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Rossby elevation waves in the presence of a critical layer

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Caillol, P., and Roger H.J. Grimshaw. 2019. "Rossby Elevation Waves in the Presence of a Critical Layer".
figshare. <https://hdl.handle.net/2134/2893>.

Abstract: In a previous paper, we investigated the solitary-wave-like development of small-amplitude Rossby waves propagating in a zonal shear current, for the particular case when the Rossby wave speed equals the mean-flow velocity at a certain latitude in the β -plane. They presented a general theory for the nonlinear critical-layer theory, and illustrated it by explicitly describing the motion of a depression solitary wave (D-wave). Here we report a continuation of that study and consider the more complex case of an elevation solitary wave (E-wave). The method involves matched asymptotic expansions between the outer flow away from the critical layer and the inner flow inside the latter, where the flows having different scalings. We showed previously that the critical-layer flow expansion diverged in the case of the E-wave on the separatrices bounding the open and closed streamlines, which led us to defer a detailed E-wave study. Thus, in this paper, we examine the motion in the additional layer located along the separatrices where this singularity is removed by using a third scaling and find that the previous undesirable distortions are removed. The evolution equation is derived and is a Korteweg-de-Vries type-equation modified by new nonlinear terms generated by the nonlinear interactions occurring in the critical layer. This equation supports a family of E-waves provided that the mean flow obeys certain conditions. The energy exchange that occurs between the mean flow and the D or E-wave during the critical-layer formation is evaluated in the quasi-steady régime assumption.

1 Introduction

This paper is a continuation of the study of the evolution of a two-dimensional, small-amplitude, long-wave neutral Rossby mode in a sheared mean flow when the Rossby wave speed is equal to the zonal-flow velocity at a critical latitude. The subsequent singularity in the Rayleigh-Kuo equation is resolved near the critical level by isolating a narrow region around it. This is the critical layer, where the singularity in the linear theory is removed by adding nonlinearity through a specific scaling. The critical-layer induced flow admits a second singularity on the separatrix due to the manner in which the integration constants are determined, that is, by averaging the viscous components of the potential-vorticity equation over the streamwise coordinate. This new singularity is then removed with an additional rescaling of the critical-layer flow valid around the separatrix.

The outcome for the E-wave is a nonlinear amplitude evolution equation having the general form (A is a rescaled amplitude):

$$\partial_T A = \mathcal{C}_X[A], \quad (1)$$

where $\mathcal{C}$ is a smooth functional of A , containing unconventional nonlinear and dispersive terms, which are generally of KdV type. $\mathcal{C}$ is made necessary by matching the $O(\epsilon^{\frac{3}{2}})$ inner streamwise velocity on the dividing streamlines. The slow space and time variables X and T are defined by

$$X = \epsilon^{\frac{1}{4}} x', \quad T = \epsilon^{\frac{5}{4}} t. \quad (2)$$

Here ϵ is a small parameter representing the dimensionless wave amplitude. $x' = x - ct$, where c is the linear wave speed. The matching of the $\partial_X^2 A$ term in the $O(\epsilon^{\frac{3}{2}})$ velocity expansion inside the critical layer between the unbounded or bounded streamlines yields this unusual scaling (see Caillol and Grimshaw (2007), which henceforth is quoted as CG). The form of $\mathcal{C}(A)$ differs between the depression wave (D-wave) discussed in CG, and the elevation wave (E-wave) which is the subject of this present paper.

The plan of the paper is as follows. In section 2, we recall the formulation of the problem and the main assumptions. The critical-layer flow is examined in section 3. In section 4, we study the flow around the separatrices and match it with the open and closed-streamline inner flows. The evolution equation is examined in section 5, and a solitary-wave solution is found whose characteristics depend on the outer flow. In section 6, we demonstrate the existence of an infinity of modes which are able to generate an elevation solitary wave for different classical mean-velocity profiles and compare the energy exchange ratios between these profiles and the D or E-waves. Section 7 offers some concluding remarks.

2 Formulation and outer flow

We consider a steady and horizontal parallel shear flow $U(y)$ in a Cartesian frame (x, y) centred at a latitude whose directions are east for x and north for y . We suppose that the fluid is confined between two rigid walls y_1, y_2 either or both of which may be at infinity. In a frame of reference moving with the wave $x' = x - ct$, we write the total streamfunction Ψ as

$$\Psi = \int (U(y) - c) dy + \epsilon \psi, \quad (3)$$

where ψ is the perturbation streamfunction. The dimensionless equation of motion is the vorticity equation on a β -plane

$$\{\partial_t + (U - c)\partial_{x'}\}\Delta\psi + \epsilon J(\Delta\psi, \psi) + (\beta - U'')\partial_{x'}\psi = \frac{1}{R}\Delta^2\psi. \quad (4)$$

R is the Reynolds number and is assumed to be very large: $1/R = \lambda\epsilon^{\frac{7}{4}}$ where λ is an $O(1)$ constant. A body force balances the diffusion of the mean flow U and will appear in the equation of the inner flow. Equation (4) is supplemented by the boundary conditions at the rigid walls,

$$\psi = 0 \quad \text{at} \quad y = y_1, y = y_2. \quad (5)$$

Here, we focus on the long-time asymptotic régime after the critical-layer formation stage characterized by an $O(\epsilon^{\frac{1}{2}})$ vorticity spreading throughout two diffusion layers located at either sides of the critical layer. This outward diffusion from the critical layer generates a distorted mean flow that matches with the undisturbed mean flow (denoted U_u) very far away from the critical layer; $U_u = (U_0 - \epsilon^{\frac{1}{2}}U_{u,1}) + \epsilon^{\frac{1}{2}}U_{u,1}$, where a body force maintains the steady flow $U_0 - \epsilon^{\frac{1}{2}}U_{u,1}$ and diffusion only acts on $\epsilon^{\frac{1}{2}}U_{u,1}$. The shear velocity on the edges of the critical layer is therefore decomposed in the form

$$U(y) = U_0(y) + \epsilon^{\frac{1}{2}}U_1(y) + \epsilon U_2(y) + \dots \quad (6)$$

$U_0(y)$ is the initial velocity profile, $U_1(y), U_2(y) \dots$ are the outcome of its interaction with the Rossby wave. The initial wave speed c_0 is slightly modified by the interaction

$$c = c_0 + \epsilon^{\frac{1}{2}}c_1 + \epsilon c_2 + \dots \quad (7)$$

The perturbation streamfunction is then expanded as

$$\psi = \psi^{(0)} + \epsilon^{\frac{1}{2}}\psi^{(1)} + \epsilon\psi^{(2)} + \epsilon^{\frac{3}{2}}\ln\epsilon\psi^{(3)} + \epsilon^{\frac{3}{2}}\psi^{(4)} \dots \quad (8)$$

2.1 The singular mode

At the leading order, substitution of (8) into (4) and assuming a separation of the coordinates of the form: $\psi^{(0)} = A^*(X, T)\phi(y)$ (A^* being the mode amplitude) yield the Rayleigh-Kuo equation

$$\mathcal{L}_0(\phi) = \partial_y^2\phi + \frac{\beta - U_0''(y)}{U_0(y) - c_0}\phi = 0. \quad (9)$$

The critical level is defined by $y = y_c$ where $U_0(y_c) = c_0$ with $U_0'(y_c) = U_0' \neq 0$ and $U_0''(y_c) = U_0'' \neq \beta$. The amplitude is redefined by $A^* = U_0' A$. On either side of the critical level, the singular mode takes a unique form

$$\phi(y) = ab_0\phi_a + \phi_b, \quad (10)$$

characteristic of a zero-phase jump nonlinear mode, while ϕ_a and ϕ_b are the Frobenius solutions of (9) around y_c (see CG for more details). Here,

$$\begin{aligned} \phi_a &\sim y - y_c, \phi_b \sim 1 + b_0(y - y_c) \ln|y - y_c| \quad \text{as} \quad y - y_c \rightarrow 0, \\ \text{where} \quad b_0 &= (U_0'' - \beta)/U_0', \bar{b}_0 = b_0/U_0', \bar{\beta} = \beta/U_0' \quad \text{and} \quad \hat{\beta} = \bar{\beta}/b_0. \end{aligned}$$

2.2 $O(\epsilon^{\frac{3}{2}})$ flow

The next-order motion is divided into two parts: advection by the additional flow U_1 (denoted by the subscript l) and dispersion (denoted by d). Thus $\psi^{(1)}$ satisfies the equation

$$\mathcal{L}_0(\psi^{(1)}) = -\mathcal{L}_1(\phi)A^* - \phi \partial_X^2 A^*, \quad (11)$$

$$\text{where } \mathcal{L}_i(\phi) = \frac{U_i(y) - c_i}{U_0(y) - c_0} \partial_y^2 \phi - \frac{U_i''(y)}{U_0(y) - c_0} \phi, \quad c_i = U_i(y_c), \quad i = 1, 2, \dots$$

The general solution for $\psi^{(1)}$ around the critical level is ($\eta = y - y_c$)

$$\begin{aligned} \psi^{(1)} = \phi_l^{(1)} A^* + \phi_d^{(1)} \partial_X^2 A^* + \phi B^* = & \left(\sum_{n=0}^{\infty} [b_{l,1,n} \ln |\eta^*| + c_{l,1,n}] \eta^n + \alpha_{l,1} \phi_a + \beta_{l,1} \phi_b \right) A^* \\ & + \left(\sum_{n=2}^{\infty} [c_{d,1,n} \eta^n] + \alpha_{d,1} \phi_a + \beta_{d,1} \phi_b \right) \partial_X^2 A^* + \phi B^*. \end{aligned} \quad (12)$$

The coefficients α (respectively β) may take different values above and below the critical layer. The jumps of the singular-Frobenius-solution coefficients $\beta_{l,d}$ are calculated by matching across the critical layer and the jumps of $\alpha_{l,d}$, by applying the Fredholm alternative to the inhomogeneous equation (11).

3 Nonlinear critical layer flow

Balancing the mean flow and the perturbation in the critical layer leads to the scaling

$$\eta = y - y_c = \epsilon^{\frac{1}{2}} Y. \quad (13)$$

The governing vorticity equation (4) then becomes

$$\{\epsilon^{\frac{3}{2}} \partial_T + (\Psi_Y \partial_X - \Psi_X \partial_Y)\} (\partial_Y^2 + \epsilon^{\frac{3}{2}} \partial_X^2) \Psi + \epsilon^{\frac{3}{2}} \beta \Psi_X = \epsilon \lambda (\partial_Y^4 \Psi + 2\epsilon^{\frac{3}{2}} \partial_X^2 \partial_Y^2 \Psi + \epsilon^3 \partial_X^4 \Psi) + \epsilon^3 \lambda F. \quad (14)$$

Here F is the (viscous) body force,

$$F = -U_0'''(y) + \epsilon^{\frac{1}{2}} U_{u,1}'''(y) = -[U_0''' + \epsilon^{\frac{1}{2}} (U_0^{IV} Y - U_{u,1}''') + \dots],$$

The expansion of the outer flow in terms of Y determines the way in which the inner expansion must proceed,

$$\Psi = \epsilon \{ \Psi^{(0)} + \epsilon^{\frac{1}{2}} \ln \epsilon \Psi^{(1)} + \epsilon^{\frac{1}{2}} \Psi^{(2)} + \dots \}.$$

We take the inviscid limit $\lambda \rightarrow 0$, and hence search for a solution in the form,

$$\Psi = \Psi_i + \lambda \Psi_v + O(\lambda^2).$$

3.1 $O(\epsilon)$

The leading-order equation is

$$(\Psi_Y^{(0)} \partial_X - \Psi_X^{(0)} \partial_Y) \Psi_{YY}^{(0)} = \lambda \partial_Y^2 \Psi_{YY}^{(0)}. \quad (15)$$

$\Psi^{(0)}$ is a more suitable cross-stream coordinate, and so we introduce the new coordinates $\xi = X$ and $S = \Psi^{(0)}$. (15) thus reduces to

$$\Psi_{i,YY}^{(0)} = \mathcal{F}^{(0)}(S),$$

where $\mathcal{F}^{(0)}$ is an arbitrary function. But then, on matching with the outer expansion, we find that

$$\mathcal{F}^{(0)} = U_0', \quad U_0' S = \Psi_i^{(0)} = U_0' \left[\frac{1}{2} Y^2 + A(X, T) \right]. \quad (16)$$

We suppose that A has a solitary-wave shape; that is, it decays rapidly as $X \rightarrow \pm\infty$ and consists of a single hump, with a minimum of $A_0 = A(0) < 0$ at $\zeta = X - VT = 0$ (ζ is a coordinate in the frame

moving with the nonlinear-wave speed). In the terminology of Redekopp (1977), this is an E-wave (wave of elevation); the opposite case of a D-wave when $A_0 > 0$ was considered in CG. The critical layer is then characterized by a zone in which streamlines are enclosed within a shape commonly called the cat's eye. These streamlines are separated from the remaining part of the flow by separatrices, on which $S = S_c = 0$. The cross-stream locations of the separatrices are thus $\pm Y_s = \pm(-2A)^{1/2}$. The cat's eye has an infinite period: the streamlines have a centre at $A = A_0, S = A_0$ and two hyperbolic points moved to $\xi = \pm\infty$ with $A = 0$ and $S = 0$. S is therefore negative inside the cat's eye and positive outside.

3.2 $O(\epsilon^{\frac{3}{2}})$

We consider first the motion outside the separatrices, where all streamlines go to infinity. The $O(\epsilon^{\frac{1}{2}})$ potential vorticity (PV): $Q^{(2)} = \Psi_{YY}^{(2)} - \beta Y$ satisfies (15). So, the inviscid PV is then given by $Q_i^{(2)} = Q(S)$. Matching with the outer expansion and applying the secularity condition on the viscous PV give

$$Q(S) = sb_0 U_0' \sqrt{2S} + U_1', \quad \text{where } s = \text{sgn}[Y]. \quad (17)$$

Integrating Q , we obtain $\psi_Y^{(2)} = b_0 \mathcal{G}(A, S) + \beta S + U_1' S_Y + \mathcal{U}^{(2)}(\xi, T)$, where

$$\mathcal{G}(A, S) = U_0' \{A \ln[\mathcal{T}(A, S)] + [S(S - A)]^{\frac{1}{2}}\} \text{ and } \mathcal{T}(A, S) = \left| \frac{S - A}{A_0} \right|^{\frac{1}{2}} + \left| \frac{S}{A_0} \right|^{\frac{1}{2}}.$$

Within the region of closed streamlines, we invoke a modified form (see Appendix B of CG) of the Prandtl-Batchelor theorem (Batchelor (1956)) to determine the interior potential vorticity. At this order, the latter, the $O(\epsilon^{\frac{1}{2}})$ PV Q_2 is a constant. Consequently, by matching the vorticity on the separatrix $S = S_c$, we get that

$$Q(S_c) = Q_2 = U_1'.$$

The additional mean flow does not possess a vorticity jump through y_c as was the case for the D-wave.

3.3 $O(\epsilon^2)$

We shall omit the study of the $O(\epsilon^2 \ln \epsilon)$ flow because it is identical to that of the D-wave (see CG). We only recall that the second-order streamfunction jump is zero:

$$[\beta_{d,1}]_{-}^{+} = 0, \quad [\beta_{l,1}]_{-}^{+} = 0,$$

and examine the next-order flow as it yields the amplitude equation. The $O(\epsilon)$ inviscid vorticity is

$$\Psi_{YY,i}^{(5)} = \bar{Q}_S \left(\Psi^{(2)} - \Pi[S_Y] \right) + \mathcal{F}^{(5)}(S), \quad (18)$$

$$\text{where } S_Y = s[2(S - A)]^{1/2} \quad \text{and} \quad \Pi[S_Y] = \int_0^\xi \frac{\partial_T A(x, T)}{S_Y(S, x, T)} dx.$$

$\Psi_{YY}^{(5)}$ becomes infinite at $S = S_c = 0$ for the E-wave since $Q_S(S) \sim S^{-1/2}$. We remark that this singularity is caused by the specific form of Q , that is due to the averaging technique on the viscous PV over an infinite period. In the same way, the Lagrangian variable employed by Caillol and Grimshaw (2004) enables one to avoid the use of the secularity condition but generates a singularity at the meeting point of the separatrices. In any case, describing the streamlines around the stagnation point must always be done carefully in critical-layer studies. For instance, Caillol and Maslowe (2007) examined the three-dimensional nonlinear critical theory applied to a barotropic and circular vortex supporting a neutral Kelvin mode. The analytical expression for the three-dimensional streamlines in the critical layer was derived and involved double integrals. The streamline computation time increased near the stagnation point (cf. the streamline chart) because this made the integrals improper. The constant $\mathcal{F}^{(5)}$ is given by this secularity condition (see CG). Integrating $\Psi_{YY}^{(5)}$ and again applying matching conditions as $Y \rightarrow \pm\infty$, we get the velocity

$$\begin{aligned} \Psi_Y^{(5)} = & \frac{1}{3}(U_0''' - \frac{1}{3}\beta b_0)(2A + S)S_Y + (U_2' - 2b_0^2 A^*)S_Y + sb_0(\frac{2}{3}\beta A + \frac{1}{9}\beta S + \mathcal{U}^{(2)})\sqrt{2S} \\ & + U_1''[S(S - A)]^{\frac{1}{2}} - s\frac{1}{2}b_0 \int_\infty^S \frac{\Pi[(w - A)^{\frac{1}{2}}]}{[2w(w - A)]^{\frac{1}{2}}} dw \\ & + b_0 U_0'(\beta_{l,1} A + \beta_{d,1} \partial_X^2 A + sb_0 \sqrt{2SA}) \ln[\mathcal{T}(A, S)] + \mathcal{U}^{(5)}(\xi, T), \end{aligned} \quad (19)$$

$$\text{where } \mathcal{U}^{(5)}(\xi, T) = (\alpha_{l,1} + \beta_{l,1}b_0 - \frac{1}{2}\bar{U}_1'')A^* + (\alpha_{d,1} + \beta_{d,1}b_0)\partial_X^2 A^*.$$

A second integration yields the streamfunction

$$\begin{aligned} \Psi^{(5)} = & (U_0''' - b_0\frac{\beta}{3})(2A + \frac{1}{2}S)\frac{S}{3} + (U_2' - 2b_0^2A^* + \frac{1}{3}sU_1''\sqrt{2S})S \\ & + \bar{b}_0(\frac{2}{3}\bar{\beta}A + \mathcal{U}^{(2)})\mathcal{G}(A, S) + \frac{\bar{b}_0\bar{\beta}}{18}\{(\frac{3}{2}A + S)[S(S - A)]^{\frac{1}{2}} + \frac{3}{2}A^2\ln[\mathcal{T}(A, S)]\} \\ & - \frac{1}{4}b_0\int_{\infty}^S \frac{dw_1}{(w_1 - A)^{\frac{1}{2}}} \int_{\infty}^{w_1} \frac{\Pi[(w_2 - A)^{\frac{1}{2}}] - w_2^{-\frac{1}{2}}A_{TX}^{-1}}{[w_2(w_2 - A)]^{\frac{1}{2}}} dw_2 - \frac{1}{4}b_0\mathcal{K}(A, S)A_{TX}^{-1} \\ & + b_0U_0'(\beta_{l,1}A + \beta_{d,1}\partial_X^2 A + B)\{S_Y\ln[\mathcal{T}(A, S)] - s\sqrt{2S}\} \\ & + \frac{1}{2}b_0^2A^*\{2[S(S - A)]^{\frac{1}{2}}\ln[\mathcal{T}(A, S)] - S + A\ln[\mathcal{T}(A, S)]^2\} + \mathcal{U}^{(5)}(\xi, T)S_Y + \Phi^{(5)}(\xi, T). \end{aligned} \quad (20)$$

Here, recalling that $A < 0$ for a E-wave, $\mathcal{K}(S, A) = -\frac{2}{Y_s}\{[A(A - S)]^{\frac{1}{2}} + S\operatorname{arctanh}\left[\left(\frac{A}{A - S}\right)^{\frac{1}{2}}\right]\}$,

with $\mathcal{K}(0, A) = -Y_s$. $\Phi^{(5)}$ is defined in CG.

The velocity within the closed streamlines is

$$\Psi_Y^{(5,c)} = \frac{1}{3}U_0'''(2A + S)S_Y + Q_5S_Y + \mathcal{U}_5(\xi, T),$$

where Q_5 is a constant. The superscript c denotes the flow inside the separatrices. The extension of the Prandtl-Batchelor theorem implies a non-constant vorticity at that order. Matching of the vorticity is not possible. However, the streamwise velocity must be continuous, which leads to two conditions; the first determines $\mathcal{U}_5$ and the second yields the evolution equation

$$\begin{aligned} (U_2'^+ + U_2'^- - 2Q_5)Y_s - 4b_0^2U_0'(1 + \frac{1}{9}\hat{\beta})AY_s + [\mathcal{U}^{(5)}]_{\pm}^+ &= b_0\int_{\infty}^{S_c} \frac{\Pi[(S - A)^{\frac{1}{2}}]}{[2S(S - A)]^{\frac{1}{2}}} dS \\ &= -2\frac{b_0}{Y_s}\int_0^{\xi} K'\left[\frac{A(x, T)}{A(\xi, T)}\right] \partial_T A(x, T) dx, \end{aligned} \quad (21)$$

where K' is the complete complementary elliptic integral of the first kind, $A(x, T)/A(\xi, T)$ is its parameter, that is, the square of the elliptic modulus (cf. Abramowitz and Stegun (1972)). $[\]_{\pm}^+$ denotes the jump across the critical layer. In order to evaluate the integral on the right-hand side of (21), we assume that the solution is an E-wave with constant speed V . This gives

$$-2b_0VY_s\{E'\left[\frac{A_0}{A}\right] - \frac{A_0}{A}K'\left[\frac{A_0}{A}\right]\},$$

where E' being the complete complementary elliptic integral of the second kind. Since this equation must be valid when $A \rightarrow 0$, by dividing (21) by Y_s and taking the limit $A \rightarrow 0$, we find Q_5 : $Q_5 = (U_2'^+ + U_2'^-)/2$. We will show in section 5 that $V = 0$. So, the right-hand side of (1) is given by

$$\mathcal{C}[A] = -2(1 + \frac{1}{9}\hat{\beta})b_0U_0'A^2 - \frac{1}{4}\{[\alpha_{l,1}]_{\pm}^+ A + [\alpha_{d,1}]_{\pm}^+\partial_X^2 A\}\frac{Y_s}{b_0}. \quad (22)$$

The expressions of the $O(\epsilon^{\frac{3}{2}})$ velocity jumps: $[\alpha_{l,1}]_{\pm}^+$ and $[\alpha_{d,1}]_{\pm}^+$ involve both the outer and inner flows and are

$$[\alpha_{l,1}]_{\pm}^+ = -\mathcal{P}\int_{y_1}^{y_2} \left(\frac{U_1''(y)}{U_0(y) - c_0} + [\beta - U_0''(y)]\frac{U_1(y) - c_1}{[U_0(y) - c_0]^2}\right)\phi^2(y) dy, \quad (23)$$

$$[\alpha_{d,1}]_{\pm}^+ = \int_{y_1}^{y_2} \phi^2(y) dy. \quad (24)$$

Here, $\mathcal{P}$ denotes the regular Cauchy part.

3.4 Redefinition of the separatrix

As in the subsection 4.5 of CG, we need to improve the description of the separatrices, since the variable $S = \Psi^{(0)}$ gives a poor approximation of the location of the separatrices. A strained coordinate is introduced as follows,

$$S = \tilde{S} + \delta\varphi^{(1)}(\tilde{S}) + \epsilon^{\frac{1}{2}}\varphi^{(2)}(\tilde{S}) + \delta^2\varphi^{(3)}(\tilde{S}) + \epsilon^{\frac{1}{2}}\delta\varphi^{(4)}(\tilde{S}) + \epsilon\varphi^{(5)}(\tilde{S}) + \dots \quad \text{where} \quad \delta = \epsilon^{\frac{1}{2}}|\ln \epsilon|. \quad (25)$$

The functions $\varphi^{(i)}$ are determined by ensuring that there is zero velocity at $\xi = 0$ in the core of the cat's eye, a property which was not previously satisfied. We also check that the velocity is zero at the stagnation points. The expressions for φ are the same to the first orders as those for the D-wave.

We obtain the new expansion:

$$\tilde{\Psi} = \epsilon U_0' \tilde{S} + \epsilon^{\frac{3}{2}} \tilde{\Psi}^{(2)} + \epsilon^2 \ln \epsilon^2 \tilde{\Psi}^{(3)} + \epsilon^2 \ln \epsilon \tilde{\Psi}^{(4)} + \epsilon^2 \tilde{\Psi}^{(5)} + \dots$$

This change of variables does not alter the values of the various jumps that we have encountered. In particular, the matching of the $O(\epsilon)$ potential vorticity still cannot be satisfied. The new expressions of the streamwise velocity are however different from the D-wave case:

$$\begin{aligned} \tilde{\Psi}_Y^{(5)} = & \frac{1}{3}[U_0''' \tilde{S}_Y - \frac{1}{3}\beta b_0(\tilde{S}_Y - s\tilde{Y}_s)](2A + \tilde{S}) - 2b_0^2 A^*(\tilde{S}_Y - s\tilde{Y}_s) + U_1''[\tilde{S}(\tilde{S} - A)]^{\frac{1}{2}} \\ & + sb_0\{\frac{2}{3}\beta A + \mathcal{U}^{(2)} + \frac{1}{9}\beta\tilde{S} + b_0 A^* \ln[\mathcal{T}(A, \tilde{S})]\}\sqrt{2\tilde{S}} - s\frac{1}{2}b_0[L(A, \tilde{S}) - L(A, \tilde{S}_c)] \\ & - (\beta A + \hat{\mathcal{U}}_2)(Q + \bar{\beta}\tilde{S}_Y - Q_2) + sb_0 U_0'(\beta_{l,1}A + \beta_{d,1}\partial_X^2 A + B)\frac{\sqrt{2\tilde{S}}}{\tilde{S}_Y}, \quad (26) \end{aligned}$$

$$\text{while in the cat's eye} \quad \tilde{\Psi}_Y^{(5,c)} = \frac{1}{3}U_0'''(2A + \tilde{S})\tilde{S}_Y - \bar{\beta}(\beta A + \hat{\mathcal{U}}_2)\tilde{S}_Y.$$

The functions $\varphi^{(5)}$ and $\varphi^{(5,c)}$ are

$$\begin{aligned} \varphi^{(5)}(S) = & \frac{1}{2}\frac{\varphi^{(2)2}}{S_Y^2} - \varphi^{(2)}\bar{Q}_2 + sb_0^2[\frac{1}{9}\hat{\beta}(2A + \tilde{S}) + 2A]Y_s S_Y \\ & + \{s\frac{1}{2}\bar{b}_0 L[A, \tilde{S}_c] - \bar{\mathcal{U}}^{(5)}\}S_Y - 2(\tilde{S} - A)\bar{\mathcal{U}}_2' - b_0(\beta_{l,1}A + \beta_{d,1}\partial_X^2 A + B)(S_Y \ln \tau - s\sqrt{2\tilde{S}}), \quad (27) \end{aligned}$$

$$\varphi^{(5,c)} = \frac{1}{2}\frac{\varphi^{(2)2}}{S_Y^2} - \bar{Q}_5 S_Y^2 - \varphi^{(2)}\bar{Q}_2 - \bar{\mathcal{U}}_5 S_Y,$$

$$\text{with} \quad \hat{\mathcal{U}}_5 = \underline{\mathcal{U}}^{(5)} + \frac{1}{2}b_0 U_0'(\beta_{l,1}A + \beta_{d,1}\partial_X^2 A + B) \ln[\frac{A}{A_0}],$$

$$\text{and} \quad L(A, S) = \int_{\infty}^S \frac{\int_0^{\xi} \frac{\partial \tau A dx}{(w-A)^{\frac{1}{2}}}}{[2w(w-A)]^{\frac{1}{2}}} dw.$$

$\varphi^{(5)} = \varphi^{(5,c)}$ on the separatrix gives again (21) and $\hat{\mathcal{U}}_5 = \mathcal{U}_5$. However, this present case is more complex because we see that there is a divergence of the expansion in section 3 on the separatrix when $\tilde{S} = \tilde{S}_c = 0$, the leading divergence appearing at the order ϵ for the vorticity. So, an improved expansion will be performed using two inner variables: $\tilde{S}$ and $\hat{S}$ in the neighbourhood of the dividing streamlines and the matching inside/outside the cat's eye will be done with this new expansion.

This second layer is not particular to this problem but arises whenever the motion around the separatrices and the related stagnation points results from a strong interaction between a mean flow and a disturbance, and when the action of viscosity is negligible. The emergence of nonlinear critical layers around a circular vortex shows the creation of such a layer whose width is ϵ around the stagnation points while using polar coordinates (Caillol and Grimshaw (2004)). We will demonstrate that this layer here has a width scaling as $\epsilon \ln \epsilon$.

Figure 1: Modifications of the cat's eye shape due to the better parametrization of the streamlines, as a function of ζ (expansion stopped at the order $\epsilon^{\frac{1}{2}}$, $\epsilon = 0.1$). (a) D-wave: --- separatrices given by $\tilde{S} = A_0$, $-\cdot-$ $\tilde{Y} = 0$, $b_0(2A_0)^{\frac{1}{2}} = -0.8$, $a = 1.5$, (b) E-wave: — separatrices given by $\tilde{S} = 0$, --- separatrices given by $\tilde{S} = 0$, $-\cdot-$ $\tilde{Y} = 0$, — — limits of the layer resolving the singularity $\tilde{S} = 0$ for $\tilde{S} = \pm 0.4A_0$, other numerical values: $\beta_{l,1} - \bar{U}_1''/b_0 = 0.5$, $\beta_{d,1} = 0$, $B = 0$, $\bar{\beta}(-2A_0)^{\frac{1}{2}} = 0.6$ and $b_0(-2A_0)^{\frac{1}{2}} = -0.8$.

4 Improved expansion near the separatrices

We have already commented that the previous inner expansion diverges; this appears first in $\tilde{\Psi}_{YY}^{(5)}$. Hence a new scaling valid in the neighbourhood of the dividing streamlines must be introduced,

$$\hat{S} = \frac{\tilde{S}}{\delta} + \frac{\mathcal{S}^{(2)}}{|\ln \epsilon|} + \epsilon^{\frac{1}{2}} \mathcal{S}^{(4)} + \frac{\epsilon^{\frac{1}{2}}}{|\ln \epsilon|} \mathcal{S}^{(5)} + \dots \quad (28)$$

The location of the separatrix is now given by $\hat{S} = 0$ which comes down to

$$\begin{aligned} \tilde{S}_c = & -\epsilon^{\frac{1}{2}} \mathcal{S}^{(2)} - \epsilon^{\frac{1}{2}} \delta \mathcal{S}^{(4)} + \epsilon (\mathcal{S}^{(2)} \mathcal{S}_{\tilde{S}}^{(2)} - \mathcal{S}^{(5)}) - \epsilon^{\frac{1}{2}} \delta^2 \mathcal{S}^{(6)} + \epsilon \delta (\mathcal{S}^{(2)} \mathcal{S}_{\tilde{S}}^{(4)} + \mathcal{S}_{\tilde{S}}^{(2)} \mathcal{S}^{(4)} - \mathcal{S}^{(7)}) \\ & + \epsilon^{\frac{3}{2}} [\mathcal{S}_{\tilde{S}}^{(2)} \mathcal{S}^{(5)} - \mathcal{S}^{(2)} (\mathcal{S}_{\tilde{S}}^{(2)2} - \mathcal{S}_{\tilde{S}}^{(5)} + \frac{1}{2} \mathcal{S}^{(2)} \mathcal{S}_{\tilde{S}\tilde{S}}^{(2)})] + \epsilon \delta^2 (\mathcal{S}^{(6)} \mathcal{S}_{\tilde{S}}^{(2)} + \mathcal{S}^{(4)} \mathcal{S}_{\tilde{S}}^{(4)} + \mathcal{S}^{(2)} \mathcal{S}_{\tilde{S}}^{(6)}) + \dots \end{aligned} \quad (29)$$

The asymptotic expansion of the streamfunction with this new variable is

$$\begin{aligned} \hat{\Psi} = & \epsilon \delta \hat{S} + \epsilon^{\frac{3}{2}} \delta \hat{\Psi}^{(2)} + \epsilon^{\frac{3}{2}} \delta \hat{\Psi}^{(4)} + \epsilon^2 \hat{\Psi}^{(5)} + \epsilon^{\frac{3}{2}} \delta^2 \hat{\Psi}^{(6)} + \epsilon^2 \delta \hat{\Psi}^{(7)} + \epsilon^{\frac{5}{2}} \hat{\Psi}^{(8)} \\ & + \epsilon^{\frac{3}{2}} \delta^{\frac{5}{2}} \hat{\Psi}^{(9)} + \epsilon^2 \delta^{\frac{3}{2}} \hat{\Psi}^{(10)} + \epsilon^{\frac{3}{2}} \delta^3 \hat{\Psi}^{(11)} + \epsilon^2 \delta^2 \hat{\Psi}^{(12)} + \epsilon^{\frac{5}{2}} \delta \hat{\Psi}^{(13)} + \epsilon^3 \hat{\Psi}^{(14)} \\ & + \epsilon^{\frac{3}{2}} \delta^{\frac{7}{2}} \hat{\Psi}^{(15)} + \epsilon^2 \delta^{\frac{5}{2}} \hat{\Psi}^{(16)} + \epsilon^{\frac{5}{2}} \delta^{\frac{3}{2}} \hat{\Psi}^{(17)} + \epsilon^{\frac{3}{2}} \delta^4 \hat{\Psi}^{(18)} + \dots \end{aligned} \quad (30)$$

The expressions of $\hat{\Psi}^{(i)}$, solutions of the equation of motion, are given in the Appendix A1. The matching of the above flow with the inner flow is realized, firstly taking the limit $\epsilon \rightarrow 0$, $\tilde{S}$ kept constant, which is equivalent to $\hat{S} \rightarrow \infty$ and secondly $\tilde{S} \rightarrow 0$. The matching on the separatrix corresponds to $\hat{S} = 0$. The first matching gives the functions $\mathcal{S}^{(i)}$. We are looking for the functions $\mathcal{S}^{(2)}$, $\mathcal{S}^{(4)}$... expressed as Taylor expansions in $\tilde{S}$. The second matching gives the flow inside the separatrices. Both matchings are reported in Appendix A2. The cat's eye flow determined from this new matching is different from that found with the plain inner expansion. As a result, we denote the cat's eye new integration constants for the velocity and streamfunction fields by a $\hat{\cdot}$ such as $\hat{U}_2$ and $\hat{\Phi}_2$.

The amplitude equation is not modified by this new separatrix definition. The addition of this third scaling $\hat{S}$ does not really increase the distortion of the streamlines in the neighborhood of the separatrices in the E-wave case, compared to the use of the strained coordinate $\tilde{S}$ (cf. Figure 1). The distortion is mainly due to the topological constraints, and the separatrix singularity (which is caused by the averaging method we have used and not by a physical process) has only a minor effect. On the other hand, the

D-wave critical layer has the motion distorted around the stagnation point with a magnitude δ in the Y -frame while the separatrices are shifted from a perfectly symmetrical critical layer with a shift of length $\epsilon^{\frac{1}{2}}$, which still remains when $|\xi| \rightarrow \infty$.

5 The amplitude equation

From (21) the amplitude A satisfies the following equation

$$\partial_T A = \mathcal{C}_X[A], \quad (31)$$

where the right-hand side is

$$\mathcal{C}[A] = -\frac{\varpi A - \chi \partial_X^2 A}{E'[\frac{A_0}{A}] - \frac{A_0}{A} K'[\frac{A_0}{A}]} \frac{Y_s}{\bar{b}_0} - 2 \frac{(1 + \frac{1}{9}\hat{\beta})}{E'[\frac{A_0}{A}] - \frac{A_0}{A} K'[\frac{A_0}{A}]} b_0 U'_0 A^2. \quad (32)$$

In general, due to the complexity of $\mathcal{C}[A]$, it would seem likely that this equation is not integrable, and needs to be solved numerically. However, it is possible to find a travelling solitary wave explicitly. Thus, we seek a solution where $A = A(\zeta)$, $\zeta = X - VT$ where $A \rightarrow 0$ as $\zeta \rightarrow \pm\infty$, so that (31) becomes

$$-VA = \mathcal{C}[A]. \quad (33)$$

Next, using (32) and separating each distinct dependence on A , we get

$$-V\{E'[\frac{A_0}{A}] - \frac{A_0}{A} K'[\frac{A_0}{A}]\} A + 4\mu r_0 A^2 + \varpi Y_s A = \chi Y_s \partial_X^2 A, \quad (34)$$

$$\text{where } Y_s = [-2A]^{\frac{1}{2}},$$

$$\text{while } \chi = -\frac{1}{4}[\alpha_{d,1}]^+/\bar{b}_0, \quad \varpi = \frac{1}{4}[\alpha_{l,1}]^+/\bar{b}_0, \quad \mu = \frac{1}{2}b_0 U'_0 \quad \text{and} \quad r_0 = 1 + \frac{1}{9}\hat{\beta}.$$

Here we recall that $A_0 = A(0)$. This can readily be integrated once more to yield

$$\partial_X \hat{A} = \pm \left\{ \frac{\hat{A}}{[\alpha_{d,1}]^+} \left[8b_0^2 r_0 Y_0 \hat{Y}_s \hat{A} - [\alpha_{l,1}]^+ \hat{A} + \frac{16}{3} \frac{\bar{b}_0 V}{Y_0 \hat{A}} \left(\hat{Y}_s \{E'[\hat{A}^{-1}] - 2K'[\hat{A}^{-1}]\} + \hat{Y}_s^3 E'[\hat{A}^{-1}] - 1 \right) \right] \right\}^{\frac{1}{2}} \quad (35)$$

where $\hat{A} = A/A_0$, $Y_0 = (-2A_0)^{\frac{1}{2}}$ and $\hat{Y}_s = \sqrt{\hat{A}}$. The third term inside the large bracket is $O(1)$ when $\hat{A} \rightarrow 0$, consistent with a finite X -period of the nonlinear wave. Thus nonlinear wave speed must be zero if we wish to have $\partial_X \hat{A} \sim O(\hat{A})$ as $\hat{A} \rightarrow 0$, needed for a solitary wave.

The behaviour of $\partial_X \hat{A}$ in $\zeta = 0$ gives the jump of $\alpha_{l,1}$

$$[\alpha_{l,1}]^+ = 8(1 + \frac{1}{9}\hat{\beta})b_0^2 Y_0. \quad (36)$$

(35) rewritten with $\hat{Y}_s$ and integrated gives

$$\int_1^{\hat{Y}_s} \frac{dr}{\sqrt{s_0(r-1)}r} = \pm \frac{1}{2} \frac{\zeta}{\zeta_0} \quad (37)$$

where $\zeta_0 = \sqrt{\frac{[\alpha_{d,1}]^+}{[\alpha_{l,1}]^+}}$ and $s_0 = \text{sign}[r_0]$. A solitary-wave solution implies that s_0 must be negative, hence $r_0 \leq 0$, and

$$\hat{A} = \text{sech}^4 \left[\frac{1}{4} \frac{\zeta}{\zeta_0} \right]. \quad (38)$$

$\hat{A}$ thus possesses a more localised profile than the KdV soliton in sech^2 , that is, it is flatter as $|\zeta| \rightarrow \infty$. The width of the hump is inversely proportional to $|A_0|^{\frac{1}{4}}$. Some typical plots of this wave and the corresponding separatrices are shown in figures 2. The condition $s_0 = -1$ requires that $8/9\beta < U''_0 < \beta$, which is quite restrictive. For instance, a constant-shear mean flow cannot support a E-wave. However, the constant-speed assumption may be too drastic; to capture more solutions from (21), it would be interesting to consider the general amplitude equation (21).

Figure 2: E-wave for $\zeta_0 = 1/2, 1, 2$: (a) A/A_0 , (b) Y_s/Y_0 .

6 Nonlinear critical layer neutral modes

We now consider the issue of the existence of such solitary-wave motions. The problem reduces to finding a solution $\phi(y)$ of the Rayleigh-Kuo equation (9), that satisfies the boundary conditions (5) in the outer flow far from the critical layer, and matches the local Frobenius solutions (10) at the edges of the critical layer. Further, in order to have a solitary-wave solution, we must also have on the one hand $r_0 \leq 0$ (see (37), a condition depending on the chosen velocity profile U_0 and β) while on the other hand, the jump $[\alpha_{l,1}]^\pm$ in (36) is fixed differently from the D-wave case. The computational method is described in section 7 of CG.

The matching of the open- and closed-streamline flows within the E-wave nonlinear critical layer does not need a distorted mean flow $U_1, U_2 \dots$ unlike the D-wave case. As the critical layer is forming in a transition stage, where the initial unsteady process is caused by vorticity diffusion from the critical layer, a viscosity-induced flow gradually spreads out from the critical layer in two regions on either side, called diffusion boundary layers. Unlike the D-wave case, this flow does not possess a mean over the streamwise direction in the diffusion boundary layers that departs from the undisturbed mean flow (denoted U_u) located far away from the critical layer. Here all mean-flow jumps such as $[U_1']^\pm, [U_1'']^\pm \dots$ are zero. Non-zero jumps can only come from $[\alpha_l]^\pm, [\alpha_d]^\pm \dots$ and $[\beta_l]^\pm, [\beta_d]^\pm \dots$. However, the first-order jumps $[\beta_{l,d,1}]^\pm$ are zero. The critical level remains still during the critical-layer development and does not move with the diffusive time scale λT ; in the same way the wave speed is constant. So, for example the $O(\epsilon^{\frac{1}{2}})$ mean velocity $< \Psi_Y >$ is $U_1(y) = U_{u,1}(y)$. This additional undisturbed flow is nevertheless necessary in order to satisfy the condition (36). For the additional undisturbed profile $U_{u,1}(y)$, we choose a shape perturbation of the flow U_0 ; that is, for instance, if we study the hyperbolic tangent mixing layer, $U_{u,1}$ will be also defined by $\tanh y$ but multiplied by a coefficient so that $U'_{u,1}(y_c) = U'_{u,1}$. The given parameters are y_c and $U'_{u,1}$. We will prefer employing the dimensionless ratio $\sigma = U'_{u,1}/(b_0 U'_0 Y_0)$ instead of $U'_{u,1}$ as the second parameter. The Cauchy part of the integral in (23) is computed by separating the different flows, namely, the outer flow and the inner flow around the critical layer described by the Frobenius solutions. A limitation of the study also arises due to the proximity of the boundaries; however, here the wall and diffusion layers interaction was not examined owing to its complexity.

The velocity profiles studied are:

- (a) Poiseuille flow: $U_0(y) = \delta u/2(y^2 - d^2)$ for $0 \leq y \leq d$; we deliberately avoid the interaction of two critical layers by taking $y \geq 0$;
- (b) A jet profile: $U_0(y) = \delta u \operatorname{sech}^2(y)$ for $0 \leq y \leq d$ with the same remark;
- (c) A laminar mixing layer: $U_0(y) = \delta u \tanh(y)$ for $-d \leq y \leq d$.

Here $\delta u > 0$ is positive, d is chosen large for (b) and (c) so that the flow at $y \sim |d|$ is nearly uniform. The boundary conditions are that $\phi = 0$ at $y = 0, d$ for (a), $\phi = 0$ at $y = 0$ for (b), ϕ decays exponentially at $y = d$ for (b) and at $y = -d$ for (c), and ϕ radiates at $y = d$ for (c).

We next comment on the numerical results obtained for these three profiles and displayed in Table 1. The condition $r_0 < 0$ leads to neutral modes that possess very short variations in β and c . σ is always negative and has a larger variation. a has large variations for the jet and the mixing layer; the product ab_0 remains nevertheless bounded and the asymptotic expansion near the critical level remains valid. $[\alpha_{d,1}]^\pm$ is positive and has a variation of only a few percents whereas $[\alpha_{l,1}]^\pm$ is negative and $[\alpha_{l,1}]^\pm/(b_0^2 Y_0)$ has very large variations. The leading-order wave induced streamwise velocity jump is in the same way as the

$U_0(y)$	y_c	$\beta/\delta u$	$c/\delta u$	σ	$\mathbf{a}$	$[\alpha_{l,1}]_{-}^{+}/(\mathbf{b}_0^2 \mathbf{Y}_0)$	$[\alpha_{d,1}]_{-}^{+}$
a	0.037	1.030	-49.9993	-1.825	1.631	-22.533	5.038
	0.040	1.020	-49.9992	-2.531	1.654	-37.227	4.381
b	0.867	0.531	0.510	-0.036	0.093	-1.059	1.280
	0.879	0.500	0.502	-0.267	1.495	-131.056	1.331
c	-0.153	0.297	-0.152	-0.053	2040.416	-593.535	27.097
	-0.130	0.286	-0.129	-0.0006	30.737	-0.097	28.583

Table 1: The second column contains the smallest and largest possible values of y_c . The high values of a for the mixing layer are attenuated by the values of b_0 respectively equal to $b_0 = -0.0005$ and -0.0319 . a single value of $[\alpha_{l,1}]_{-}^{+}$ stands for each value of y_c .

Figure 3: Energy absorption ratio as a function of $\beta/\delta u$, $\epsilon = 0.001$, $\epsilon^{1/2}/\nu = 1/3$. (a) $1000 \times \tau_{Ek} / (\epsilon^{1/2} A_0)^{1/2}$ for Poiseuille ($\sigma = -1$ cross, in x -axis, we have in fact $x = \beta - 9.9$) and Couette ($\sigma = 1$ diamond) flows (b) $\tau_{Ek} / (\epsilon^{1/2} A_0)^{1/2}$ for the mixing layer ($\sigma = -2$ small star and $\sigma = 1$ square) and the jet ($\sigma = -1$ diamond, $\sigma = 1$ cross and $\sigma = 2$ triangle) flows.

mean flow U_0 direction ($[u]^\pm = U_0' \{[\alpha_{l,1}]^\pm A + [\alpha_{d,1}]^\pm \partial_X^2 A\}$), the expression between the curly brackets being positive. The passage of the nonlinear wave is therefore easily observed by a local acceleration of the current. When $|r_0| \gg 1$, $U_0'' \sim \beta$ and the neutral mode is quasi-regular, which happens in the lines 4 and 5. This section demonstrates that whatever the velocity profile $U_0(y)$, it is possible to find a very small set in c such as singular neutral modes exist and an undisturbed flow $U_{u,1}$ giving the right second-order streamwise-velocity jump through the critical layer.

6.1 Energetic considerations

After integrating equation (1) with respect to the spatial coordinate ζ , one obtains:

$$\partial_T \langle A \rangle = \lim_{l \rightarrow \infty} (2l)^{-1} \int_{-l}^{+l} A_T(\zeta, T) d\zeta = 0.$$

This relation shows that the volume of the solitary wave is conserved. If (1) is first multiplied by A and then integrated, an energy law follows:

$$\partial_T \langle A^2 \rangle = 0.$$

Thus within this present asymptotic expansion, the solitary wave does not exchange energy with the mean current. Instead the motion is adiabatic. If we want to consider energetic exchanges between the solitary wave and the mean flow in the present long-wave scaling, then we have to relax the inviscid assumption. As we saw in section 5 of CG, the motion evolves with the very slow variable $\nu = 1/(2\sqrt{\lambda T})$ in the quasi-steady régime assumption and this exchange is negligible. However, we can ascertain the amount of energy that the mean flow has received in the transition stage before reaching this régime. The initial kinetic energy of the mean flow is $\int_{y_1}^{y_2} U_0(y)^2 / 2 dy$; when the quasi-steady régime is attained, the kinetic energy has become $\int_{y_1}^{y_2} \left(U_0(y) + \epsilon^{\frac{1}{2}} [U_1(y) - U_{u,1}(y)] + O(\epsilon) \right)^2 / 2 dy$. We can evaluate the leading-order energy absorption ratio of the mean current by:

$$\tau_{E_k} = 2\epsilon^{\frac{1}{2}} \frac{\int_{y_1}^{y_2} U_0(y) [U_1(y) - U_{u,1}(y)] dy}{\int_{y_1}^{y_2} U_0^2(y) dy}. \quad (39)$$

But here for the E-wave, we get

$$\tau_{E_k} = 0. \quad (40)$$

Hence, on this X -average, the E-wave does not exchange energy with the mean flow.

We now comment on the evolution of τ_{E_k} as a function of $\beta/|\delta u|$ (cf. Figure 3) for a D-wave. τ_{E_k} is determined by numerically computing (39) by separating the diffusion layer flow and the critical-layer flow. In Figure 3(a), we have plotted $1000 \times \tau_{E_k} / (\epsilon A_0)^{\frac{1}{2}}$ related to some neutral modes of Couette and Poiseuille flows as the energy exchange is very weak for these profiles. Constant-shear branches correspond to both branches plotted in CG (Figure 5(b)) for the mean flow $U_0(y) = \delta u y$, $\sigma = U_{u,1}' / (b_0(y_{c0}) U_0'(y_{c0}) Y_\infty) = 1$. τ_{E_k} is negative so the mean flow loses energy at the expense of the neutral mode. Both branches are bounded at large β by the limitation L3 (see CG) and at small β by L4. For the first branch of the parabolic profile $\sigma = -1$ characterized by $\beta \sim 0$ (Figure 5 (b) in CG), $\tau_{E_k} / (\epsilon A_0)^{\frac{1}{2}}$ is of order 10^{-5} and we have decided not to show on this figure, while the second branch with $\beta \sim 10$ is displayed and $\tau_{E_k} / (\epsilon A_0)^{\frac{1}{2}} = O(10^{-4})$. τ_{E_k} is negative again and increases with β . As β tends the minimum value of the set, τ_{E_k} falls rapidly, $\tau_{E_k} / (\epsilon A_0)^{\frac{1}{2}} \simeq -0.0016$ for $\beta/|\delta u| \simeq 9.95$ (for the D-wave $\delta u < 0$). This is a behaviour that we can expect. Indeed, a mean flow submitted to a weak rotation gradient is more unstable with respect to Rossby waves than if it is submitted to a large rotation gradient. So, if β is small, the mean flow is more likely to give its energy to Rossby waves whose amplitudes grow up, so $\tau_{E_k} < 0$, but if β is large, the mean flow can earn energy from waves and $\tau_{E_k} > 0$. As the instability of a shear flow is linked furthermore to the presence of an inflexion point at high Reynolds number, absent for both these profiles, the weakness of the energy exchange is not surprising. The mode amplitude is slowly increasing so that A_0 can be chosen large since $\tau_{E_k} / (\epsilon A_0)^{\frac{1}{2}}$ is small.

In Figure 3(b), we observe the evolution of τ_{E_k} for the profiles (b) and (c). We have already noticed in CG that a positive σ is favourable for the existence of (b) and (c) solitary-wave neutral modes, which is again visible here. We have plotted three branches of (b) related to the values: $\sigma = -1$, 1 and $\sigma = 2$. The branch $\sigma = -1$ is very short for $\beta/|\delta u| \sim 1$ and τ_{E_k} is small, increasing with β between

$\tau_{E_k}/(\epsilon A_0)^{\frac{1}{2}} = -0.024$ to 0.012 . This branch is limited by L2 for larger β values and by L3 for smaller β 's. On the contrary, when $\sigma > 0$, the two other branches have a larger extent. They are limited for small β by the limitation L3 and for large β by L3 and L4. In their interior, they are limited by L2 between $\beta/\delta u = 0.521$ and $\beta/\delta u = 0.887$ for $\sigma = 1$ and by L3 between $\beta/\delta u = 0.442$ and $\beta/\delta u = 0.814$ for $\sigma = 2$. The lower limit for $\sigma = 1$ ($\beta/\delta u = 0.521$, $\tau_{E_k} = 0.0205$) is given by a type-II D-wave (see CG). τ_{E_k} depends on the sign of σ but does not practically depend on the value of σ . The small- β part has a slightly positive $\tau_{E_k}/(\epsilon A_0)^{\frac{1}{2}}$ (ranges between 0 and 0.02) and the large- β part has a negative τ_{E_k} , between -0.03 and -0.23 for $\sigma = 1$, and -0.03 and -0.76 for $\sigma = 2$. In this last case, τ_{E_k} falls rapidly around $\beta/\delta u = 1.1$ (not shown) and the maximal amplitude of the mode is substantially increasing, its initial value must be sufficiently small to allow for the saturation of the wave through the formation of the critical layer, otherwise the Rossby wave blows up and the perturbation method is no longer valid. We have also plotted two branches of (c) for $\sigma = -2$ and $\sigma = 1$. The branch $\sigma = -2$ is divided into three small parts between which the solitary wave condition L2 is not satisfied and where the energy exchange is very weak. The second branch for $\sigma = 1$ has a larger extent and is bounded by L4 at small β and L2 at large β . $\tau_{E_k}/(\epsilon A_0)^{\frac{1}{2}}$ varies between -0.01 and 0.16 . That would imply that the mixing layer predominantly tends to receive its energy from the neutral mode. The initial value of the Rossby wave amplitude must be large enough to allow for the formation of the critical layer. In the case of the jet profile, the initial wave amplitude must be all the larger so since β is small, whereas for large β , the initial amplitude must be small enough to enable the wave to receive its energy from the mean flow while the critical layer is forming and large enough to allow for the formation of the latter. The profile shape thus has a great effect on the energetic exchange.

Let us now consider the jump through the critical layer of the longitudinal momentum flux. Equation (14) is integrated with respect to Y , Y taking a value between $-\infty$ and $+\infty$, then averaged over X , and after discarding terms of order higher than $O(\lambda)$ we get

$$- \langle \Psi_X \Psi_Y \rangle_Y + \epsilon^{\frac{3}{2}} \langle \Psi_{TY} \rangle + \kappa = \epsilon \lambda \langle \partial_Y^3 \Psi \rangle - \epsilon^{\frac{5}{2}} \lambda (U_0'' - \epsilon^{\frac{1}{2}} U_{u,1}'').$$

Here κ gathers all averages of the form $\langle g_X \rangle$, and is thus of $O(\lambda)$; indeed if g was inviscid the average would be zero. At the leading order $\kappa = \langle (\Psi_Y^2)_X \rangle + \epsilon^{\frac{3}{2}} \beta \langle \Psi_{XY^{-1}} \rangle$, and so for a finite-period critical layer, $\kappa = 0$. Integrating between $Y = -\infty$ and $+\infty$ yields the momentum jump driven by the viscous force that is applied over the whole width of the critical layer and is attenuated by the body force, the beta-effect and the inertia of the motion:

$$-[\langle \Psi_X \Psi_Y \rangle]_{\pm}^+ = \epsilon \lambda [\langle \Psi_{TY} \rangle]_{\pm}^+ - \epsilon^{\frac{5}{2}} \lambda \int_{-\infty}^{\infty} U_0'' - \epsilon^{\frac{1}{2}} U_{u,1}'' dY - \epsilon^{\frac{3}{2}} [\langle \Psi_T \rangle]_{\pm}^+ - \int_{-\infty}^{\infty} \kappa dY \quad (41)$$

At the leading order $O(\epsilon^{\frac{5}{2}})$, using (17) and $[U_1']_{\pm}^+ = -2b_0 U_0' Y_{\infty}$ (cf. equation (33) in CG), we obtain

$$-[\langle \Psi_X^{(0)} \Psi_Y^{(2)} \rangle]_{\pm}^+ - [\langle \Psi_X^{(2)} \Psi_Y^{(0)} \rangle]_{\pm}^+ = \lambda [U_1']_{\pm}^+$$

hence

$$[\langle (u - c)v \rangle]_{\pm}^+ = -2\lambda b_0 U_0' Y_{\infty} \epsilon^{\frac{3}{2}}. \quad (42)$$

In the quasi-steady régime, the momentum flux is zero for the E-wave but varies slowly with λT for the D-wave in such a way that it is proportional to the inverse Reynolds number and to the square root of the maximal nonlinear wave amplitude ϵA_0 :

$$[\langle (u - c)v \rangle]_{\pm}^+ = 2 \frac{\sqrt{2}}{R} (\beta - U_0'') (\epsilon A_0)^{\frac{1}{2}}. \quad (43)$$

Its direction is given by the sign of $\beta - U_0''$. Note the very subtle rôle of viscosity. It is considered very small throughout this study but nevertheless intervenes to define univoquely the critical-layer flow for open (secularity condition) and closed-streamline (PB theorem) motions, and appears stringent for characterizing the momentum and energy exchanges as was previously outlined by Troitskaya (1991) for the internal-wave critical layer. The X -averaged momentum equation of the outer flow in the frame moving with the wave speed c is

$$\partial_t \langle u \rangle = -\frac{d}{dy} \langle (u - c)v \rangle + f \langle v \rangle - \langle p_x \rangle - 2(u - c) \langle u_x \rangle + \lambda \epsilon^{\frac{7}{4}} (\Delta \langle u \rangle + F).$$

At the leading order, we find

$$\partial_t \langle u \rangle \simeq 2\sqrt{2} \lambda \frac{(U_0'' - \beta)(A_0)^{\frac{1}{2}}}{\delta_{\text{CL}}} \epsilon^{\frac{9}{4}}. \quad (44)$$

Here $\delta_{\text{CL}} = O(2(2|A_0|\epsilon)^{\frac{1}{2}})$ is the width of the nonlinear critical layer. However $\partial_t \langle u \rangle \equiv \epsilon^{\frac{1}{2}} \partial_t U_1 = O(\lambda \epsilon^{\frac{7}{4}})$; we have established in a different way than in CG that the critical-layer induced mean flow evolves on the time-scale λT . If $U_0'' > (<) \beta$, the mean flow is accelerated (absorbed) around $y = y_c$ provided $c > 0$ with a rate dependent of $(A_0)^{1/2}$, whereas a naive approach would have estimated a dependence on A_0^2 . We note that the second-type D-wave is always characterized by $U_0'' < \beta$. The neutral modes which induce an acceleration are well separated from those who induce a deceleration for the studied velocity profiles. The Poiseuille and Couette profiles only lead to modes with $U_0'' < \beta$, both induce a deceleration at the critical level. In figure 6 in CG concerning the mixing-layer profile, the neutral modes on the first branch for which β is small have either $U_0'' > \beta$ and $c < 0$ or $U_0'' < \beta$ and $c > 0$, and so are fed by the current (figure 6 (b)), whereas $U_0'' < \beta$ and $c < 0$ for the modes on the second branches in 6 (a) and (b), and so are absorbed. In figure 7 in CG for the jet profile, the neutral modes in the allowed band with small β have $U_0'' > \beta$ and the second part for larger values of β has $U_0'' < \beta$. Here, the acceleration occurs in the first part and inversely for the second branch.

7 Conclusion

We have analytically examined the motion generated by a Rossby elevation wave, resulting from the nonlinear-critical-layer interaction between a free Rossby mode and a shear flow in the long-wave régime. As discussed in CG, there are two classes of waves, E-waves and D-waves, which are not only different in their geometrical shapes. Here we have shown that the flow inside the E-wave critical layer is not described by a single scaling as is usually the case in the nonlinear critical-layer theory, but by two scalings. Indeed, the long-wavelength assumption leads to a blow-up of the inner flow around the separatrices which bound the closed and open streamlines. These separatrices are not symmetrical with respect to the critical level but depart from a perfect symmetry with a shift of $\delta Y = \delta = \epsilon^{\frac{1}{2}} |\ln \epsilon|$ at the place where the solitary wave has its maximum amplitude. Moreover, around them, a second layer of width δ is necessary to introduce where the second scaling is valid in order to absorb this second singularity. However, whereas the vorticity and velocity become discontinuous on the separatrices respectively at the orders $O(\epsilon)$ and $O(\epsilon^2 \ln \epsilon)$ when described by the first-order streamfunction S , these distortions vanish in that layer, where we recall that ϵ measures the wave amplitude on the outer layer. This leads us to suggest that such distortions may be the result of a bad choice of variables to describe the severe rearrangement of streamlines, and consequently of the potential vorticity isolines.

Our analysis of the existence of singular neutral modes reveals an infinity of possible modes, but only for very small β -domains. The existence of an E-wave is strongly constrained by the solitary-wave condition applied to the amplitude equation. The constant-speed assumption for a E-wave may be too stringent. A more complete study of the integro-differential equation (21) is required.

Also, we have shown that E-waves do not exchange energy and longitudinal momentum with the mean flow in the quasi-steady régime. But, in contrast, the D-wave critical-layer induced energy exchange is proportional to the square root of the quasi-steady-régime D-mode maximal amplitude and is hence larger than a standard wave/mean flow interaction. The momentum exchange is independent on the rescaled maximal amplitude A_0 and evolves with the slow diffusive time $\lambda T = t/(\epsilon^{\frac{1}{2}} R)$. After examining a few classical mean velocity profiles, we can conclude that there does not exist a general behaviour concerning energy exchange between a D-wave neutral mode and the mean flow, the choice of the profile is stringent.

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Appendix

A Flow around the separatrix

A1 Integration of the equation of motion

A1.1 Orders $\epsilon^{\frac{3}{2}}$ to ϵ^2

The three first terms of (30) obey

$$(\tilde{S}_Y^2 \hat{\Psi}_{\tilde{S}\hat{S}}^{(i)})_\xi = 0. \quad (\text{A1})$$

According to the required matching with the flow $\tilde{\Psi}$, the only possible solutions are

$$\hat{\Psi}^{(2)} = \mathcal{Z}^{(2)}(\tilde{S}, A), \quad \hat{\Psi}^{(4)} = \mathcal{Z}_1^{(4)}(\tilde{S}, A)\hat{S} + \mathcal{Z}_2^{(4)}(\tilde{S}, A) \quad \text{and} \quad \hat{\Psi}^{(5)} = \mathcal{Z}^{(5)}(\tilde{S}, A).$$

Then, at the order $\epsilon^{\frac{3}{2}}\delta^2$, we have

$$(\tilde{S}_Y^2 \hat{\Psi}_{\tilde{S}\hat{S}}^{(6)})_\xi = -\beta \frac{A_\xi}{\tilde{S}_Y} - [\tilde{S}_Y^2 (\hat{\Psi}_{\tilde{S}\hat{S}}^{(2)} + U'_0 \mathcal{S}_{\tilde{S}\hat{S}}^{(2)} + 2\mathcal{Z}_{1,\tilde{S}}^{(4)})]_\xi - (\hat{\Psi}_{\tilde{S}}^{(2)} + U'_0 \mathcal{S}_{\tilde{S}}^{(2)} + \mathcal{Z}_1^{(4)})_\xi. \quad (\text{A2})$$

We will write $\hat{\Psi}^{(6)}$ in the form

$$\hat{\Psi}^{(6)} = \frac{1}{2} \mathcal{Z}_0^{(6)}(\tilde{S}, A)\hat{S}^2 + \mathcal{Z}_1^{(6)}(\tilde{S}, A)\hat{S} + \mathcal{Z}_2^{(6)}(\tilde{S}, A),$$

where $\mathcal{Z}_0^{(6)}(\tilde{S}, A) = \beta/\tilde{S}_Y - \rho_{\tilde{S}\hat{S}} - 2\mathcal{Z}_{1,\tilde{S}}^{(4)} - [\rho_{\tilde{S}} + \mathcal{Z}_1^{(4)} - \mathcal{Z}_3^{(6)}(\tilde{S})]/\tilde{S}_Y$, $\rho = \hat{\Psi}^{(2)} + U'_0 \mathcal{S}^{(2)}$ and $\mathcal{Z}_3^{(6)}(\tilde{S})$ is an integration constant. $\hat{\Psi}^{(7)}$ and $\hat{\Psi}^{(8)}$ verify (A1), as a result they are simply expressed by

$$\hat{\Psi}^{(7)} = \mathcal{Z}_1^{(7)}(\tilde{S}, A)\hat{S} + \mathcal{Z}_2^{(7)}(\tilde{S}, A) \quad \text{and} \quad \hat{\Psi}^{(8)} = \mathcal{Z}_2^{(8)}(\tilde{S}, A).$$

These solutions have the same forms at either side of the separatrix.

A1.2 Orders $\epsilon^{\frac{3}{2}}\delta^{\frac{5}{2}}$ and $\epsilon^2\delta^{\frac{3}{2}}$

$\hat{\Psi}_{\tilde{S}\hat{S}}^{(9)}$ is given by (A1) and $\epsilon^{\frac{3}{2}}\delta^{\frac{1}{2}}\tilde{S}_Y^2\hat{\Psi}_{\tilde{S}\hat{S}}^{(9)}$ must match with $\epsilon^{\frac{3}{2}}sb_0U'_0\sqrt{2\tilde{S}}$ when $\hat{S}$ tends to infinity. As a result,

$$\tilde{S}_Y^2 \hat{\Psi}_{\tilde{S}\hat{S}}^{(9)} = \mathcal{Z}^{(9)}(\hat{S}) = sb_0U'_0\sqrt{2\hat{S}},$$

and integrating twice with respect to $\hat{S}$

$$\hat{\Psi}^{(9)} = s \frac{b_0U'_0}{15\tilde{S}_Y^2} \sqrt{2\hat{S}}^5 + \mathcal{Z}_1^{(9)}(\tilde{S}, A)\hat{S} + \mathcal{Z}_2^{(9)}(\tilde{S}, A).$$

The equivalent streamfunction inside the separatrices is

$$\hat{\Psi}^{(9,c)} = \mathcal{Z}_1^{(9,c)}(\tilde{S}, A)\hat{S} + \mathcal{Z}_2^{(9,c)}(\tilde{S}, A).$$

$\hat{\Psi}_{\tilde{S}\hat{S}}^{(10)}$ is given by $(\tilde{S}_Y^2 \hat{\Psi}_{\tilde{S}\hat{S}}^{(10)})_\xi = \tilde{S}_Y (\tilde{S}_Y \hat{\Psi}_\xi^{(2)} - \partial_T A) \hat{\Psi}_{\tilde{S}\hat{S}}^{(9)}$.

The presence of the order $\epsilon^{\frac{3}{2}}\delta^{\frac{5}{2}}$ leads to the appearance of the orders $\epsilon^2\delta^{\frac{3}{2}}$, $\epsilon^{\frac{5}{2}}\delta^{\frac{1}{2}}$..., and of the singularity $1/\hat{S}^{\frac{1}{2}}$ in the open-streamline flow if $\tilde{S}_Y \hat{\Psi}_\xi^{(2)} - \partial_T A$ is not cancelled. As a result,

$$\hat{\Psi}^{(2)}(\xi, \tilde{S}, T) = \Pi[\tilde{S}_Y] + f(\tilde{S}). \quad (\text{A3})$$

If we assume $\partial_T A = -V\partial_X A$, then $\hat{\Psi}^{(2)}(\tilde{S}, A) = V\tilde{S}_Y$. We choose $\hat{\Psi}^{(2,c)} = \hat{\Psi}^{(2)}$. Matching with the inner flow leads to $\hat{\Psi}^{(10)} = \hat{\Psi}^{(10,c)} = 0$.

A1.3 Orders $\epsilon^{\frac{3}{2}}\delta^3$ and $\epsilon^2\delta^2$

$\hat{\Psi}_{\hat{S}\hat{S}}^{(11)}$ and $\hat{\Psi}_{\hat{S}\hat{S}}^{(12)}$ satisfy equations with long nonhomogeneous parts which are useless to display explicitly.

$$(\tilde{S}_Y^2 \hat{\Psi}_{\hat{S}\hat{S}}^{(11)})_{\xi} = -[\tilde{S}_Y^2 (2\hat{\Psi}_{\hat{S}\hat{S}}^{(6)} + U_0' \mathcal{S}_{\hat{S}\hat{S}}^{(4)} + \hat{\Psi}_{\hat{S}\hat{S}}^{(4)})]_{\xi} - (\hat{\Psi}_{\hat{S}}^{(6)} + \hat{\Psi}_{\hat{S}}^{(4)} + U_0' \mathcal{S}_{\hat{S}}^{(4)})_{\xi} \\ + f_{11}(\rho_{\hat{S}}, \rho_{\hat{S}\hat{S}}, \hat{\Psi}_{\hat{S}}^{(2)}, \hat{\Psi}_{\hat{S}\hat{S}}^{(2)}, \mathcal{Z}_{1,\hat{S}}^{(4)}, \varphi^{(i)}, \varphi_{\hat{S}}^{(i)}, \dots) \quad (\text{A4})$$

$$(\tilde{S}_Y^2 \hat{\Psi}_{\hat{S}\hat{S}}^{(12)})_{\xi} = -(\mathcal{Z}_1^{(7)} + U_0' \mathcal{S}_{\hat{S}}^{(5)})_{\xi} - [\tilde{S}_Y^2 (2\mathcal{Z}_{1,\hat{S}}^{(7)} + U_0' \mathcal{S}_{\hat{S}\hat{S}}^{(5)})]_{\xi} \\ + f_{12}(\rho_{\hat{S}}, \rho_{\hat{S}\hat{S}}, \rho_{\hat{S}T}, \rho_{\hat{S}\hat{S}T}, \hat{\Psi}_{\hat{S}}^{(2)}, \hat{\Psi}_{\hat{S}\hat{S}}^{(2)}, \hat{\Psi}_{\hat{S}}^{(5)}, \hat{\Psi}_{\hat{S}\hat{S}}^{(5)}, \varphi^{(i)}, \varphi_{\hat{S}}^{(i)}, \dots) \quad (\text{A5})$$

We will write them more simply $f_{11}(\bullet, \bullet)$ and $f_{12}(\bullet, \bullet)$, we note that they do not depend on $\hat{S}$.

A1.4 orders $\epsilon^{\frac{5}{2}}\delta$ to $\epsilon^{\frac{5}{2}}\delta^{\frac{3}{2}}$

$\hat{\Psi}^{(13)}$ and $\hat{\Psi}^{(14)}$ obey (A1). General solutions are

$$\hat{\Psi}^{(13)} = \mathcal{Z}_1^{(13)}(\tilde{S}, A)\hat{S} + \mathcal{Z}_2^{(13)}(\tilde{S}, A) \text{ and } \hat{\Psi}^{(14)} = \mathcal{Z}^{(14)}(\tilde{S}, A).$$

$\hat{\Psi}_{\hat{S}\hat{S}}^{(15)}$, $\hat{\Psi}_{\hat{S}\hat{S}}^{(16)}$ and $\hat{\Psi}_{\hat{S}\hat{S}}^{(17)}$ satisfy equations with nonhomogeneous terms depending on $\hat{\Psi}^{(9)}$

$$(\tilde{S}_Y^2 \hat{\Psi}_{\hat{S}\hat{S}}^{(15)})_{\xi} = -2(\tilde{S}_Y^2 \hat{\Psi}_{\hat{S}\hat{S}}^{(9)})_{\xi} - \hat{\Psi}_{\hat{S}\xi}^{(9)}, \\ (\tilde{S}_Y^2 \hat{\Psi}_{\hat{S}\hat{S}}^{(16)})_{\xi} = -2\mathcal{S}_{\hat{S}\xi}^{(2)} \mathcal{Z}^{(9)}(\hat{S}) + (\mathcal{Z}_{2,\xi}^{(4)} - \frac{1}{2}b_0\partial_T A)\bar{\mathcal{Z}}_{\hat{S}}^{(9)}(\hat{S}), \quad (\text{A6}) \\ \text{and } (\tilde{S}_Y^2 \hat{\Psi}_{\hat{S}\hat{S}}^{(17)})_{\xi} = [\hat{\Psi}_{\xi}^{(5)} - \frac{\mathcal{S}_T^{(2)}}{\tilde{S}_Y} - \hat{\Psi}_{\hat{S}}^{(2)}\mathcal{S}_{\xi}^{(2)} - \partial_T(\bar{\beta}A + \bar{\mathcal{U}}_2)]\bar{\mathcal{Z}}_{\hat{S}}^{(9)}(\hat{S}),$$

which can be easily integrated; $\mathcal{Z}_2^{(4)}$ will be determined in the following subsection and will cancel the singularity in $\hat{\Psi}_{\hat{S}\hat{S}}^{(16)}$ due to $\mathcal{Z}_{\hat{S}}^{(9)}$, so $\hat{\Psi}_{\hat{S}\hat{S}}^{(15)}$ and $\hat{\Psi}_{\hat{S}\hat{S}}^{(16)}$ are simply

$$\tilde{S}_Y^2 \hat{\Psi}_{\hat{S}\hat{S}}^{(15)} = -2\tilde{S}_Y^2 \hat{\Psi}_{\hat{S}\hat{S}}^{(9)} - \hat{\Psi}_{\hat{S}}^{(9)} + \mathcal{Z}^{(15)}(\hat{S}, \tilde{S}) \quad \tilde{S}_Y^2 \hat{\Psi}_{\hat{S}\hat{S}}^{(16)} = -2\mathcal{S}_{\hat{S}}^{(2)} \mathcal{Z}^{(9)}(\hat{S}) + \mathcal{Z}^{(16)}(\hat{S}, \tilde{S}).$$

In order to avoid another singularity in $\hat{\Psi}_{\hat{S}\hat{S}}^{(17)}$, we choose $\hat{\Psi}^{(5)} = \int_{-\infty}^{\xi} \frac{\mathcal{S}_Y^{(2)}}{\tilde{S}_Y} + \hat{\Psi}_{\hat{S}}^{(2)}\mathcal{S}_{\xi}^{(2)} + \partial_T(\bar{\beta}A + \bar{\mathcal{U}}_2) dx + Cst$. If we assume that the pattern moves with a nonlinear speed V , then $\hat{\Psi}^{(5)}$ is easily expressed as $\hat{\Psi}^{(5)} = \partial_{TX^{-1}}(\bar{\beta}A + \bar{\mathcal{U}}_2)$. $\hat{\Psi}_{\hat{S}\hat{S}}^{(17)}$ hence becomes $\tilde{S}_Y^2 \hat{\Psi}_{\hat{S}\hat{S}}^{(17)} = \mathcal{Z}^{(17)}(\hat{S}, \tilde{S})$.

A2 Matchings with the inner flow and the flow inside the separatrices

A2.1 Order $\epsilon^{\frac{3}{2}}$

Matchings $\hat{S} \rightarrow \infty$:

Matching the streamfunction, we get a relation between $\mathcal{S}^{(2)}$ and $\tilde{\Psi}^{(2)}$

$$U_0' \mathcal{S}^{(2)} = \tilde{\Psi}^{(2)} - \hat{\Psi}^{(2)} - \mathcal{Z}_1^{(4)} \tilde{S} - \frac{1}{2} \mathcal{Z}_0^{(6)} \tilde{S}^2 - s \frac{b_0 U_0'}{15 \tilde{S}_Y^2} \sqrt{2\tilde{S}}^5 + O(\tilde{S}^3), \quad (\text{A7})$$

and inside the separatrices

$$U_0' \mathcal{S}^{(2,c)} = \tilde{\Psi}^{(2,c)} - \hat{\Psi}^{(2)} - \mathcal{Z}_1^{(4,c)} \tilde{S} - \frac{1}{2} \mathcal{Z}_0^{(6,c)} \tilde{S}^2 + O(\tilde{S}^3).$$

Around $\tilde{S} = 0$, the streamfunctions outside and inside the separatrices are given by

$$\tilde{\Psi}^{(2)} = \Phi^{(2)} + s(\mathcal{U}^{(2)} - \hat{\mathcal{U}}_2 - \frac{1}{3}\beta A + \frac{1}{2}b_0 U_0' A \ln |\frac{A}{A_0}|)Y_s \\ + [U_1' Y_s + s(\mathcal{U}^{(2)} - \hat{\mathcal{U}}_2 - \beta A) - sb_0 U_0' \ln |\frac{A}{A_0}|Y_s^2] \frac{\tilde{S}}{Y_s} - s \frac{1}{2} [\mathcal{U}^{(2)} - \hat{\mathcal{U}}_2 - \frac{1}{2}\beta Y_s^2] \frac{\tilde{S}^2}{Y_s^3} - s \frac{b_0 U_0'}{30A} \sqrt{2\tilde{S}}^5 + O(\tilde{S}^3),$$

$$\tilde{\Psi}^{(2,c)} = \hat{\Phi}_2(\xi, T) - s\frac{1}{3}\beta AY_s + (U'_1 + s\frac{1}{2}\beta Y_s)\tilde{S} + s\frac{1}{4}\beta\frac{\tilde{S}^2}{Y_s} + O(\tilde{S}^3).$$

We match the inner and improved expansions of the streamfunction. It can be easily shown that the related $O(\epsilon)$ velocity and $O(\epsilon^{\frac{1}{2}})$ vorticity are then matched. The non-integer powers of $\tilde{S}$ in each $S^{(i)}$ must be discarded in order to keep a regular expansion to higher orders. (A7) expanded following $\tilde{S}$ yields $\rho(0) = \tilde{\Psi}^{(2)}(0)$ at the order 0, and $\rho_{\tilde{S}}(0) = \tilde{\Psi}_{\tilde{S}}^{(2)}(0) - \mathcal{Z}_1^{(4)}(0)$ at the first order, $\mathcal{Z}_3^{(6)} = U'_1$ at the second and third orders that are redundant, and at the fourth order, we have (all functions are evaluated at $\tilde{S} = 0$)

$$\tilde{\Psi}_{\tilde{S}\tilde{S}\tilde{S}\tilde{S}}^{(2)} = \rho_{\tilde{S}\tilde{S}\tilde{S}\tilde{S}} - 3\mathcal{Z}_{0,\tilde{S}\tilde{S}}^{(6)} - \frac{5}{Y_s^2}\mathcal{Z}_{0,\tilde{S}}^{(6)} + \frac{8}{Y_s^4}\mathcal{Z}_0^{(6)} + 4\mathcal{Z}_{1,\tilde{S}\tilde{S}\tilde{S}}^{(4)}. \quad (\text{A8})$$

The matchings of the vorticity at the orders $\epsilon^2|\ln \epsilon|$ and ϵ^2 gives two other relationships between $\mathcal{Z}_1^{(4)}$ and ρ ; the higher-order unknowns do not intervene, cancellations occur by using (A4) and (A5):

$$\int f_{11}(\bullet, \bullet) dx = -\tilde{\Psi}_{Y_Y}^{(4)}(0), \quad (\text{A9})$$

$$\int f_{12}(\bullet, \bullet) dx + S^{(2)}(\mathcal{Z}_0^{(6)} + 2\tilde{S}_Y^2\mathcal{Z}_{0,\tilde{S}}^{(6)} + \tilde{S}_Y^2\hat{\Psi}_{\tilde{S}\tilde{S}}^{(11)}) = U'_2. \quad (\text{A10})$$

We obtain three coupled integro-differential equations solving $\mathcal{Z}_1^{(4)}$, ρ and their derivatives evaluated in $\tilde{S} = 0$. The solutions are:

$$\begin{aligned} U'_0 S^{(2)} &= \Phi^{(2)} - \hat{\Psi}^{(2)} + s(\mathcal{U}^{(2)} - \hat{\mathcal{U}}_2 + \frac{1}{2}b_0 U'_0 A \ln|\frac{A}{A_0}| - \frac{1}{3}\beta A)Y_s \\ &\quad + [U'_1 Y_s + s(\mathcal{U}^{(2)} - \hat{\mathcal{U}}_2 - \beta A) - sb_0 U'_0 Y_s^2 \ln|\frac{A}{A_0}|]\frac{\tilde{S}}{Y_s} + \frac{1}{2}\rho_{\tilde{S}\tilde{S}}\tilde{S}^2 + O(\tilde{S}^3), \end{aligned} \quad (\text{A11})$$

$$U'_0 S^{(2,c)} = \hat{\Phi}_2 - \hat{\Psi}^{(2)} - s\frac{1}{3}\beta Y_s A + (U'_1 + s\frac{1}{2}\beta Y_s)\tilde{S} + \frac{1}{2}\rho_{\tilde{S}\tilde{S}}^c \tilde{S}^2 + O(\tilde{S}^3), \quad (\text{A12})$$

$$\mathcal{Z}_1^{(4)} = O(\tilde{S}^4). \quad (\text{A13})$$

We take the simplest solution: zero for $\mathcal{Z}_1^{(4)}$ and its three first derivatives in $\tilde{S} = 0$ and solve for $\rho_{\tilde{S}\tilde{S}}$ from (A9), $\rho_{\tilde{S}\tilde{S}\tilde{S}}$ from (A10) and $\rho_{\tilde{S}\tilde{S}\tilde{S}\tilde{S}}$ from (A8). We remark that $\rho_{\tilde{S}\tilde{S}} = \rho_{\tilde{S}\tilde{S}}^c$, $\rho_{\tilde{S}\tilde{S}\tilde{S}} = \rho_{\tilde{S}\tilde{S}\tilde{S}}^c$ and $\rho_{\tilde{S}\tilde{S}\tilde{S}\tilde{S}} = \rho_{\tilde{S}\tilde{S}\tilde{S}\tilde{S}}^c$.

Matching on the separatrix:

The matching of the streamfunction is trivial since by definition $\hat{\Psi}^{(2)} = \tilde{\Psi}^{(2,c)}$. The $O(\epsilon)$ velocity is: $\hat{\Psi}_Y^{(2)}(A, \tilde{S}) = \tilde{S}_Y(\mathcal{Z}_1^{(4)} + \rho_{\tilde{S}})$. The matching of the $O(\epsilon)$ velocity on the separatrix $\hat{\Psi}_Y(\hat{S} = 0, \tilde{S} = \tilde{S}_c)$ is simply $\hat{\Psi}_Y^{(2)}(A, 0)$, that yields: $\mathcal{S}_S^{(2)}(0) = \mathcal{S}_S^{(2,c)}(0)$, which is equivalent to

$$\hat{\mathcal{U}}_2 = \mathcal{U}^{(2)} + \frac{1}{2}b_0 U'_0 A \ln[\frac{A}{A_0}],$$

as for the potential vorticity matching: $\mathcal{Z}_3^{(6,c)} = \mathcal{Z}_3^{(6)} = Q_2 = U'_1$. The equality of the boundary (29) between closed and open streamlines also gives $\hat{\Phi}_2 = \Phi^{(2)}$.

A2.2 Order $\epsilon^2|\ln \epsilon|$

Matching $\hat{S} \rightarrow \infty$:

The matchings of streamfunction, velocity and vorticity yield

$$\begin{aligned} U'_0 \mathcal{S}^{(4)} + \mathcal{Z}_2^{(4)} + \mathcal{Z}_1^{(6)}\tilde{S} + O(\tilde{S}^2) &= -\tilde{\Psi}^{(4)}, \\ \tilde{S}_Y(\mathcal{Z}_1^{(6)} + U'_0 \mathcal{S}_{\tilde{S}}^{(4)} + \mathcal{Z}_{2,\tilde{S}}^{(4)}) + \dots + O(\tilde{S}) &= -\tilde{\Psi}_Y^{(4)}, \end{aligned} \quad (\text{A14})$$

$$(\mathcal{Z}_1^{(6)} + U'_0 \mathcal{S}_{\tilde{S}}^{(4)} + \mathcal{Z}_{2,\tilde{S}}^{(4)} + \tilde{S}_Y^2 (2\mathcal{Z}_{1,\tilde{S}}^{(6)} + U'_0 \mathcal{S}_{\tilde{S}\tilde{S}}^{(4)} + \mathcal{Z}_{2,\tilde{S}\tilde{S}}^{(4)} + \hat{\Psi}_{\tilde{S}\tilde{S}}^{(11)}) + \dots + O(\tilde{S})) = \int f_{11}(\bullet, \bullet) dx + \dots + O(\tilde{S}) = -\tilde{\Psi}_{YY}^{(4)} \quad (\text{A15})$$

... means that a group of terms only containing the functions φ^i is not given because it presents no interests for the matching. The r.h.s. of (A15) comes from (A4). (A15) simplified by using (A4) gives (A9). (A14) to the two first orders in $\tilde{S}$ yields

$$\mathcal{Z}_2^{(4)}(0) + U'_0 \mathcal{S}^{(4)}(0) = -\tilde{\Psi}^{(4)}(0), \quad (\text{A16})$$

$$\mathcal{Z}_{2,\tilde{S}}^{(4)}(0) + U'_0 \mathcal{S}_{\tilde{S}}^{(4)}(0) + \tilde{\mathcal{Z}}_1^{(6)}(0) = -\tilde{\Psi}_{\tilde{S}}^{(4)}(0). \quad (\text{A17})$$

The unknown functions are: $\mathcal{Z}_1^{(6)}$, $\mathcal{Z}_2^{(4)}$ and $\mathcal{S}^{(4)}$. We choose: $\mathcal{Z}_1^{(6)}(0) = 0$ and $\mathcal{Z}_{2,\tilde{S}}^{(4)}(0) = 0$.

From the behaviour of $\tilde{\Psi}^{(4)}$ at the neighbourhood of $\tilde{S} = 0$, we deduce the expressions of $\mathcal{S}^{(4)}$ and $\mathcal{S}^{(4,c)}$

$$U'_0 \mathcal{S}^{(4)} = \frac{1}{2} b_0 A \mathcal{U}^{(2)} - \Phi^{(4)} - \mathcal{Z}_2^{(4)}(0) + s(\tilde{\mathcal{U}}_4 - \mathcal{U}^{(4)} - \frac{1}{2} b_0 Q_2 A) Y_s + s(\tilde{\mathcal{U}}_4 - \mathcal{U}^{(4)} - \frac{1}{2} b_0 Q_2 A + s \frac{1}{2} \beta b_0 A Y_s) \frac{\tilde{S}}{Y_s} + O(\tilde{S}^2),$$

$$U'_0 \mathcal{S}^{(4,c)} = \frac{1}{2} b_0 A \hat{\mathcal{U}}_2 - \hat{\Phi}_4 - \mathcal{Z}_2^{(4,c)}(0) + (\frac{1}{2} \beta b_0 A - Q_4) \tilde{S} + O(\tilde{S}^2).$$

Matching on the separatrix:

The $O(\epsilon |\ln \epsilon|)$ vorticity on the separatrix $\Psi_{YY}(\hat{S} = 0, \tilde{S} = \tilde{S}_c)$ is

$$\mathcal{Z}_1^{(6)}(0) + U'_0 \mathcal{S}_{\tilde{S}}^{(4)}(0) + \mathcal{Z}_{2,\tilde{S}}^{(4)}(0) + [2\mathcal{Z}_{1,\tilde{S}}^{(6)}(0) + U'_0 \mathcal{S}_{\tilde{S}\tilde{S}}^{(4)}(0) + \mathcal{Z}_{2,\tilde{S}\tilde{S}}^{(4)}(0) + \hat{\Psi}_{\tilde{S}\tilde{S}}^{(11)}(0)] Y_s^2 + \dots \quad (\text{A18})$$

The matching comes down from (A15) to the continuity of $\tilde{\Psi}_{YY}^{(4)}$ which is satisfied. The continuity of the $O(\epsilon^{\frac{3}{2}} |\ln \epsilon|)$ velocity $\hat{\Psi}_Y(\hat{S} = 0, \tilde{S} = \tilde{S}_c)$ is assured by the continuity of $\tilde{\Psi}_Y^{(4)}$ (cf. equation (A17)).

$$s[\mathcal{Z}_1^{(6)}(0) + U'_0 \mathcal{S}_{\tilde{S}}^{(4)}(0) + \mathcal{Z}_{2,\tilde{S}}^{(4)}(0)] Y_s + \dots,$$

which implies $\mathcal{S}_{\tilde{S}}^{(4)}(0) = \mathcal{S}_{\tilde{S}}^{(4,c)}(0)$, (29) comes down to the equality $\mathcal{S}^{(4)}(0) = \mathcal{S}^{(4,c)}(0)$. Moreover, the $O(\epsilon^2 |\ln \epsilon|)$ streamfunction: $\hat{\Psi}(\hat{S} = 0, \tilde{S} = \tilde{S}_c) = \mathcal{Z}_2^{(4)}(0)$ must be continuous through the separatrix and we therefore have

$$\hat{\Phi}_4 = \Phi^{(4)}, \quad \hat{\mathcal{U}}_4 = \frac{1}{2} b_0 U'_0 (\beta_{l,1} A + \beta_{d,1} \partial_X^2 A + B), \quad Q_4 = 0.$$

A2.3 Order ϵ^2

Matching $\hat{S} \rightarrow \infty$:

The streamfunction matching gives

$$\tilde{\Psi}^{(5)} = U'_0 \mathcal{S}^{(5)} + \hat{\Psi}^{(5)} + \mathcal{S}^{(2)} \mathcal{Z}_1^{(4)} + (\mathcal{Z}_0^{(6)} \mathcal{S}^{(2)} + \mathcal{Z}_1^{(7)}) \tilde{S} + s \frac{b_0 U'_0}{3 \tilde{S}_Y^2} \mathcal{S}^{(2)} \sqrt{2 \tilde{S}}^3 + O(\tilde{S}^2), \quad (\text{A19})$$

as for the velocity

$$\tilde{\Psi}_Y^{(5)} = \tilde{S}_Y (\mathcal{Z}_1^{(7)} + \hat{\Psi}_{\tilde{S}}^{(5)} + U'_0 \mathcal{S}_{\tilde{S}}^{(5)} + \mathcal{Z}_0^{(6)} \mathcal{S}^{(2)}) + s \frac{b_0 U'_0}{\tilde{S}_Y} \mathcal{S}^{(2)} \sqrt{2 \tilde{S}} + \dots + O(\tilde{S}), \quad (\text{A20})$$

and the vorticity

$$\begin{aligned} \frac{s b_0}{\sqrt{2 \tilde{S}}} \{ \tilde{\Psi}^{(2)} - \Pi[\tilde{S}_Y] \} + \mathcal{F}^{(5)}(\tilde{S}) &= s b_0 U'_0 \frac{\mathcal{S}^{(2)}}{\sqrt{2 \tilde{S}}} + \mathcal{Z}^{(16)}(\tilde{S}) + \tilde{S}_Y^2 (\hat{\Psi}_{\tilde{S}\tilde{S}}^{(5)} + 2 \mathcal{Z}_0^{(6)} \mathcal{S}_{\tilde{S}}^{(2)} + 2 \mathcal{Z}_{1,\tilde{S}}^{(7)} + U'_0 \mathcal{S}_{\tilde{S}\tilde{S}}^{(5)} + \hat{\Psi}_{\tilde{S}\tilde{S}}^{(12)}) \\ &+ \hat{\Psi}_{\tilde{S}}^{(5)} + \mathcal{Z}_1^{(7)} + U'_0 \mathcal{S}_{\tilde{S}}^{(5)} + \mathcal{S}^{(2)} (\mathcal{Z}_0^{(6)} + 2 \tilde{S}_Y^2 \mathcal{Z}_{0,\tilde{S}}^{(6)} + \tilde{S}_Y^2 \hat{\Psi}_{\tilde{S}\tilde{S}}^{(11)}) + \dots + O(\tilde{S}). \end{aligned} \quad (\text{A21})$$

We can check that the singularity present in (18) has vanished. The powers in $\tilde{S}^{\frac{1}{2}}$ in $\mathcal{F}^{(5)}$ must be balanced by $\mathcal{Z}^{(16)}$ so that

$$\mathcal{Z}^{(16)}(\hat{S}) = s(U_1'' - \frac{b_0}{2} U_1') \sqrt{2 \tilde{S}}.$$

The expansions of $\tilde{\Psi}^{(5)}$ around $\tilde{S} = 0$ are

$$\begin{aligned}\tilde{\Psi}^{(5)} = & \Phi^{(5)} + 2AU_2' + \frac{4}{3}(U_0''' - \frac{1}{3}b_0\beta - 3b_0^2U_0')A^2 - \frac{1}{8}b_0^2U_0'(\hat{\beta} + \ln|\frac{A}{A_0}|)A^2 \ln|\frac{A}{A_0}| \\ & - (\bar{U}^{(2)} + \frac{1}{2}b_0A \ln|\frac{A}{A_0}|)(\beta A + \hat{U}_2) + \frac{1}{2}b_0(L(A, 0) + \frac{1}{2}\partial_{TX^{-1}}A)Y_s + \frac{1}{2}U_0'(\bar{\beta}A + \bar{U}_2)^2 \\ & - \frac{1}{4}b_0 \int_{\infty}^0 \frac{dw_1}{(w_1 - A)^{\frac{1}{2}}} \int_{\infty}^{w_1} \frac{\Pi[(w_2 - A)^{\frac{1}{2}}] - w_2^{-\frac{1}{2}}\partial_{TX^{-1}}A}{[w_2(w_2 - A)]^{\frac{1}{2}}} dw_2 \\ & + \left(\frac{2}{3}(U_0''' - \frac{1}{3}\beta b_0)A - \bar{\beta}(\beta A + \hat{U}_2) - U_2'\right)\tilde{S} + O(\tilde{S}^{\frac{3}{2}}), \quad (\text{A22})\end{aligned}$$

$$\tilde{\Psi}^{(5,c)} = \hat{\Phi}_5 + 2Q_5A + \frac{1}{2}U_0'(\bar{\beta}A + \bar{U}_2)^2 - (\bar{\beta}A + \bar{U}_2)\hat{U}_2 - [Q_5 + \bar{\beta}(\beta A + \hat{U}_2)]\tilde{S} + O(\tilde{S}^2)$$

with unknowns $\hat{\Psi}^{(5)}$, $\mathcal{S}^{(5)}$ and $\mathcal{Z}_1^{(7)}$. (A19) yields

$$U_0'\mathcal{S}^{(5)}(0) + \hat{\Psi}^{(5)}(0) = \tilde{\Psi}^{(5)}(0),$$

$$U_0'\mathcal{S}_{\tilde{S}}^{(5)}(0) + \hat{\Psi}_{\tilde{S}}^{(5)}(0) + \mathcal{Z}_1^{(7)}(0) = s \frac{\tilde{\Psi}_Y^{(5)}(0)}{Y_s} - \mathcal{Z}_0^{(6)}(0)\mathcal{S}^{(2)}(0). \quad (\text{A23})$$

$\mathcal{S}^{(5)}$ is then expressed by

$$\begin{aligned}U_0'\mathcal{S}^{(5)} = & \Phi^{(5)} - \hat{\Psi}^{(5)} + 2AU_2' + \frac{4}{3}(U_0''' - \frac{1}{3}b_0\beta - 3b_0^2U_0')A^2 + \frac{1}{2}U_0'(\bar{\beta}A + \bar{U}_2)^2 \\ & - \frac{1}{8}b_0^2U_0'(\hat{\beta} + \ln|\frac{A}{A_0}|)A^2 \ln|\frac{A}{A_0}| - (\bar{U}^{(2)} + \frac{1}{2}b_0A \ln|\frac{A}{A_0}|)(\beta A + \hat{U}_2) \\ & - \frac{1}{4}b_0 \int_{\infty}^0 \frac{dw_1}{(w_1 - A)^{\frac{1}{2}}} \int_{\infty}^{w_1} \frac{\Pi[(w_2 - A)^{\frac{1}{2}}] - w_2^{-\frac{1}{2}}\partial_{TX^{-1}}A}{[w_2(w_2 - A)]^{\frac{1}{2}}} dw_2 + \frac{1}{2}b_0[\frac{1}{2}\partial_{TX^{-1}}A + L(A, 0)]Y_s \\ & - \left(\mathcal{Z}_0^{(6)}(0)\mathcal{S}^{(2)}(0) + \mathcal{Z}_1^{(7)}(0) + \beta(\bar{\beta}A + \bar{U}_2)\right)\tilde{S} + O(\tilde{S}^2) \quad (\text{A24})\end{aligned}$$

We remark that the $\tilde{S}^{\frac{3}{2}}$ -terms present in $\tilde{\Psi}^{(5)}$ are nonlonger in $\mathcal{S}^{(5)}$.

$$U_0'\mathcal{S}^{(5,c)} = \hat{\Phi}_5 - \hat{\Psi}^{(5,c)} + 2Q_5A + \frac{1}{2}U_0'(\bar{\beta}A + \bar{U}_2)^2 - (\bar{\beta}A + \bar{U}_2)\hat{U}_2 - [\beta(\bar{\beta}A + \bar{U}_2) + \mathcal{Z}_0^{(6)}(0)\mathcal{S}^{(2)}(0) + \mathcal{Z}_1^{(7,c)}(0)]\tilde{S} + O(\tilde{S}^2) \quad (\text{A25})$$

(29) yields $\mathcal{S}^{(5)}(0) = \mathcal{S}^{(5,c)}(0)$ and $\mathcal{S}_{\tilde{S}}^{(5)}(0) = \mathcal{S}_{\tilde{S}}^{(5,c)}(0)$. The first equality gives

$$\begin{aligned}\hat{\Phi}_5 = & \Phi^{(5)} + 2(U_2' - Q_5)A + \frac{4}{3}(U_0''' - \frac{1}{3}\beta b_0 - 3b_0^2U_0')A^2 - \frac{1}{2}b_0(\frac{1}{4}\beta + \frac{1}{4}b_0U_0' \ln|\frac{A}{A_0}|)A^2 \ln|\frac{A}{A_0}| \\ & - \frac{1}{4}b_0 \int_{\infty}^0 \frac{dw_1}{(w_1 - A)^{\frac{1}{2}}} \int_{\infty}^{w_1} \frac{\Pi[(w_2 - A)^{\frac{1}{2}}] - w_2^{-\frac{1}{2}}\partial_{TX^{-1}}A}{[w_2(w_2 - A)]^{\frac{1}{2}}} dw_2 + \frac{1}{2}b_0(\frac{1}{2}\partial_{TX^{-1}}A + L(A, 0))Y_s \quad (\text{A26})\end{aligned}$$

and the second relationship $\mathcal{Z}_1^{(7,c)}(0) = \mathcal{Z}_1^{(7)}(0)$.

Matching on the separatrix:

The vorticity matching gives

$$[\hat{\Psi}_{\tilde{S}\tilde{S}}^{(5)}(0) + 2\mathcal{Z}_0^{(6)}(0)\mathcal{S}_{\tilde{S}}^{(2)}(0) + 2\mathcal{Z}_{1,\tilde{S}}^{(7)}(0) + U_0'\mathcal{S}_{\tilde{S}\tilde{S}}^{(5)}(0) + \hat{\Psi}_{\tilde{S}\tilde{S}}^{(12)}(0)]Y_s^2 + \hat{\Psi}_{\tilde{S}}^{(5)} + \mathcal{Z}_1^{(7)}(0) + U_0'\mathcal{S}_{\tilde{S}}^{(5)}(0) + \dots$$

which is equivalent according to (A21) to have the continuity of

$$\mathcal{S}^{(2)}(\mathcal{Z}_0^{(6)} + 2\tilde{S}_Y^2\mathcal{Z}_{0,\tilde{S}}^{(6)} + \tilde{S}_Y^2\hat{\Psi}_{\tilde{S}\tilde{S}}^{(11)}) - U_2'.$$

$\mathcal{Z}_0^{(6)}$ and $\mathcal{Z}_{0,\tilde{S}}^{(6)}$ are continuous. After (A4), the above vorticity is not distorted provided $\mathcal{S}_{\tilde{S}\tilde{S}}^{(4)} = \mathcal{S}_{\tilde{S}\tilde{S}}^{(4,c)}$.

(A14) expanded to the order $\tilde{S}^2$ does not give $\mathcal{S}_{\tilde{S}\tilde{S}}^{(4)}(0)$ but the relationship (A9). Indeed, all new functions $\mathcal{S}^{(4)}$, $\mathcal{Z}_1^{(6)}$ and $\mathcal{Z}_2^{(4)}$ and their derivatives vanish at that order and we have an extra condition on $\mathcal{Z}_1^{(4)}$ and ρ . To the order $\tilde{S}^3$, we have a relationship linking $\mathcal{S}_{\tilde{S}\tilde{S}}^{(4)}(0)$, $\mathcal{S}_{\tilde{S}\tilde{S}\tilde{S}}^{(4)}(0)$, $\mathcal{Z}_{2,\tilde{S}\tilde{S}\tilde{S}}^{(4)}(0)$ and $\mathcal{Z}_{1,\tilde{S}}^{(6)}(0)$, $\mathcal{Z}_{1,\tilde{S}\tilde{S}}^{(6)}(0)$. We can thus arbitrarily choose $\mathcal{S}_{\tilde{S}\tilde{S}}^{(4)}(0)$ and $\mathcal{S}_{\tilde{S}\tilde{S}}^{(4,c)}(0)$ so that they may be continuous on the separatrix. The velocity on the separatrix is

$$s\left(\frac{\rho_{\tilde{S}}(0)}{2A} - \rho_{\tilde{S}\tilde{S}}(0)\right)\mathcal{S}^{(2)}(0)Y_s + s[\mathcal{Z}_1^{(7)}(0) + \hat{\Psi}_{\tilde{S}}^{(5)}(0) + U_0'\mathcal{S}_{\tilde{S}}^{(5)}(0)]Y_s + \dots$$

and the $O(\epsilon^2)$ streamfunction $\Psi(\hat{S} = 0, \tilde{S} = \tilde{S}_c)$ is $\hat{\Psi}^{(5)}(0) - \mathcal{S}^{(2)}(0)\hat{\Psi}_{\tilde{S}}^{(2)}(0)$. Matching the velocity is equivalent to impose the continuity of $\tilde{\Psi}_Y^{(5)}$ after (A23), which again yields the amplitude equation (34). The streamfunction is trivially continuous.

The continuity of the streamfunction, streamwise velocity and vorticity at the order $\epsilon^{\frac{5}{2}} \ln^2 \epsilon$ presents no difficulties to demonstrate. At the order $\epsilon^{\frac{5}{2}} |\ln \epsilon|$, we only show how the vorticity singularity vanishes.

A2.4 Order $\epsilon^{\frac{5}{2}} |\ln \epsilon|$

Matching $\hat{S} \rightarrow \infty$:

The vorticity matching leads to

$$sb_0 U_0' \frac{\mathcal{S}^{(4)}}{\sqrt{2\tilde{S}}} + \tilde{S}_Y^2 (2\mathcal{S}_{\tilde{S}}^{(2)} \hat{\Psi}_{\tilde{S}\tilde{S}}^{(11)} + \mathcal{Z}_{2,\tilde{S}\tilde{S}}^{(7)} + \hat{\Psi}_{\tilde{S}\tilde{S}}^{(19)} + 2\mathcal{S}_{\tilde{S}}^{(4)} \mathcal{Z}_0^{(6)} + 2\hat{\Psi}_{\tilde{S}\tilde{S}}^{(12)}) \\ + U_0'(\mathcal{Z}_{2,\tilde{S}}^{(7)} + \hat{\Psi}_{\tilde{S}}^{(12)}) = -Q_{\tilde{S}}[\Phi^{(4)} - \frac{1}{2}\bar{b}_0 \partial_{TX^{-1}} A - \frac{1}{2}\bar{b}_0 A(\bar{\beta}\tilde{S} + \mathcal{U}^{(2)})] + \bar{\beta}\mathcal{U}^{(4)}. \quad (\text{A27})$$

The singularity in $1/\sqrt{\tilde{S}}$ must be cancelled; after the expression of $\mathcal{S}^{(4)}$, that yields $\mathcal{Z}_2^{(4)}(0) = b_0/2\partial_{TX^{-1}} A$, which eliminates at the same time the singularity in $\hat{\Psi}_{\tilde{S}\tilde{S}}^{(16)}$ (cf. equation (A6)).