϵ^2 , the expression of the harmonics $p = 0, 2n$ is

$$H_{2,p} = -\frac{1}{2} \lambda^3 a_1^{s2} \left(\frac{1}{2} \frac{J_0(\lambda\eta)}{J_1(\lambda\eta)^3} J_n(\lambda\eta)^2 - J_{n-1}(\lambda\eta) \frac{J_n(\lambda\eta)}{J_1(\lambda\eta)^2} + \frac{(n - \frac{1}{2})}{\lambda\eta} \frac{J_n(\lambda\eta)^2}{J_1(\lambda\eta)^2} \right) \\ + \lambda^3 a_1^{s2} \xi_{2,p} \frac{J_{2n}(\lambda\eta)}{J_1(\lambda\eta)} + \lambda^3 a_1^{s2} \xi'_{2,p} \frac{Y_{2n}(\lambda\eta)}{J_1(\lambda\eta)}. \quad (\text{B4})$$

The expansion of these harmonics is

$$H_{2,p}^s = -\epsilon^{-\frac{3}{2}} \frac{\hat{\eta}_0^4}{16\hat{\eta}^3} \left(1 + \epsilon^{\frac{1}{2}} \frac{1}{2} \frac{\hat{\eta}}{\eta_c} - \epsilon Q^s(\mu_i) \frac{\hat{\eta}^2}{\eta_c^2} \right), \quad (\text{B5})$$

with

$$Q^s(\mu_i) = -\frac{1}{4} (\rho_{n+1} + \rho_{n-1})^2 - n^2 - \mu_i^2 + \rho_{n-1}^2 + 2n\rho_{n+1} - 4\rho_0(\rho_p \xi_{2,p}^s + \varphi_p \xi'_{2,p}),$$

$$\varphi_p = \mu_i \frac{Y_p(\mu_i)}{J_n(\mu_i)}.$$

B2 The inner flow

$f_3^s(\theta)$ defined in 6.1 by

$$f_3^s(\theta) = \frac{s\hat{\eta}_0^4}{16\eta_c^2} \left(1 + 2\mu_i^2 - 3n^2 + Q^s(\mu_i) + 16a_{2,j} \frac{J_n(\mu_i) + \sigma_j Y_n(\mu_i)}{J_0(\mu_i)} \cos n\theta + (1 + 2\mu_i^2 - 3n^2 + Q^s(\mu_i)) \cos 2n\theta \right).$$

The coefficients of the series modelling the flow within the cat's eyes are

$$C_3 = (n^2 + \frac{\mu_i^2}{3} - n\rho_{n-1} + \frac{1}{2}\rho_{n-1}^2 - \frac{Q^i(\mu_i) + Q^{i+1}(\mu_i)}{4}) \frac{\hat{\eta}_0^4}{4\eta_c^2}, \quad (\text{B6})$$

$$A_{n,3} + B_{n,3} = -(a_{2,i} \frac{J_n(\mu_i) + \sigma_i Y_n(\mu_i)}{J_0(\mu_i)} + a_{2,i+1} \frac{J_n(\mu_i) + \sigma_{i+1} Y_n(\mu_i)}{J_0(\mu_i)} - \frac{\mu_i^2}{3}) \frac{\hat{\eta}_0^4}{\eta_c^2}, \quad (\text{B7})$$

$$A_{2n,3} + B_{2n,3} = (n^2 - n\rho_{n-1} + \frac{1}{3}\mu_i^2 + \frac{1}{2}\rho_{n-1}^2 - \frac{Q^i(\mu_i) + Q^{i+1}(\mu_i)}{4}) \frac{\hat{\eta}_0^4}{4\eta_c^2}. \quad (\text{B8})$$

The coefficients $B_{p,2}$ are determined by the following linear relationships:

$$\sum_{p=1}^{\infty} p I_{np,0} B_{np,2} = \frac{Q^{i+1}(\mu_i) - Q^i(\mu_i)}{16n\eta_c} \hat{\eta}_0^3, \\ \sum_{p=1}^{\infty} p I_{np,n} B_{np,2} = (a_{2,i+1}(J_n(\mu_i) + \sigma_{i+1} Y_n(\mu_i)) - a_{2,i}(J_n(\mu_i) + \sigma_i Y_n(\mu_i))) \frac{\hat{\eta}_0^3}{2n\eta_c J_0(\mu_i)}, \\ \sum_{p=1}^{\infty} p I_{np,2n} B_{np,2} = \frac{Q^{i+1}(\mu_i) - Q^i(\mu_i)}{32n\eta_c} \hat{\eta}_0^3, \\ \sum_{p=1}^{\infty} p I_{np,nq} B_{np,2} = 0, \quad q > 2,$$

with $I_{np,nq} = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 + \cos n\theta} \cos np\theta \cos nq\theta d\theta$.

$p I_{np,nq}$ has an identical asymptotic behaviour in $1/p$ and the convergence of the system is again very slow.

C Mean flow adjustment

C1 $\eta_c = 1$

The order ϵ^2 coefficients related to the expansion of the subsection 4.3 are

$$\begin{aligned}
C_3 &= \left(\zeta_{1,0} - \xi_{2,0} + \frac{5}{12}\lambda^2 + 2(Z_{2,0} + I_{2,0}(1 - \ln 2)) \frac{\omega'_{1c}}{\hat{\eta}_0} \right) \frac{\hat{\eta}_0^4}{4}, \\
A_{2,3} + B_{2,3} &= \left(a_2 - b_2 + \frac{\lambda^2}{6} + (Z_{2,2} + I_{2,2}(1 - \ln 2) - \frac{1}{2}) \frac{\omega'_{1c}}{\hat{\eta}_0} \right) \hat{\eta}_0^4, \\
A_{4,3} + B_{4,3} &= \left((24 - \lambda^2)\xi_{2,4} - \zeta_{2,4} + \frac{5\lambda^2}{12} + 4(Z_{2,4} + I_{2,4}(1 - \ln 2)) \frac{\omega'_{1c}}{\hat{\eta}_0} \right) \frac{\hat{\eta}_0^4}{4}, \\
A_{2q,3} + B_{2q,3} &= (Z_{2,2q} + I_{2,2q}(1 - \ln 2)) \hat{\eta}_0^3 \omega'_{1c}, \quad q > 2.
\end{aligned}$$

The coefficients $B_{p,2}$ are determined by the following linear relationships:

$$\begin{aligned}
\sum_{p=1}^{\infty} p I_{2p,0} B_{2p,2} &= \left(\frac{5}{12}\lambda^2 - \xi_{2,0} - \zeta_{1,0} + 2(Z_{2,0} + I_{2,0}(1 - \ln 2)) \frac{\omega'_{1c}}{\hat{\eta}_0} \right) \frac{\hat{\eta}_0^3}{8}, \\
\sum_{p=1}^{\infty} p I_{2p,2} B_{2p,2} &= \left(a_2 + b_2 + \frac{\lambda^2}{6} + (Z_{2,2} + I_{2,2}(1 - \ln 2) - \frac{1}{2}) \frac{\omega'_{1c}}{\hat{\eta}_0} \right) \frac{\hat{\eta}_0^3}{4}, \\
\sum_{p=1}^{\infty} p I_{2p,4} B_{2p,2} &= \left((24 - \lambda^2)\xi_{2,4} + \zeta_{4,2} + \frac{5\lambda^2}{12} + 4(Z_{2,4} + I_{2,4}(1 - \ln 2)) \frac{\omega'_{1c}}{\hat{\eta}_0} \right) \frac{\hat{\eta}_0^3}{16}, \\
\sum_{p=1}^{\infty} p I_{2p,2q} B_{2p,2} &= (Z_{2,2q} + I_{2,2q}(1 - \ln 2)) \frac{\hat{\eta}_0^2}{4} \omega'_{1c}, \quad q > 2,
\end{aligned}$$

with $Z_{2p,2q} = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 + \cos 2p\theta} \ln[1 + \sqrt{1 + \cos 2p\theta}] \cos 2q\theta d\theta$.

C2 $\eta_c \neq 1$

The contributions due the additional mean flow are for the $O(\epsilon^2)$ Fourier coefficients (B6-B8)

$$\begin{aligned}
C_3 &= -(Z_{n,0} + (1 - \ln 2)I_{n,0})(\omega'_{1c} - \omega'_{1c}) \frac{\hat{\eta}_0^3}{2}, \\
A_{n,3} + B_{n,3} &= -(Z_{n,2} + (1 - \ln 2)I_{n,2} - \frac{1}{2})(\omega'_{1c} - \omega'_{1c}) \hat{\eta}_0^3, \\
A_{2n,3} + B_{2n,3} &= -(Z_{n,4} + (1 - \ln 2)I_{n,4})(\omega'_{1c} - \omega'_{1c}) \hat{\eta}_0^3, \\
A_{nq,3} + B_{nq,3} &= -(Z_{n,2q} + (1 - \ln 2)I_{n,2q})(\omega'_{1c} - \omega'_{1c}) \hat{\eta}_0^3.
\end{aligned}$$