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Kriging-Weighted Laplacian Kernels

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Abstract—Laplacian kernels, which are widely used as sharpening filters in image processing, are isotropic and tend to over-highlight fine details with a sharp discontinuity in images. To address this issue, this paper introduces a method that integrates anisotropic averaging with the Laplacian kernels. The proposed method can also be useful as a new type of image convolution for designing convolutional neural networks.

Index Terms—Image sharpening, convolution, Laplacian operators, geostatistics, ordinary kriging.

I. INTRODUCTION

Image convolution, which is the process of adding values of a pixel and its neighbors weighted by a filter kernel to every pixel, is not only a general well-adopted procedure in image processing but also plays a critical role in the design of convolutional neural networks [1], [2] and most widely used in deep learning [3], [4]. The purpose of convolution is to modify the spatial characteristics of an image, resulting in, such as smoothing, sharpening, enhancing, or edge highlighting. In general, the convolution of a kernel k and an image $f(x, y)$ can be expressed as

$$g(x, y) = k \otimes f(x, y) = \sum_{m=-a}^a \sum_{n=-b}^b k(m, n) f(x - m, y - n), \quad (1)$$

where $g(x, y)$ is the output image, and $\otimes$ denotes the convolution operator.

Laplacian kernels are widely applied for image sharpening. The Laplacian operator, which takes partial derivatives along the two spatial axes of an image $f(x, y)$, is defined as [5]

$$\begin{aligned} \nabla^2 f &= \frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2} \\ &= f(x + 1, y) + f(x - 1, y) + f(x, y + 1) + \\ &\quad f(x, y - 1) - 4f(x, y), \end{aligned} \quad (2)$$

which highlights sharp intensity transitions and reduces the effect of regions having slowly varying gray levels, resulting in the following 3×3 Laplacian filters

$$\mathbf{L}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

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where the left matrix is based on Eq. 2, and the right is an extension of Eq. 2 to include the diagonal elements.

To rectify the effect of featureless image background while sharpness is still reserved can be obtained by subtracting the Laplacian image from the original, giving [5]

$$f^*(x, y) = f(x, y) - \nabla^2 f. \quad (3)$$

Because $\mathbf{L}_1$ and $\mathbf{L}_2$ are derived from the second derivative of an image, which highlights regions of rapid intensity change, both kernels are sensitive to noise. To address this issue, it is suggested to Gaussian-smooth the image before applying the Laplacian. This procedure is known as the Laplacian of Gaussian (LoG), whose function is expressed as [6]

$$\Delta^2 G(x, y) = \frac{1}{\pi\sigma^4} \left(1 - \frac{x^2 + y^2}{2\sigma^2} \right) \exp \left(-\frac{x^2 + y^2}{2\sigma^2} \right), \quad (4)$$

where σ is the standard deviation,

All $\mathbf{L}_1$, $\mathbf{L}_2$, and LoG kernels yield isotropic results. Because an isotropic operator applies the rate of change equally in all directions of an image, it carries no particular sensitivity or bias toward different spatial orientations. An anisotropic topography has directional attributes that may be different at all points or directions of an image. Effort was spent on addressing this issue by minimizing the overall anisotropy produced by Laplacian kernels [7], or using local Laplacian kernels [8], [9].

As an attempt to consider anisotropic properties of an image convolved with Laplacian kernels, this paper introduces the use of kriging weights as a secondary kernel whose derivation is based on the statistical properties of an anisotropic semivariogram. Unlike having fixed values of coefficients for a kernel, the kriging weights vary according to changes in image semivariograms, giving different effects of convolution in images of different spatial statistics.

II. KRIGING-WEIGHTED LAPLACIAN OPERATORS

Let $z(a)$ be a random function of spatial location a , the ordinary kriging of geostatistics holds the following intrinsic hypothesis [10]:

$$E[z(a)] = \mu. \quad (5a)$$

$$\text{Var}[z(a) - z(a + h)] = E[\{z(a) - z(a + h)\}^2], \quad (5b)$$

$$= 2\gamma(h), \quad (5c)$$

where $E[\cdot]$, μ , and $\gamma(\cdot)$ denote expected value, mean, and semivariogram of the random function, respectively; and a and $a + h$ are any two spatial locations separated by distance h .

Let $z(a_i)$ and $z(a_j)$ be the values of two data points at two spatial locations a_i and a_j , respectively. It was shown that [11]

$$\gamma(a_i - a_j) = \gamma(a_i - a) + \gamma(a_j - a) - \text{Cov}[z(a_i) - z(a), z(a_j) - z(a)]. \quad (6)$$

To derive kriging weights in the setting of the Laplacian operators, let a_0 be the location of $f(x, y)$, a_1 of $f(x-1, y-1)$, a_2 of $f(x, y-1)$, a_3 of $f(x+1, y-1)$, a_4 of $f(x-1, y)$, a_5 of $f(x+1, y)$, a_6 of $f(x-1, y+1)$, a_7 of $f(x, y+1)$, and a_8 of $f(x+1, y+1)$, which are shown in Fig. 1.

a_1	a_2	a_3
a_4	a_0	a_5
a_6	a_7	a_8

Fig. 1: A 3×3 window and spatial locations of pixels for computing kriging estimate of pixel at center.

The value of $f(a_0)$ can be predicted using the best linear unbiased estimator of ordinary kriging as [11]

$$\hat{f}(a_0) = \sum_{i=1}^8 w_i f(a_i), \quad (7)$$

where w_i , $i = 1, \dots, 8$ are kriging weights that can be optimally determined by minimizing the estimation variance $\sigma^2(a_0)$:

$$\sigma^2(a_0) = \text{Var}[\hat{f}(a_0) - f(a_0)]. \quad (8)$$

Using Eq. 7, the estimate is done via the following constrained optimization

$$\sigma^2(a_0) = \text{Var} \left[\sum_{i=1}^8 w_i (f(a_i) - f(a_0)) \right], \quad (9)$$

which is subject to $\sum_{i=1}^8 w_i = 1$. Based on the following proof [11]

$$\text{Var} \left[\sum_i w_i z(a_i) \right] = \sum_i \sum_j w_i w_j \text{Cov}[z(a_i), z(a_j)], \quad (10)$$

Eq. 9 can be rewritten as

$$\sigma^2(a_0) = \sum_{i=1}^8 \sum_{j=1}^8 w_i w_j \text{Cov}[f(a_i) - f(a_0), f(a_j) - f(a_0)]. \quad (11)$$

Using Eq. 6,

$$\begin{aligned} \sigma^2(a_0) &= \sum_{i=1}^8 \sum_{j=1}^8 w_i w_j [\gamma(a_i - a_0) + \gamma(a_j - a_0) - \gamma(a_i - a_j)] \\ &= 2 \sum_{i=1}^8 w_i \gamma(a_i - a_0) - \sum_{i=1}^8 \sum_{j=1}^8 w_i w_j \gamma(a_i - a_j). \end{aligned} \quad (12)$$

The Lagrangian function for ordinary kriging estimate of $f(a_0)$ can be constructed as

$$L(w_1, \dots, w_8; \lambda) = \sigma^2(a_0) + 2\lambda \left(\sum_{i=1}^8 w_i - 1 \right), \quad (13)$$

where λ is a Lagrange multiplier.

Let $\gamma(a_i - a_j)$ now be denoted as $\gamma_{i,j}$. To minimize the variance of estimation, all first derivatives of the Lagrangian function with respect to w_i , $i = 1, \dots, 8$, and λ must be set to zero, which are

$$\frac{\partial L(w_1, \dots, w_8; \lambda)}{\partial w_i} = 2\gamma_{i,0} - 2\gamma_{i,j} + 2\lambda = 0, \quad (14a)$$

$$\frac{\partial L(w_1, \dots, w_8; \lambda)}{\partial \lambda} = \sum_{i=1}^8 w_i - 1 = 0. \quad (14b)$$

The above Lagrangian method results in the determination of unique kriging weights by solving the following set of simultaneous equations:

$$\begin{cases} \sum_{i=1}^8 w_i \gamma_{1,i} - \lambda = \gamma_{1,0} \\ \sum_{i=1}^8 w_i \gamma_{2,i} - \lambda = \gamma_{2,0} \\ \dots \\ \sum_{i=1}^8 w_i \gamma_{8,i} - \lambda = \gamma_{8,0} \\ \sum_{i=1}^8 w_i = 1 \end{cases} \quad (15)$$

To consider anisotropy in the semivariogram model, the experimental semivariogram measured in the horizontal direction of an image can be expressed as

$$\gamma(\Delta x) = \frac{1}{2N(\Delta x)} \sum_{\Delta x} [f(x, y) - f(x + \Delta x, y)]^2, \quad (16)$$

where $N(\Delta x)$ is the number of pixel pairs separated by Δx .

Likewise, for the vertical direction, the experimental semivariogram can be defined as

$$\gamma(\Delta y) = \frac{1}{2N(\Delta y)} \sum_{\Delta y} [f(x, y) - f(x, y + \Delta y)]^2, \quad (17)$$

where $N(\Delta y)$ is the number of pixel pairs separated by Δy .

For a matrix, the main diagonal is the line of elements running from the top-left to bottom-right corners; whereas the antidiagonal is the line of elements running from the top-right to bottom-left corners. Let ΔD^+ and ΔD^- indicate $(\Delta x, \Delta y)$ measured in the directions of the main diagonal and antidiagonal of an image, respectively. The experimental semivariograms measured in the main diagonal and antidiagonal directions can be obtained, respectively, as

$$\gamma(\Delta D^+) = \frac{1}{2N(\Delta D^+)} \sum_{\Delta D^+} [f(x, y) - f(x + \Delta x, y + \Delta y)]^2, \quad (18)$$

where $N(\Delta D^+)$ is the number of pixel pairs separated by $(\Delta x, \Delta y)$ in the main-diagonal direction, and

TABLE I: Derivation of a 3×3 anisotropic kriging-weighted kernel

$$\mathbf{A} = \begin{bmatrix} 0 & \gamma_{1,2}(\Delta x) & \gamma_{1,3}(\Delta x) & \gamma_{1,4}(\Delta y) & \gamma_{1,5}(\Delta D^+) & \gamma_{1,6}(\Delta y) & \gamma_{1,7}(\Delta D^+) & \gamma_{1,8}(\Delta D^+) & 1 \\ & 0 & \gamma_{2,3}(\Delta x) & \gamma_{2,4}(\Delta D^-) & \gamma_{2,5}(\Delta D^+) & \gamma_{2,6}(\Delta D^-) & \gamma_{2,7}(\Delta y) & \gamma_{2,8}(\Delta D^+) & 1 \\ & & 0 & \gamma_{3,4}(\Delta D^-) & \gamma_{3,5}(\Delta y) & \gamma_{3,6}(\Delta D^-) & \gamma_{3,7}(\Delta D^-) & \gamma_{3,8}(\Delta y) & 1 \\ & & & 0 & \gamma_{4,5}(\Delta x) & \gamma_{4,6}(\Delta y) & \gamma_{4,7}(\Delta D^+) & \gamma_{4,8}(\Delta D^+) & 1 \\ & & & & 0 & \gamma_{5,6}(\Delta D^-) & \gamma_{5,7}(\Delta D^-) & \gamma_{5,8}(\Delta y) & 1 \\ & & & & & 0 & \gamma_{6,7}(\Delta x) & \gamma_{6,8}(\Delta x) & 1 \\ & & & & & & 0 & \gamma_{7,8}(\Delta x) & 1 \\ & & & & & & & 0 & 1 \\ & & & & & & & & 0 \end{bmatrix}$$

$$\mathbf{w} = [w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6 \quad w_7 \quad w_8 \quad -\lambda]^T$$

$$\mathbf{b} = [\gamma_{1,0}(\Delta D^+) \quad \gamma_{2,0}(\Delta y) \quad \gamma_{3,0}(\Delta D^-) \quad \gamma_{4,0}(\Delta x) \quad \gamma_{5,0}(\Delta x) \quad \gamma_{6,0}(\Delta D^-) \quad \gamma_{7,0}(\Delta y) \quad \gamma_{8,0}(\Delta D^+) \quad 1]^T$$

 Fig. 2: Original and filtered images using Laplacian (L_2), LoG, and kriging-weighted Laplacian (ω_2) kernels, where first, second, and third rows are “Lena”, “Bridge”, and “Nature” images, respectively.

$$\gamma(\Delta D^-) = \frac{1}{2N(\Delta D^-)} \sum_{\Delta D^-} [f(x, y) - f(x - \Delta x, y + \Delta y)]^2, \quad (19)$$

where $N(\Delta D^-)$ is the number of pixel pairs separated by $(\Delta x, \Delta y)$ in the antidiagonal direction.

Using the form of Eq. 15, the kriging weights can be determined by solving for $\mathbf{w}$ in the following ordinary kriging system represented in matrix form as

$$\mathbf{A}\mathbf{w} = \mathbf{b}, \quad (20)$$

where $\mathbf{A}$, $\mathbf{w}$, and $\mathbf{b}$ are given in Table I. It can be seen that $\mathbf{A}$ and $\mathbf{b}$ contain terms of statistical distances that incorporate anisotropy, spatial correlation, and clustering information; and therefore provide the set of weights for estimation according to particular pattern of spatial continuity [12].

As a result, the 3×3 kriging-weighted kernel for Laplacian filter L_2 is of variable coefficients, depending on the

TABLE II: Measures of image sharpness and relative visual quality.

Attributes	Original	L_1	ω_1	L_2	ω_2	LoG
"Lena" image						
Sharpness	15.8020	15.9595	20.4523	22.4898	20.1075	17.1046
$ \Delta_r $	0	0.0742	0.0409	0.1031	0.0129	0.0554
"Bridge" image						
Sharpness	7.5394	7.6270	8.6489	13.7154	9.7726	9.1889
$ \Delta_r $	0	0.2205	0.0643	0.1763	0.0142	0.1265
"Nature" image						
Sharpness	10.8730	12.4206	15.5980	25.6434	11.1216	15.5762
$ \Delta_r $	0	0.0693	0.0030	0.4656	0.0082	0.1801

anisotropic semivariogram of an image, where, being similar to the Laplacian kernel, the weight at the kernel center is set to -1 to impose that the sum of all elements of the kernel is zero so that the convolution of a homogeneous region is zero. Likewise, the kriging-weighted kernel for Laplacian filter $\mathbf{L}_1$ can be constructed by considering only $f(a_2)$, $f(a_4)$, $f(a_5)$, and $f(a_7)$ for the estimate of $f(a_0)$ to compute corresponding w_1^* , w_2^* , w_3^* , and w_4^* , respectively, $\sum_{i=1}^4 w_i^* = 1$; whereas weights for the diagonal elements are set to zero. The corresponding kriging system yields $\mathbf{A}^*$, $\mathbf{w}^*$, and $\mathbf{b}^*$ of size 5×5 , 5×1 , and 5×1 , respectively. Thus, kriging-weighted kernels for $\mathbf{L}_1$ and $\mathbf{L}_2$ are defined, respectively, as

$$\mathbf{K}_1 = \begin{bmatrix} 0 & w_1^* & 0 \\ w_2^* & -1 & w_3^* \\ 0 & w_4^* & 0 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} w_1 & w_2 & w_3 \\ w_4 & -1 & w_5 \\ w_6 & w_7 & w_8 \end{bmatrix}.$$

Because convolution is commutative: $\alpha \otimes \beta = \beta \otimes \alpha$, and associative: $\alpha \otimes (\beta \otimes \kappa) = (\alpha \otimes \beta) \otimes \kappa$, where α , β , and κ are kernels, multistage filtering therefore can be performed in a single operation. Instead of having an image convolved with a Laplacian kernel first, then the result convolved with the kriging-weighted kernel, the convolution of an image with the kriging-weighted Laplacian kernel can be carried out as

$$g(x, y) = \omega_i \otimes f(x, y), \quad (21)$$

where ω_i is a hybrid filter or kriging-weighted Laplacian kernel that is defined as

$$\omega_i = \mathbf{K}_i \otimes \mathbf{L}_i, \quad i = 1, 2. \quad (22)$$

III. RESULTS

Fig. 2 shows the 225×225 "Lena" (a), 195×259 "bridge" (e), and 182×278 "nature" (i) images used to illustrate the performance of the kriging-weighted Laplacian kernels. For the the LoG, $\sigma = 0.5$ and the filter size is 5×5 . Fig. 2 also shows images filtered by Laplacian L_2 , LoG, and kriging-weighted Laplacian ω_2 kernels. For the "Lena" image, it can be seen that while both L_2 and LoG sharpened the image but also highlighted pixels of high changes in intensity, yielding undesirable visual effects such as pixels of the eyes and hair; the ω_2 kernel provided a sharpened image with a much better visual appearance. Similar observations can be made for the "bridge" image, particularly with the suspension cables. For the "nature" image, the L_2 kernel yielded most dominantly

noise-sensitive results, while the ω_2 kernel could balance both sharpness and noise suppression.

A measure of sharpness in grayscale images using gradients [13], and the perception-based image quality evaluator (PIQE) [14], which is a no-reference image-quality metric, were used to evaluate both sharpness and quality of the images filtered by the Laplacian (L_1 and L_2), LoG, and kriging-weighted Laplacian (ω_1 and ω_2) filters. The PIQE score is inversely correlated to the perceptual quality of an image in the range [0, 100]. A lower score indicates higher perceptual image quality. Here, the absolute relative difference in quality between the original and a filtered image is defined as $|\Delta_r| = |[PIQE(f) - PIQE(f^*)]| / PIQE(f)$, where f and f^* are original and filtered images, respectively.

Table II shows the measure of sharpness and $|\Delta_r|$ of the three original and corresponding images filtered by the Laplacian (L_1 and L_2), LoG, and kriging-weighted Laplacian (ω_1 and ω_2) kernels. All filtered images increased the sharpness of the three original images. While being able to increase sharpness in images, both ω_1 and ω_2 provided results that have the most similar naturally visual appearance to the original images by avoiding over-highlighting fine details with a sharp discontinuity. Particularly, as seen in Fig. 2 (b) and (j) and corresponding results shown in Table II, L_2 yielded highest measures of image sharpness but resulted in most unnatural visual expression. The LoG filter highlights the fine details to a lesser degree than the Laplacian filters.

IV. CONCLUDING REMARKS

A geostatistical approach for incorporating anisotropic information into the isotropic Laplacian filters has been presented. Results obtained from the examples illustrate the effective performance of the kriging-weighted Laplacian kernels. Kriging-weighted kernels of difference sizes can be derived for convolving with other types of filters, and used as an adaptive method by considering local directional semivariograms in an image. Furthermore, kriging-weighted kernels can be applied as a stand-alone filters for image smoothing and enhancement.

Code availability: MATLAB codes implemented in this study are available at: <https://sites.google.com/view/tuan-d-pham/codes>, under the name "Kriging-weighted Laplacian kernels".

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