**Novel solitary pulses for a variable-coefficient derivative
nonlinear Schrödinger equation**

K. W. Chow and L. P. Yip

Department of Mechanical Engineering, University of Hong Kong

Pokfulam, Hong Kong

Fax: (852) – 2858 – 5415; Email: kwchow@hkusua.hku.hk

R. H. J. Grimshaw

Department of Mathematical Sciences, Loughborough University

Loughborough LE11 3TU, United Kingdom

Fax: (44) – 1509 – 223 969 Email: r.h.j.grimshaw@lboro.ac.uk

Submission date: February 2007

Keywords: variable coefficient Chen Lee Liu equation; derivative nonlinear
Schrödinger equation

PACS numbers: 42.81.Dp; 42.65.Tg; 05.45.Yv

JPSJ Classification: 02; 03; 47

ABSTRACT

A derivative nonlinear Schrödinger equation with variable coefficient is considered. Special exact solutions in the form of a solitary pulse are obtained by the Hirota bilinear transformation. The essential ingredients are the identification of a special chirp factor and the use of wavenumbers dependent on time or space. The inclusion of damping or gain is necessary. The pulse may then undergo broadening or compression. Special cases, namely, exponential and algebraic dispersion coefficients, are discussed in detail. The case of exponential dispersion also permits the existence of a 2-soliton. This provides a strong hint for special properties, and suggests that further tests for integrability need to be performed. Finally, preliminary results on other types of exact solutions, e.g. periodic wave patterns, are reported.

1. Introduction

The nonlinear Schrödinger equation (NLS) has been demonstrated to be widely applicable to many phenomena involving the dynamics of wave packets.¹⁾ For situations where higher order nonlinear effects need to be restored, a family of derivative nonlinear Schrödinger equations (DNLS) has been investigated.^{2–4)} Notable examples are the Chen – Lee – Liu equation (CLL),

$$iA_t + A_{xx} + iAA^*A_x = 0, \quad (1)$$

and the Kaup – Newell equation (KN),

$$iB_t + B_{xx} + i(B^2B^*)_x = 0. \quad (2)$$

There exists a gauge transformation relating these two equations and their variants.⁵⁾ In fact further transformations demonstrate that envelope equations of the form will also arise

$$i\psi_t + \psi_{xx} + |\psi|^2\psi + i\psi\psi^*\psi_x + i\psi^2\psi_x^* + |\psi|^4\psi = 0. \quad (3)$$

In practice, the coefficients of the various terms in (1 – 3) will not be unity, and properties of solutions will depend on their signs and magnitude. The analytical structures of these equations, in terms of the scattering formulation and the associated transformations, have been studied intensively.^{6, 7)}

The goal of the present work is to extend consideration to DNLS equations with variable coefficients. In particular, we shall identify special versions of CLL with variable coefficients which possess exact soliton solutions. In fact NLS equations with variable coefficients have already been studied in

the literature.^{8 – 11)} A variety of ingenious similarity transformations has been introduced. The solitary pulse can undergo broadening or compression, with or without the presence of damping / gain. One objective here is to determine if these phenomena will continue to exist for DNLS.

A remark on the potential applications of these derivative NLS models is in order. In fluid mechanics, multiple scale techniques applied to the dynamics of a wave packet, in a reference frame moving with the group velocity, will yield the NLS equation at the leading order. Waves of still larger amplitude will require consideration of higher order nonlinear effects. Successive terms in a perturbation expansion will then generate higher order NLS equations.^{12, 13)} An analytically intriguing, and physically important, example is the case of a marginal state for modulational instability. Since the plane wave of the conventional NLS model becomes unstable as the product of the coefficients of the nonlinear and dispersive terms changes from negative to positive, marginal instability occurs when this product is positive but small in magnitude. The usual scaling of NLS breaks down in this regime and a modified perturbation expansion yields an equation of the form of (3).^{14 – 17)} Given the close relations among the various versions of the derivative NLS equation, it seems plausible to study at first a variable coefficient version of CLL (1).

Most existing works on DNLS equations focus exclusively on equations with constant coefficients. For wave propagation through an inhomogeneous medium, variable coefficients will arise. A simple and relevant example will be

the properties of short optical pulses along a fiber with spatially increasing / decreasing dispersion.

The analytical technique employed here to treat these inhomogeneous DNLS equations is the Hirota bilinear method, which has been proven to be effective. The bilinear form for a constant coefficient CLL equation (1) has been obtained earlier in the literature.¹⁸⁾ Our contributions here are, (a) to extend the consideration to a variable coefficient CLL equation, and (b) to obtain exact solutions for special circumstances.

The plan of the present paper can now be explained. We first derive the appropriate bilinear forms after we isolate a special chirp factor (Section 2). For special cases of exponential and algebraic dispersion coefficients, explicit expressions for a 1-soliton are worked out in detail (Section 3). The remarkable fact is that a 2-soliton also exists for the case of an exponential dispersion coefficient, providing suggestive indications of integrability (Section 4). Furthermore, the component solitons can display novel properties resulting from interactions. Finally we provide a preliminary report on other types of wave patterns, e.g. periodic waves (Section 5). Discussions and brief conclusions are presented in Section 6.

2. Bilinear Forms

Consider the derivative nonlinear Schrödinger equation (DNLS) of the Chen – Lee – Liu type:

$$i \frac{\partial A}{\partial t} + \beta \frac{\partial^2 A}{\partial x^2} + K A^2 A^* + i \gamma A A^* \frac{\partial A}{\partial x} + i \Gamma A = 0, \quad (4)$$

where, in anticipation of the results derived, we assume $\beta = \beta(t)$, γ being constant, and $K(x, t)$ will eventually be given as a linear function of x with time dependent coefficients. We have also incorporated a gain / loss for the system through the function $\Gamma(t)$. Although we implicitly consider (4) with t and x as time and space respectively, one must note that t and x stand for space and retarded time respectively for optical applications.

A preliminary version of adopting the Hirota method to variable coefficient evolution equations has been given in our earlier works,^{19, 20)} where we discussed the idea of ‘nonlinearity management’ for the conventional NLS. Here the ideas of chirp and time dependent wavenumber are demonstrated to be equally applicable to a variable-coefficient CLL equation, and thus probably to many other integrable, higher order envelope equations too.

To be precise, we first separate the chirp of the solitary pulse from the main wave form by the transformation:

$$A = \exp \left[\frac{i m_2 x^2}{2} \right] \psi, \quad m_2 = m_2(t). \quad (5)$$

By direct substitution, the quadratic term in the DNLS equation for ψ can be eliminated if m_2 satisfies

$$\frac{1}{2} \frac{d m_2}{d t} + \beta m_2^2 = 0, \quad (6)$$

which implies that ψ will be governed by

$$i\frac{\partial\psi}{\partial t} + \beta\frac{\partial^2\psi}{\partial x^2} + 2i\beta m_2 x \frac{\partial\psi}{\partial x} + im_2\beta\psi + K\psi^2\psi^* + i\Gamma\psi + i\gamma\psi\psi^* \left(\frac{\partial\psi}{\partial x} + im_2 x\psi \right) = 0.$$

By defining

$$\psi = \frac{G}{F}, \quad (7)$$

where *both* G and F can be complex, the equation for ψ can be rewritten as *three* bilinear equations:

$$[iD_t + \beta D_x^2 + 2i\beta m_2 x D_x + i(m_2\beta + \Gamma)]G \cdot F = 0, \quad (8)$$

$$\beta D_x^2 F \cdot F^* = \frac{i\gamma}{2} D_x G \cdot G^* + (K - \gamma m_2 x)GG^*, \quad (9)$$

$$\beta D_x F \cdot F^* = \frac{i\gamma GG^*}{2}. \quad (10)$$

D is the Hirota bilinear operator and we recall that $*$ stands for the complex conjugate.

A 1-soliton can be obtained by looking for waves with time dependent wavenumber:

$$G = \exp[xh_1 + h_0],$$

$$F = 1 + n \exp[x(h_1 + h_1^*) + h_0 + h_0^*],$$

where h_1, h_0, n are functions of t .

Some bilinear identities on functions with time dependent wavenumbers are given in Appendix 1. Manipulations of such identities show that this

sequence of reasoning will lead to a 1-soliton for the variable coefficient CLL equation (4).

From the first bilinear equation, the time dependent wavenumber h_1 is given by

$$\frac{dh_1}{dt} + 2\beta m_2 h_1 = 0,$$

while h_0 is obtained from

$$i \frac{dh_0}{dt} + \beta h_1^2 + i(m_2 \beta + \Gamma) = 0,$$

but Γ must be determined from below.

The cubic term from the first bilinear equation dictates that the function n must be given by

$$n = z_0 m,$$

where $z_0 = \xi + i\eta$ is a complex constant and $m = m(t)$ is a real function. The precise form is determined by the last bilinear equation as

$$\beta m(h_1 + h_1^*) = \frac{\gamma}{4\eta}.$$

The second bilinear equation identifies the nonlinear coefficient K through

$$\beta m(z_0 + z_0^*)(h_1 + h_1^*)^2 = \frac{i\gamma(h_1 - h_1^*)}{2} + K - \gamma m_2 x.$$

Except for the last two terms on the right, all expressions are functions of t . Hence the special coefficient for the cubic nonlinearity, K , must be a linear function in x with time dependent coefficients for a 1-soliton of this type to exist.

The loss / gain coefficient Γ is then found from

$$\frac{1}{m} \frac{\partial m}{\partial t} = 2(m_2 \beta + \Gamma).$$

On knowing Γ , the function h_0 , the component independent of the spatial coordinate in the phase, can be determined from the governing equations earlier.

3. Special Cases: Exponential and Algebraic Dispersion Coefficients

The formulation of the previous section can be applicable in principle to arbitrary $\beta(t)$, with γ constant, and K being a linear function in x . Quite remarkable properties can arise for concrete, specific choices of the dispersion coefficient, and the aim here is to display several examples.

Special Case (A)

First we take (δ real)

$$\beta = \exp(-\delta t), \tag{11}$$

On taking the integration constant of (6), the differential equation for the chirp factor, to be zero, one gets

$$m_2 = -\frac{\delta \exp(\delta t)}{2}. \quad (12)$$

m_2 is thus free of any singularity. The remaining parameters can be found as

$$\begin{aligned} h_1 &= r \exp(\delta t), \quad r = p + iq, \quad m = \frac{\gamma}{4\eta(r + r^*)}, \quad z_0 = \xi + i\eta, \\ \Gamma &= \frac{\delta}{2}, \quad h_0 = \frac{ir^2 \exp(\delta t)}{\delta}, \quad K = \gamma \exp(\delta t) \left[\frac{\xi p}{\eta} + q - \frac{\delta x}{2} \right]. \end{aligned} \quad (13)$$

Figure 1 shows the evolution of this bright 1-soliton. As expected, in the presence of damping, or with positive values of δ and Γ as defined by (4) and (13), attenuation in the intensity of the wave should be observed. The amplitude of the pulse remains relatively constant, but the width (or area enclosed) decreases steadily. Furthermore, the pulse appears to approach an asymptote, or a limiting position, instead of propagating indefinitely with a finite velocity. This is in marked contrast with many soliton systems.

Special Case (B)

We again take

$$\beta = \exp(-\delta t),$$

but we now choose a nonzero constant of integration for the chirp factor m_2 .

This leads to

$$m_2 = \frac{1}{1 - 2 \exp(-\delta t)/\delta}, \quad (14)$$

while the other parameters are given as

$$\begin{aligned}
h_1 &= \frac{r}{1 - 2\exp(-\delta t)/\delta}, \quad r = p + iq, \quad m = \frac{\gamma}{8\eta p} \left(\exp(\delta t) - \frac{2}{\delta} \right), \\
z_0 &= \xi + i\eta, \quad h_0 = -\frac{ir^2}{2(1 - 2\exp(-\delta t)/\delta)} - \frac{1}{2} \log \left[\exp(\delta t) - \frac{2}{\delta} \right], \\
\Gamma &= \frac{\delta}{2}, \quad K = \frac{\gamma(p\xi/\eta + q + x)}{1 - 2\exp(-\delta t)/\delta}.
\end{aligned} \tag{15}$$

Analytically m_2 will reach a singularity in finite time for positive δ . Figure 2 shows the evolution of the bright 1-soliton. Remarkably, the pulse still suffers compression even though we have chosen a positive value of δ , i.e. a value corresponding to the damping regime. At the same time the pulse also increases in amplitude, and we believe that the area enclosed must have decreased due to the presence of damping. This process appears to persist up to the moment the chirp factor becomes infinite.

Special Case (C)

We now consider a medium with an algebraic dispersion relation

$$\beta = \frac{\delta}{t^2}, \tag{16}$$

where we will choose the constant of integration to be zero for the chirp factor m_2 . This leads to

$$m_2 = -\frac{t}{2\delta}, \tag{17}$$

which in turn yields the other parameters as

$$\begin{aligned}
h_1 &= rt, & m &= \frac{\gamma t}{8\delta\eta p}, & r &= p + iq, & z_0 &= \xi + i\eta, \\
\Gamma &= \frac{1}{t}, & h_0 &= ir^2\delta t - \frac{1}{2}\log(t), & K &= \gamma t\left(\frac{p\xi}{\eta} + q - \frac{x}{2\delta}\right).
\end{aligned} \tag{18}$$

Figure 3 illustrates the evolution of the bright 1-soliton. Due to the form of the dispersion coefficient in (16), obviously the results should be taken to be valid only for time $t > 0$. Unlike the previous cases, the damping is not constant but decays algebraically with time. The pulse just undergoes attenuation without any irregular behavior.

4. 2-soliton

Following the standard procedure, a 2-soliton solution can in principle be obtained by considering:

$$\begin{aligned}
G &= \exp[xh_1 + h_0] + \exp[xH_1 + H_0] + N_1 \exp[x(h_1 + H_1 + H_1^*) + h_0 + H_0 + H_0^*] \\
&+ N_2 \exp[x(h_1 + h_1^* + H_1) + h_0 + h_0^* + H_0],
\end{aligned}$$

$$\begin{aligned}
F &= 1 + m_{11} \exp[x(h_1 + h_1^*) + h_0 + h_0^*] + m_{12} \exp[x(h_1 + H_1^*) + h_0 + H_0^*] \\
&+ m_{21} \exp[x(h_1^* + H_1) + h_0^* + H_0] + m_{22} \exp[x(H_1 + H_1^*) + H_0 + H_0^*] \\
&+ M \exp[x(h_1 + h_1^* + H_1 + H_1^*) + h_0 + h_0^* + H_0 + H_0^*],
\end{aligned}$$

provided that the bilinear equations are indeed satisfied. (h_1, h_0) , (H_1, H_0) are expressions for the 1-soliton case with parameters r, R respectively.

To avoid the complexity associated with a singularity in the chirp factor, we consider the extension for special case (A) only:

$$\beta = \exp(-\delta t), \quad m_2 = -\frac{\delta \exp(\delta t)}{2}.$$

The corresponding parameters h_1, h_0, H_1, H_0 are now given as

$$\begin{aligned} h_1 &= r \exp(\delta t), & h_0 &= \frac{ir^2 \exp(\delta t)}{\delta}, \\ H_1 &= R \exp(\delta t), & H_0 &= \frac{iR^2 \exp(\delta t)}{\delta}. \end{aligned} \quad (19)$$

As the algebra is rather tedious, we focus on the case of real r and R only. In that case, we have successfully proven that the bilinear expansions do truncate at this level. The parameters $N_1, N_2, m_{11}, m_{12}, m_{21}, m_{22}, M$ are then independent of x and t and are given as:

$$\begin{aligned} N_1 &= -\frac{i\gamma(r-R)^2}{8R(r+R)^2}, & N_2 &= -\frac{i\gamma(r-R)^2}{8r(r+R)^2}, \\ m_{11} &= \frac{i\gamma}{8r}, & m_{22} &= \frac{i\gamma}{8R}, & m_{12} &= \frac{i\gamma r}{2(r+R)^2}, & m_{21} &= \frac{i\gamma R}{2(r+R)^2}, \\ M &= -\frac{\gamma^2(r-R)^4}{64rR(r+R)^4}, & \Gamma &= \frac{\delta}{2}, & K &= -\frac{1}{2}x\gamma\delta\exp(\delta t). \end{aligned} \quad (20)$$

Figure 4 shows the evolution of the bright 2-soliton solution of special case (A) with typical values of the parameters. Similar to the 1-soliton solution of special case (A), both solitons decay by getting narrower, as dictated by the presence of damping. However, one component of the pair undergoes a periodic modulation of the amplitude in the direction of t . Furthermore, the position of the crests of this component seems to pursue an oscillatory path, instead of a monotonic route found in many classical systems. The crests do indeed get closer as they propagate. Similar to Special Case (A) above, both pulses approach limiting values for large times.

5. Periodic solutions

The presence of damping or amplification (or nonzero Γ in (4)) is necessary for the existence of a 1-soliton in the previous examples. We now illustrate the difficulty, and subtlety, for this class of nonlinear evolution equations by obtaining the periodic solutions of some variable coefficient CLL without loss / gain. However, the long wave limit of these periodic patterns is a plane wave rather than a 1-soliton.

For periodic waves, the bilinear forms must be generalized to

$$(iD_t + \beta D_x^2 + 2i\beta m_2 x D_x + i\beta m_2 + i\Gamma - C)G \cdot F = 0,$$

$$(\beta D_x^2 - C)F \cdot F^* = \frac{i\gamma}{2} D_x G \cdot G^* + (K - \gamma m_2 x)GG^*$$

$$\beta D_x F \cdot F^* = \frac{i\gamma}{2} GG^*.$$

C can be an arbitrary function of time. The Jacobi theta functions will be the convenient tools in the intermediate calculations. Basically they are Fourier series with exponentially decaying coefficients. Unlike the bilinear forms for NLS, where a single theta function will satisfy the bilinear equations, the periodic waves for CLL require a combination of theta functions. More precisely, after a little experimentation, one possible combination is

$$\begin{aligned} G &= a_0 \exp(ivx - i\Omega)(\theta_1(\zeta) + \theta_2(\zeta)), & \Omega &= \Omega(t), \\ F &= \exp(ivx)(\theta_3(\zeta) + i\theta_4(\zeta)), & \theta_3^2(0)\zeta &= \xi = xh_1(t) + h_0(t). \end{aligned} \tag{21}$$

The precise forms of the four Jacobi theta functions are given in Appendix 2. The Hirota derivatives of theta functions are handled by methods described in our earlier works,²¹⁾ and hence the presentation here will be brief. Basically the chirp factor m_2 , and the time dependent wavenumber h_1 will still satisfy the same equations as Sections 3, 4. However, h_0 , Γ and a_0 are now given by

$$\frac{dh_0}{dt} = -\frac{k^2\beta h_1}{1+\sqrt{1-k^2}}, \quad \Gamma = 0, \quad a_0 = \left(\frac{2k\beta h_1}{\gamma}\right)^{\frac{1}{2}}.$$

k is the modulus of the Jacobi elliptic function. γ must also be constant, and the nonlinear coefficient K must be given as

$$K - \gamma m_2 x = -2\beta k(1 - \sqrt{1-k^2})h_1^2.$$

Finally the first bilinear equation will yield two constraints for determining C and the derivative of Ω (and hence Ω). Omitting the algebra we just highlight the physically important quantity, namely, the intensity $|A|^2$,

$$|A|^2 = \frac{\left(\frac{2k^2\beta h_1}{\gamma}\right) \left((1-k^2)^{\frac{1}{4}} \text{sn}(\xi) + \text{cn}(\xi)\right)^2}{\left((1-k^2)^{\frac{1}{2}} + \text{dn}^2(\xi)\right)},$$

where we have converted the theta functions to the more common Jacobi elliptic functions. ξ is defined by (21). In the long wave limit (k tending to 1), both cn and dn approach the hyperbolic secant and hence the intensity does not show any spatial dependence, i.e. it becomes a plane wave with time dependent

amplitude. The proper form of the solitary wave in the absence of damping or gain ($\Gamma = 0$) remains an open question.

6. Conclusions

The Chen – Lee – Liu equation is studied as a representative example from a family of derivative nonlinear Schrödinger equations. The main difference between the current work and those in the literature is that the coefficients are allowed to be functions of space and time. We identify the special forms of these variable coefficients which permit exact, solitary pulse solutions. The inclusion of damping or gain is necessary and the pulse then undergoes broadening or compression. The novel features of the Hirota method employed here are the special chirp factors and a time-dependent wavenumber. Special cases, e.g. an exponential dispersion coefficient, permit a 2-soliton solution, and further standard machinery like the Painlevé tests should be applied to study integrability. Preliminary calculations of other wave forms, e.g., period patterns, suggest that large classes of variable coefficient partial differential equations of the DNLS family are solvable. Further works would definitely be fruitful.

Acknowledgement

Partial financial support has been provided by the Research Grants Council through the contract HKU 7123/05E.

Appendix 1

The bilinear identities for Hirota derivatives can be obtained from simple, straightforward differentiation. Examples are:

$$D_z \exp[t\xi_1(z) + \xi_2(z)] \cdot \exp[t\eta_1(z) + \eta_2(z)] \\ = \{t[\xi_1'(z) - \eta_1'(z)] + \xi_2'(z) - \eta_2'(z)\} \cdot \exp\{t[\xi_1(z) + \eta_1(z)] + \xi_2(z) + \eta_2(z)\},$$

$$D_z \exp(a) \cdot m(z) \exp(b) = m \left[D_z \exp(a) \cdot \exp(b) - \frac{1}{m} \frac{\partial m}{\partial z} \exp(a + b) \right].$$

Appendix 2

The theta functions $\theta_n(x)$, $n = 1, 2, 3, 4$ in terms of the parameter q (the nome) are defined by^{22, 23)}

$$\theta_1(x) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)x, \quad \theta_2(x) = 2 \sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos(2n+1)x,$$

$$\theta_3(x) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nx, \quad \theta_4(x) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nx, \quad 0 < q < 1.$$

Relationships between the theta and elliptic functions are

$$\operatorname{sn}(u) = \frac{\theta_3(0)\theta_1(z)}{\theta_2(0)\theta_4(z)}, \quad \operatorname{cn}(u) = \frac{\theta_4(0)\theta_2(z)}{\theta_2(0)\theta_4(z)}, \quad \operatorname{dn}(u) = \frac{\theta_4(0)\theta_3(z)}{\theta_3(0)\theta_4(z)},$$

$$z = \frac{u}{\theta_3^2(0)}, \quad k = \frac{\theta_2^2(0)}{\theta_3^2(0)}. \quad (\text{A.1})$$

Arguments of the theta and elliptic functions are related by a scale factor. The modulus of the elliptic functions, k , is related to the theta constants by (A.1).

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Figure captions

Figure 1: Evolution of the bright 1-soliton solution of special case (A) with parameters $\delta = p = q = \xi = \eta = \gamma = 1$.

Figure 2: Evolution of the bright 1-soliton solution of special case (B) with parameters $\delta = 0.1, p = q = \xi = \eta = \gamma = 1$.

Figure 3: Evolution of the bright 1-soliton solution of special case (C) with parameters $\delta = p = q = \xi = \eta = \gamma = 1$.

Figure 4: Evolution of the bright 2-soliton solution of special case (A) with parameters $\delta = \gamma = 1, r = 1, R = 2$.

Figure 1

Figure 1 – Evolution of the bright 1-soliton solution of special case (A) with parameters $\delta = p = q = \xi = \eta = \gamma = 1$.

Figure 2

Figure 2 – Evolution of the bright 1-soliton solution of special case (B) with parameters $\delta = 0.1, p = q = \xi = \eta = \gamma = 1$.

Figure 3

Figure 3 – Evolution of the bright 1-soliton solution of special case (C) with parameters $\delta = p = q = \xi = \eta = \gamma = 1$.

Figure 4

Figure 4 – Evolution of the bright 1-soliton solution of special case (A) with parameters $\delta = \gamma = 1, r = 1, R = 2$.