This item was submitted to [Loughborough's Research Repository](#) by the author.
Items in Figshare are protected by copyright, with all rights reserved, unless otherwise indicated.

Homoclinic intersections and Mel'nikov method for perturbed sine-Gordon equation

PLEASE CITE THE PUBLISHED VERSION

LICENCE

CC BY-NC-ND 4.0

REPOSITORY RECORD

Rothos, Vassilios M.. 2019. "Homoclinic Intersections and Mel'nikov Method for Perturbed Sine-gordon Equation". figshare. <https://hdl.handle.net/2134/689>.

Homoclinic Intersections and Mel'nikov Method for Perturbed sine - Gordon Equation*

Vassilios M. Rothos[†]
Department of Mathematical Sciences
Loughborough University
Loughborough LE11 3TU UK
E-mail: V.M.Rothos@lboro.ac.uk

Abstract

We describe and characterize rigorously the homoclinic structure of the perturbed sine-Gordon equation under periodic boundary conditions. The existence of invariant manifolds for a perturbed sine-Gordon equation is established. Mel'nikov method, together with geometric analysis are used to assess the persistence of the homoclinic orbits under bounded and time-periodic perturbations.

1 Introduction

Homoclinic orbits have long been identified as a possible source of chaos in nonlinear finite-dimensional dynamical systems. For example, when an integrable 2-dimensional system, containing a smooth connection between the invariant manifolds of a saddle fixed point is perturbed, this connection is generally broken and the invariant manifolds intersect transversally. This yields isolated homoclinic orbits of the perturbed system, near which one expects to find shift map embedding and chaos.

In the theory of finite-dimensional dynamical systems, a standard method to convert numerical experiments of chaotic dynamics into a rigorous mathematical description is to construct Smale horseshoes in a neighborhood of a homoclinic orbit e.g (Wiggins 1988) and then use these horseshoes to establish the existence of an invariant set on which the motion is topologically equivalent to a Bernoulli shift on a number of symbols. Such constructions begin with the existence, persistence and breaking of homoclinic orbits, which in turn are obtained through Mel'nikov analysis.

However, in the case of infinite dimensional dynamical systems the situation is more complicated (Temam 1997). As is well-known, realistic physical systems are usually modeled by nonlinear PDEs, whose chaotic behavior is generally very difficult to study analytically. In the last 10 years, D.W.McLaughlin and collaborators have studied the periodic spectral transform of integrable soliton systems and constructed representations of global objects such as homoclinic orbits and whiskered tori, extending the Mel'nikov analysis to the NLS equation

*Dynamical Systems-An International Journal, 2001 (in press)

[†]On leave from Mathematics Institute, University of Warwick, Coventry CV4 7AL, U.K

(McLaughlin and Overman 1995, McLaughlin and Shatah 1998), as well as a system of coupled NLS (Rothos and Bountis 1998, Rothos 1999). Recently, G. Haller has been developing an alternative perspective, under certain conditions, he proved the existence of manifolds of multi-pulse Silnikov type orbits homoclinic to the critical torus of NLS equation. He has applied to finite dimensional discretizations of the perturbed NLS equation (Haller 1998) and to the PDE case (Haller 1999a, Haller 1999b and Haller *et al.* 1999c).

We consider the perturbed sine-Gordon equation (SG)

$$u_{tt} - c^2 u_{xx} - \sin u = \varepsilon(bg(t, u) - au_t) \quad (1)$$

with even periodic spatially boundary conditions, ε a small perturbation parameter, $0 < \varepsilon \ll 1$ and $a, b > 0$. The perturbation term $g(t, u)$ is a smooth nonlinear function satisfying $g(t, 0) = 0$, $\partial_u g(t, 0) = 0$ and g is time-periodic with period T .

The unperturbed sine-Gordon equation

$$u_{tt} - c^2 u_{xx} - \sin u = 0 \quad (2)$$

is an infinite-dimensional Hamiltonian system (Temam, 1997) and for the particular range of parameter $c \in (1/2, 1)$ there are two exactly unstable modes. We denote by $\mathbb{H}^1$ the Sobolev space of functions of x which are L -periodic, even and square integrable with square integrable first derivative on $[0, L)$

$$\mathbb{H}^1 = \left\{ \mathbf{u} = (u, u_t) : \mathbf{u}(-x) = \mathbf{u}(x) = \mathbf{u}(x + L), \quad \int_0^L |\partial_x \mathbf{u}|^2 dx < \infty \right\} \equiv \mathcal{X} \quad (3)$$

The system of sine-Gordon models the phase difference between two superconducting layers in a Josephson junction, (Pederson, 1982). Equation (1) can also be thought of as a model for the continuum limit of a chain of coupled pendula, including damping and driving, with $u(x, t)$ the angle from the vertical down position of the pendulum at the point x along the chain.

On the phase space $\mathbb{H}^1$, the Hamiltonian of the equation (1) is given by

$$H : \mathbb{H}^1 \longrightarrow \mathbb{R}, \quad H(\mathbf{u}) = \int_0^L \left(\frac{1}{2}(u_t^2 + c^2 u_x^2) + (1 + \cos u) \right) dx$$

with an infinite number of commuting constants of motion. The common level sets of these constants of motion are generically infinite dimensional tori of maximal dimension. These constants of the motion have linearly independent gradients, at certain critical tori, which become linearly dependent when the critical tori have dimension lower than maximal (but otherwise arbitrary). Critical tori can be either stable or unstable. In the unstable case, the constant level sets have a saddle structure in a neighborhood of the critical torus. An unstable critical torus has an unstable manifold of infinite dimension. If, a phase point on this manifold approaches the critical torus as $t \rightarrow \pm\infty$, such a phase point is said to lie on an orbit which is homoclinic to the critical torus.

Bishop *et al.* 1986 have studied the sine-Gordon attractor numerically for $g(t) = \Gamma \cos \omega t$ and the parameter region $0.1 \leq \varepsilon \Gamma, \varepsilon a \leq 1$. In that region the so-called “breather” solutions (i.e localized in space and periodic in time) interact and qualitatively the dynamics appears to be well described by a four mode truncation, (Kovacic and Wiggins 1992).

Ercolani *et al.*, 1990 have explored the existence of homoclinic orbits for the unperturbed system. To this end, they used the fact that system (1) admits a Lax pair representation and derived explicit formulas for all homoclinic orbits in terms of Bäcklund transformations. In these formulas, the use of the imaginary double points of the spectrum of Lax operator is very important, since it describes the homoclinic orbits in infinite-dimensional space.

However, it is very important to study the behavior of the trajectories of the perturbed problem (1). Especially, it is interesting to ask: under which perturbations do the homoclinic orbits persist and how does one establish their persistence? In the finite-dimensional system, the method which is most often used to answer such questions is the one established by Mel'nikov 1963.

In this paper, we extend and apply this geometric method for the perturbed system (1), to establish the persistence of homoclinic orbits through the transversal intersection of invariant manifolds of singular fixed point. In section 2, we describe the symplectic structure and briefly outline the theory of the $\varepsilon = 0$ (integrable) case. In fact, in the integrable setting, the associated spectral problem has been used to classify all instabilities and to construct representations of their homoclinic orbits and we derive an analytic expression for the gradient of the Floquet discriminant. Section 3 is devoted to describe the invariant manifolds for the perturbed system (1). We study the spectrum of the linearized system in the neighborhood of the uniform solution $u = (0, 0)$. The linearized operator admits two pair of real eigenvalues $\lambda_0^\pm, \lambda_1^\pm$ and infinite number of imaginary eigenvalues $\lambda_j^\pm = \pm ia_j$, $j \geq 2$, thus the uniform solution $(0, 0)$ of the sine-Gordon equation in the whole space becomes a singular point of type saddle-center. We construct the corresponding Poincaré map for the time-periodic perturbed system (1) and analyze the dynamics on the perturbed manifolds.

We derive, in section 4, a formula for the distance between the codimension two center-stable and unstable manifold of the singular point of the perturbed system. We prove the existence of persistent homoclinic orbit to the saddle-focus point $O_\varepsilon = O + O(\varepsilon)$ for the system (1) with $g(t) = \Gamma \cos \omega t$, this problem has been studied numerically by Bishop *et al.* 1986.

The derivation of the distance between the manifolds is an extension of Mel'nikov methods in infinite dimensional space. We prove the convergence of the Mel'nikov integral. The existence of simple zeros of the distance function implies a transversal intersection of those manifolds, as long as certain additional non-degeneracy conditions are satisfied. We apply the finite dimensional reduction method in order to establish the existence of homoclinic orbits for the perturbed SG equation. We derive the second condition which allow us to measure the distance between the perturbed manifolds on the region of invariant plane II. The simple zeros of Melnikov integral and the second condition correspond to orbits that are asymptotic to the x -independent, time-periodic orbit originated from the point O_ε in backward time following the curve which lies in the unstable manifold and in forward time following the curves on the stable manifold. Finally we note that the geometrical method, which was applied to prove the persistence of homoclinic orbits for nearly-integrable double-discrete sine-Gordon systems, see Rothos 2001.

2 Discussion of the Integrable System

Hamiltonian Formulation

Consider the nonlinear wave equation (1) on a segment of length L with even spatial periodic boundary conditions in a symplectic space $\mathbb{H}^1$ with scalar product $\langle \cdot, \cdot \rangle$, endowed by a weakly symplectic form Ω given by

$$\Omega(\mathbf{u}, \mathbf{u}^1) = \frac{1}{L} \int_0^L (u_t^1 u - u_t u^1) dx, \quad \forall \mathbf{u}, \mathbf{u}^1 \in \mathbb{H}^1 \quad (4)$$

Let us define a domain $\mathcal{P} \subset \mathbb{H}^1$ and a function $f \in C^1(\mathcal{P})$ and let $\nabla_\alpha f \in \mathbb{H}^1$ be the gradient of f with respect to the inner product in $\mathbb{H}^1$

$$Df(\alpha)w := \frac{1}{L} \int_0^L \nabla_\alpha f(\alpha(x))w(x) dx$$

Denoting by

$$H(\mathbf{u}) = \int_0^L \left(\frac{1}{2}(u_t^2 + c^2 u_x^2) + (1 + \cos u) \right) dx \in C^1(\mathbb{H}^1)$$

the Hamiltonian vector field V_H with Hamiltonian H and the map $V_H : \mathcal{P} \rightarrow \mathbb{H}^1$ such that $V_H = J\nabla H(\mathbf{u})$, the equation (1) takes the form

$$\mathbf{u}_t = J\nabla H(\mathbf{u}) \quad (5)$$

with $\nabla H(\mathbf{u}) = \left(\frac{\delta H}{\delta u}, \frac{\delta H}{\delta u_t} \right)$. Defining the Poisson bracket

$$\{ \mathcal{F}, \mathcal{G} \} = \int_0^L \left(\frac{\delta \mathcal{F}}{\delta u} \frac{\delta \mathcal{G}}{\delta u_t} - \frac{\delta \mathcal{F}}{\delta u_t} \frac{\delta \mathcal{G}}{\delta u} \right) dx \quad (6)$$

we note that the evolution of any functional $\mathcal{F}$, under the SG flow, is governed by

$$\frac{d\mathcal{F}}{dt} = \{ \mathcal{F}, H \} \quad (7)$$

Obviously, the Hamiltonian H is conserved by the SG flow. The SG equation admits an infinite family of conserved functionals in involution with respect to the Poisson bracket (6). This fact allows the SG to be solved with the inverse scattering transform eg. (Faddeev and Takhtajan 1987).

Lax Pair and Homoclinic Orbits

The phase space of (1) can be described in terms of the spectrum of the linear operator (for a detailed description see Ercolani *et al.* 1990):

$$L^{(x)}(\mathbf{u}, \zeta) = -ic\sigma_2 \frac{d}{dx} + \frac{i}{4}w\sigma_1 - \frac{1}{16\zeta}e^{iu\sigma_3} - \zeta I \quad (8)$$

The SG equation arises as the compatibility conditions of the following Lax pair of linear operators

$$L^{(x)}(\mathbf{u}, \zeta)\phi = 0, \quad L^{(t)}(\mathbf{u}, \zeta)\phi = 0 \quad (9)$$

where

$$L^{(t)}(\mathbf{u}, \zeta) = -i\sigma_2 \frac{d}{dt} + \frac{i}{4} \mathbf{w} \sigma_1 + \frac{1}{16\zeta} e^{iu\sigma_3} - \zeta I \quad (10)$$

where σ_i denote the Pauli matrices, I is the identity matrix, $\mathbf{w} = (cu_x + u_t)$, $\mathbf{u}(x, t) = (u, u_t)$ the potential and $\zeta \in \mathbb{C}$ denotes the spectral parameter.

The spectrum of $L^{(x)}$, denoted by

$$\sigma(L^{(x)}) := \{\zeta \in \mathbb{C} : L^{(x)}\phi = 0, |\phi| < \infty, \forall x\}, \quad (11)$$

characterizes the solution of the SG, the potential $\mathbf{u}$ also satisfies the SG and is of spatial period L . To achieve the spectral analysis of $L^{(x)}$, we use Floquet Theory as follows:

Let us consider the fundamental matrix $M(x, x_0; \mathbf{u}, \zeta)$ of $L^{(x)}$ as:

$$L^{(x)}(\mathbf{u}, \zeta)M = 0, \quad M(x_0, x_0; \mathbf{u}, \zeta) = I \quad (12)$$

and the Floquet discriminant

$$\Delta(\mathbf{u}, \zeta) := \text{trace} M(x_0 + L, x_0; \mathbf{u}, \zeta)$$

The spectrum of $L^{(x)}$ is given by the following condition

$$\sigma(L^{(x)}) := \left\{ \zeta \in \mathbb{C} : \Delta(\mathbf{u}, \zeta) \in \mathbb{R}, \quad |\Delta(\mathbf{u}, \zeta)| \leq 2 \right\} \quad (13)$$

where the discriminant Δ is known to be analytic in both its arguments and is invariant along solutions of the SG equation:

$$\frac{d}{dt} \Delta(\mathbf{u}(t), \zeta) = 0$$

Thus for each ζ , Δ is a sine-Gordon constant of the motion Poisson commutes with the integrable Hamiltonian H . Moreover, by varying ζ , we generate all of the sine-Gordon constants. Thus, the level sets $M_{\mathbf{u}}$ are defined through Δ :

$$M_{\mathbf{u}} = \left\{ \mathbf{v} \in \mathbb{H}^1 : \Delta(\mathbf{u}, \zeta) = \Delta(\mathbf{v}, \zeta), \quad \forall \zeta \in \mathbb{C} \right\}$$

We have the following elements of the $\sigma(L^{(x)})$ which determine the nonlinear mode content of solutions of sine-Gordon equations and the dynamical stability of these modes:

As an example let us compute the spectrum for the spatially and temporally uniform solution $\mathbf{u} = (0, 0)$. In this case, the first system of (9) has constant coefficients and is readily solved. The Floquet discriminant is given by

$$\Delta(\mathbf{u}, \zeta) = 2\cos \left[\left(\zeta + \frac{1}{16\zeta} \right) \frac{L}{c} \right] \quad (14)$$

Making use of the definition above, the continuous spectrum is given by the entire real axis as well as the curve $|\zeta|^2 = \frac{1}{16}$ in the complex ζ -plane. The simple periodic spectrum is given by

$$\zeta = \frac{1}{2} \left(\frac{jc\pi}{L} \pm \sqrt{\left(\frac{jc\pi}{L} \right)^2 - \frac{1}{4}} \right), \quad j \in \mathbb{Z} \quad (15)$$

where $\zeta = \zeta_{\pm}^s$ satisfy the conditions

$$\Delta(\mathbf{u}, \zeta) \Big|_{\zeta^s = \zeta_{\pm}^s} = \pm 2, \quad \frac{d\Delta}{d\zeta} \Big|_{\zeta^s = \zeta_{\pm}^s} \neq 0$$

Each of these points is a double point embedded in the continuous spectrum and becomes complex if

$$0 \leq \left(\frac{2\pi jc}{L} \right)^2 \leq 1 \quad (16)$$

The condition (16) is exactly the same as the condition for linearized instability (see section 3 below) and the complex double point is given by

$$\zeta^d = \frac{1}{4} \exp[i\beta], \quad \text{with} \quad \beta = \tan^{-1} \frac{L}{2j\pi c} \sqrt{1 - \left(\frac{2jc\pi}{L} \right)^2}, \quad \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \quad (17)$$

where ζ^d satisfies the following conditions

$$\Delta(\mathbf{u}, \zeta) \Big|_{\zeta^d} = \pm 2, \quad \frac{d\Delta}{d\zeta} \Big|_{\zeta^d} = 0, \quad \frac{d^2\Delta}{d\zeta^2} \Big|_{\zeta^d} \neq 0$$

Making use of the fact that $(0, 0)$ satisfies the SG equation and the spectrum $\sigma(L^{(x)})$ is invariant in t , provides a countable number of constants of motion. Let $u(x, t)$ denote a solution of the SG equation which is periodic in x with period L and $U(x, t)$ denote one of the Bäcklund transform of u . Then their Floquet discriminant satisfies

$$\Delta(\zeta, U) = \Delta(\zeta, u), \quad \forall \zeta \in \mathbb{C}$$

that is U and u lie on the same iso-spectral set in $\mathbb{H}^1$. When $u \in \mathbb{H}_{NQ}^1$ (the N -phase quasiperiodic waves), one component of the level set $M_{\mathbf{u}}$ is an N -torus $\mathbb{T}^N = M_{\mathbf{u}} \times \mathbb{H}_{NQ}^1$. Let $u \in \mathbb{T}^N$ have a fully complex double point ζ_d , as an example we return to the case $O = (0, 0)$ with $\zeta_d = \frac{1}{4}e^{i\beta}$. Let $U \in M_{\mathbf{u}}$ as defined by

$$U(x, t) = 4 \tan^{-1} \left(\tan \beta \cos(xc \cos \beta) \operatorname{sech}(t \sin \beta) \right) \quad (18)$$

Then

- i. $U \notin \mathbb{T}^N$
- ii. U is homoclinic to $\mathbb{T}^N$, i.e there exists $U^{\pm} \in \mathbb{T}^N$ such that $\|U(x, t) - U^{\pm}(x, t)\| \rightarrow 0$ as $t \rightarrow \pm\infty$.

The expression (18) is homoclinic to the uniform solution $(0, 0)$:

$$U(x, t) = O[\exp(-\sqrt{1 - (2\pi c/L)^2} |t|)], \quad \text{as } t \rightarrow \pm\infty \quad (19)$$

The solution (18) is a homoclinic orbit of the integrable SG equation and is related to the famous soliton type solution known as the “breather”, in the sense that it may be called a “spatial breather”, since it is localized in t and periodic in space.

The gradient of Floquet Discriminant

Let us consider the special solution of the unperturbed sine-Gordon, $u = (0, 0)$. In this case the solutions of system (9) are given by

$$f^\pm(x, t) = \exp[\pm i(\frac{k(\zeta)x}{c} + \lambda(\zeta)t)] \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \quad (20)$$

where $k(\zeta) = \zeta + \frac{1}{16\zeta}$, $\lambda(\zeta) = \zeta - \frac{1}{16\zeta}$. For these solutions the spectrum of $L^{(x)}$ has a double complex point at $\zeta_d = \nu = \frac{1}{4}\exp[i\beta]$.

Let $m(x, x', t, \zeta, O) = [f_{ij}]$ denote the fundamental matrix of (9) at (ζ, O) , ψ the general solution of Lax pair system and $U(x, t)$ the homoclinic orbit. Let $M(x, x', t, \zeta, U)$ denote the fundamental matrix of (9) at $(\zeta, U(x, t))$. Then, for $\zeta^2 \neq \nu^2$, M , m are related by

$$M(x, t, \zeta, U) = G(x, t; \zeta, \nu) m(x, x', t, \zeta, O) G^{-1}(x, t; \zeta, \nu) \quad (21)$$

where the matrix G is given by

$$G(x, t; \zeta, \nu) = \begin{pmatrix} -\nu\psi_1\psi_2^{-1} & \zeta \\ -\zeta & \nu\psi_2\psi_1^{-1} \end{pmatrix} \quad (22)$$

Using the fact that the homoclinic orbit (18) is generated from O with a composition of two Bäcklund transformation (at $\zeta = \nu$ and $\zeta = -\nu^*$). It follows after some manipulation that the fundamental matrix M evaluated on U given by

$$M(x, t, \zeta, U) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \quad (23)$$

where

$$\begin{aligned} M_{11} &:= \frac{i}{2\sin 2\beta} \left[(e^{-2i\beta} f_{22} - e^{2i\beta} f_{11}) + \left(f_{12} \frac{\psi_2}{\psi_1} - f_{21} \frac{\psi_1}{\psi_2} \right) \right] \\ M_{12} &:= \frac{-i}{2\sin 2\beta} \left[\frac{\psi_2}{\psi_1} (f_{11} - f_{22}) - e^{2i\beta} f_{12} \left(\frac{\psi_2}{\psi_1} \right)^2 + e^{-2i\beta} f_{21} \right] \\ M_{21} &:= \frac{-i}{2\sin 2\beta} \left[\frac{\psi_1}{\psi_2} (f_{11} - f_{22}) + e^{2i\beta} f_{21} \left(\frac{\psi_1}{\psi_2} \right)^2 - e^{-2i\beta} f_{12} \right] \\ M_{22} &:= \frac{i}{2\sin 2\beta} \left[(e^{-2i\beta} f_{11} - e^{2i\beta} f_{22}) - \left(f_{12} \frac{\psi_2}{\psi_1} - f_{21} \frac{\psi_1}{\psi_2} \right) \right] \end{aligned} \quad (24)$$

and ψ_1, ψ_2 denotes any fixed solutions of linear system (9). At this point we derive the expressions of the gradient of Floquet discriminant Δ .

Lemma 2.1 (Rothos 2000) *The gradient of Δ with respect to u, u_t evaluated on the homoclinic solutions U is given by*

$$\begin{aligned} \frac{\delta\Delta}{\delta u}(U, \zeta_d) &= -cD_x \left[\frac{\delta\Delta}{\delta u_t}(U) \right] + \frac{ie^{-i\beta}}{4c} \left\{ \left(M_{12}^L M_{11}^2 + (M_{22}^L - 1) M_{11} M_{12} - M_{12}^2 \right) \exp[iU] \right. \\ &\quad \left. + \left(M_{21}^L M_{22}^2 - M_{12}^L M_{21}^2 + (M_{11}^L - M_{22}^L) M_{21} M_{22} \right) \exp[-iU] \right\} \end{aligned}$$

$$\frac{\delta \Delta}{\delta u_t}(U, \zeta_d) = -\frac{i}{4c} \left(\frac{1}{2 \sin 2\beta} \right)^2 \left[E \frac{\psi_2}{\psi_1} + H \frac{\psi_1}{\psi_2} \right] \quad (25)$$

where $M_{ij}^L = M_{ij}(L)$, $M_{ij} = M_{ij}(x)$ (cf. (24)), $F_{\pm} = (2c)^{-1} L \cos \beta \pm \beta$

$$\begin{aligned} E &= \frac{1}{A_1} \left[2(\cos F_+ - 2\cos F_-) \sinh 2t + \sin F_- \sin 2x \right. \\ &\quad \left. + i(\sinh 2t \sin F_+ - 2(\cos F_+ - 2\cos F_-) \sin 2x) \right] \\ H &= \frac{1}{A_2} \left[- (2\sinh 2t \cos F_- + \sin 2x \sin 2F_-) \right. \\ &\quad \left. + i(\sinh 2t \sin F_- - 2\cos F_- \sin 2x) \right] \end{aligned} \quad (26)$$

and

$$\begin{aligned} F_{\pm} &= \frac{\cos \beta}{2c} L \pm \beta, \quad \beta = \tan^{-1} \frac{L}{2\pi c} \sqrt{1 - \left(\frac{2c\pi}{L} \right)^2}, \quad x := \frac{\cos \beta}{2c} x, \quad t := \frac{\sin \beta}{2} t \\ A_1 &= 2(\cos^2 x \cosh^2 t - \sin^2 x \sinh^2 t), \quad A_2 = 2(\cos^2 x \sinh^2 t - \sin^2 x \cosh^2 t) \end{aligned}$$

3 The geometry of the perturbed system

In this section, we study the invariant manifolds for the perturbed sine-Gordon equation. There exists a two dimensional subspace $\tilde{\mathcal{X}}$

$$\tilde{\mathcal{X}} = \left\{ u \in \mathbb{H}^1 : u_x = 0 \right\}$$

such that the set $\Pi = \tilde{\mathcal{X}} \cap \mathbb{H}^1$ is invariant under the flow of the perturbed system. The manifold Π is real symplectic with a nondegenerate 2-form Ω_{Π} which arises from the restriction of the symplectic structure to Π . For $\varepsilon = 0$, the system (1) restricted to Π becomes a 1 degree-of-freedom completely integrable Hamiltonian system, pendulum system, for which we know its dynamics. Furthermore, the uniform solution $O = (0, 0)$ is a saddle type for the reduced pendulum system. On the whole space this uniform solution becomes a singular point of type saddle-center. The breather solution $U(x, t)$ of the full system approaches this set when $t \rightarrow \pm\infty$. Starting with the space Π , we are in a position to follow the trajectories of (1) and study the behavior of its solution in the infinite dimensional space.

The linearized system of the perturbed sine-Gordon in the neighborhood of the solution $O = (0, 0) \in S$ takes the following form

$$\mathbf{r}_t = \mathcal{L}_0 \mathbf{r} + \mathcal{K}(\mathbf{r}) \quad (27)$$

where $\mathbf{r} = (r, r_t)$

$$\mathcal{L}_0 = \begin{pmatrix} 0 & 1 \\ c^2 \partial_x^2 + 1 & 0 \end{pmatrix}, \quad \mathcal{K}(\mathbf{r}) = \begin{pmatrix} 0 \\ -\varepsilon a r_t - \tilde{G}(t, r; \varepsilon) \end{pmatrix}$$

and the function $\tilde{G}$ is defined as follows:

$$\tilde{G} = \varepsilon g(t, O + \mathbf{r}) + \sin r - r = O(r^3) \quad (28)$$

We can also combine the two equations (27) in a single equation,

$$r_{tt} - c^2 r_{xx} - r + \varepsilon a r_t = \tilde{G}(t, r) \quad (29)$$

Let $\varepsilon = 0$, (29) becomes

$$r_{tt} - c^2 r_{xx} - r = 0, \quad c^2 \in \left(\frac{1}{4}, 1\right) \quad (30)$$

Substituting $r = \hat{r}(t)e^{ik_j x}$ with $k_j = \frac{2\pi j}{L}$, and j an arbitrary integer we obtain from (30)

$$\hat{r}'' - (1 - c^2 k_j^2) \hat{r} = 0$$

This shows that the j -th mode grows exponentially if $c^2(2\pi j/L)^2 < 1$, $j = 0, 1$. Using now the even spatial periodic boundary condition we see that our problem has two unstable directions, corresponding to the unstable directions of the saddle point $(0, 0)$ as shown in Fig. 1: The zeroth mode in the plane Π $\alpha_0 = 1$, which is the most unstable in the sense that it grows at the fastest rate and $j = 1$, $\alpha_1 = \sqrt{1 - (2\pi c/L)^2}$ off the subspace Π . Hence, the saddle point $O = (0, 0)$ has a 2-dimensional unstable manifold and a codimension 2 center-stable manifold. If $c^2(2\pi j/L)^2 > 1$, we find infinite number imaginary eigenvalues $\lambda_j = \pm i\alpha_j = \pm i\sqrt{c^2(2\pi j/L)^2 - 1}$, $j \geq 2$, cf. Fig. 2.

In order to study the local behavior of solutions of the nonlinear system (27) we need to analyze the spectrum of the operator

$$\mathcal{L}_\varepsilon = \begin{pmatrix} 0 & 1 \\ c^2 \partial_x^2 + 1 & -\varepsilon a \end{pmatrix}$$

We consider the eigenvalue problem $\mathcal{L}_\varepsilon e = \lambda e$ for the eigen-pairs $\{e(x), \lambda\}$. Using Fourier expansions of $e(x)$ in $\exp(ik_j x)$ leads to a quadratic expression for the eigenvalues λ :

$$\lambda^2 + \lambda \varepsilon a - (1 - c^2 k_j^2) = 0, \quad k_j = 2\pi j/L, \quad j = 0, 1, 2, \dots$$

1. For $j = 0$:

$$\lambda_{0,\varepsilon}^\pm = -\frac{\varepsilon a}{2} \pm \sqrt{\left(\frac{\varepsilon a}{2}\right)^2 + 1},$$

2. For $j = 1$:

$$\lambda_{1,\varepsilon}^\pm = -\frac{\varepsilon a}{2} \pm \sqrt{\left(\frac{\varepsilon a}{2}\right)^2 + \sigma_1^2}, \quad \sigma_1 = \sqrt{1 - c^2 k_1^2},$$

3. For $j = 2, 3, \dots$:

$$\lambda_j^\pm = -\frac{\varepsilon a}{2} \pm i\sqrt{c^2 k_j^2 - 1 - \left(\frac{\varepsilon a}{2}\right)^2}$$

while for $j \geq 2$, the eigenvalues come in complex conjugate pairs.

The time-periodic perturbed sine-Gordon equation (1) can be rewritten as:

$$\mathbf{u}_t = \mathcal{L}_0 \mathbf{u} + \mathbf{B}(\mathbf{u}) + \varepsilon (\mathbf{C} \mathbf{u} + g(t, \mathbf{u})) \quad (31)$$

with

$$\mathcal{L}_0 := \begin{pmatrix} 0 & 1 \\ c^2 \partial_x^2 & 0 \end{pmatrix}, \quad \mathbf{B}(u) := \begin{pmatrix} 0 \\ \sin u \end{pmatrix}, \quad \mathbf{C} := \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}$$

Figure 1: (a) Homoclinic orbit and (b) the geometry of the unperturbed system.

Figure 2: The spectrum of the linearized operator $\mathcal{L}_0$ in the complex plane.

We assume that $g : \mathbb{H}^1 \times \mathbb{S}^1 \rightarrow \mathbb{H}^1$ is C^∞ T -periodic in time where $\mathbb{S}^1 = \mathbb{R}/T$. The associated extended autonomous system in a cylinder in the space $\mathbb{H}^1 \times \mathbb{S}^1$

$$\begin{aligned} \dot{u} &= \mathcal{L}u + B(u) + \varepsilon(Cu + g(\theta, u)) \\ \dot{\theta} &= 1 \end{aligned} \tag{32}$$

has a smooth flow F_ε^t .

Suppose that linearized system of (31) with respect to the singular point $p_0 = O$

$$\mathbf{r}_t = \mathcal{L}_\varepsilon \mathbf{r} + \varepsilon D_u g(t, u) \tag{33}$$

with

$$\mathcal{L}_\varepsilon := \begin{pmatrix} 0 & 1 \\ c^2 \partial_x^2 + 1 & -\varepsilon a \end{pmatrix}$$

has a T -periodic function $r(t, \varepsilon)$, such that $r(t, \varepsilon) = O(\varepsilon)$.

For $\varepsilon > 0$, the operator $e^{T\mathcal{L}_\varepsilon}$ has a spectrum consisting in two pair of real eigenvalues for $j = 0, 1$ and infinite number of complex eigenvalues for $j \geq 2$, $\text{spec}(e^{T\mathcal{L}_\varepsilon})_{j \geq 2}$, in such a way:

$$\mu_1 \varepsilon \leq \text{dist}\left(\text{spec}(e^{T\mathcal{L}_\varepsilon})_{j \geq 2}, |z| = 1\right) \leq \mu_2 \varepsilon \quad (34)$$

with $\mu_1, \mu_2 > 0$.

We denote by $P^\varepsilon : \mathbb{H}^1 \rightarrow \mathbb{H}^1$ the Poincaré map for the flow F_ε^t such that $P^0(p_0) = p_0$ and p_ε the fixed points of P^ε correspond to periodic orbits of F_ε^t , near to $p_0 = O : p_\varepsilon = p_0 + O(\varepsilon)$.

Indeed, by (33) we have:

$$r(t, \varepsilon) = e^{t\mathcal{L}_\varepsilon} r(0, \varepsilon) + \int_0^t \varepsilon e^{(t-\xi)\mathcal{L}_\varepsilon} D_u g(\xi) d\xi \quad (35)$$

and $r(T, \varepsilon) = r(0, \varepsilon)$, $\|r(t, \varepsilon)\| \leq \mu_3 \varepsilon$.

We seek a solution $u(t, \varepsilon)$ of the system (1) such that:

$$u(t, \varepsilon) = e^{t\mathcal{L}_\varepsilon} u(0, \varepsilon) + \int_0^t \varepsilon e^{(t-\xi)\mathcal{L}_\varepsilon} D_u g(\xi) d\xi + \int_0^t \varepsilon e^{(t-\xi)\mathcal{L}_\varepsilon} (B(u(\xi, \varepsilon)) - u(\xi, \varepsilon)) d\xi \quad (36)$$

From (35) and (36) we obtain:

$$u(t, \varepsilon) - r(t, \varepsilon) = e^{t\mathcal{L}_\varepsilon} (u(0, \varepsilon) - r(0, \varepsilon)) + \int_0^t \varepsilon e^{(t-\xi)\mathcal{L}_\varepsilon} (B(u(\xi, \varepsilon)) - u(\xi, \varepsilon)) d\xi \quad (37)$$

Thus,

$$\|u(t, \varepsilon) - r(t, \varepsilon)\|_{\mathbb{H}^1} \leq \mu_4 \varepsilon + \mu_5 \int_0^t \|u(\xi, \varepsilon)\|_{\mathbb{H}^1} d\xi$$

using $\|r(t, \varepsilon)\|_{\mathbb{H}^1} \leq \mu_3 \varepsilon$. By Gronwall's estimate we obtain $\|u(t, \varepsilon)\|_{\mathbb{H}^1} \leq \mu_6 \varepsilon$.

We consider now the set B_ε as

$$B_\varepsilon = \left\{ r(t, \varepsilon) : \|r(t, \varepsilon) - r(0, \varepsilon)\|_{\mathbb{H}^1} < \varepsilon \right\}$$

and the map

$$\begin{aligned} P^\varepsilon : B_\varepsilon &\rightarrow \mathbb{H}^1 \\ u(0, \varepsilon) &\rightarrow u(T, \varepsilon) \end{aligned}$$

We seek a fixed point of P^ε . From (37) $u(0, \varepsilon)$ is a fixed point of P^ε if and only if it is a fixed point of the following map

$$\mathbb{G}_\varepsilon : B_\varepsilon \rightarrow \mathbb{H}^1$$

with

$$\mathbb{G}_\varepsilon(u(0, \varepsilon)) = r(0, \varepsilon) + L_\varepsilon^{-1} \int_0^T e^{(T-\xi)\mathcal{L}_\varepsilon} (B(u(\xi, \varepsilon)) - u(\xi, \varepsilon)) d\xi \quad (38)$$

where $L_\varepsilon = (\text{Id} - e^{T\mathcal{L}_\varepsilon})$ is invertible operator and the condition

$$\|L_\varepsilon^{-1}\|_{\mathbb{H}^1} \leq \mu_7 \varepsilon^{-1} \quad (39)$$

is inferred from (34). The conditions (38) and (39) entail

$$\|\mathbb{G}_\varepsilon(u(0, \varepsilon)) - r(0, \varepsilon)\|_{\mathbb{H}^1} \leq \varepsilon$$

which implies that for $\varepsilon \ll 1$ $\mathbb{G}_\varepsilon$ maps B_ε to itself. Also, $\mathbb{G}_\varepsilon$ is a contraction since, for ε small enough

$$\begin{aligned} \|D\mathbb{G}_\varepsilon(u(0, \varepsilon))\|_{\mathbb{H}^1} &= \left\| L_\varepsilon^{-1} \int_0^T e^{(T-\xi)\mathcal{L}_\varepsilon} D(B(u(\xi, \varepsilon) - u(\xi, \varepsilon))) \circ DF_\xi^\varepsilon(u(0, \varepsilon)) d\xi \right\|_{\mathbb{H}^1} \\ &\leq \mu_8 \varepsilon < 1 \end{aligned}$$

Thus, $\mathbb{G}_\varepsilon$ has a unique fixed point on B_ε . There is an analogous result for the Poincaré map $P_{t_0}^\varepsilon$ with the section $\mathbb{H}^1 \times \{t_0\}$ in $\mathbb{H}^1 \times \mathbb{S}^1$.

In the following Lemma, we describe the geometry of the invariant manifolds for the non-autonomous perturbed sine-Gordon equation (1). Its proof based on the invariant manifolds theory (Hirsch *et al.* 1977) and the smoothness of the flow operator F_ε^t .

Lemma 3.1 *Let $p_\varepsilon(t_0)$ denote the unique fixed point of $P_{t_0}^\varepsilon$. Corresponding to the spectrum of $e^{t\mathcal{L}_\varepsilon}$, there are unique invariant manifolds $W^{\text{cs}}(p_\varepsilon(t_0))$ (center-stable) and $W^{\text{u}}(p_\varepsilon(t_0))$ (unstable manifold) for the map $P_{t_0}^\varepsilon$ such that they are invariant under the $P_{t_0}^\varepsilon$, tangent to the eigenspace of $e^{t\mathcal{L}_\varepsilon}$ respectively at $p_\varepsilon(t_0)$ and C^r -close as $\varepsilon \rightarrow 0$ to the homoclinic orbit $U(x, t)$ of the sine-Gordon equation (1) for t_s, t_u such that $t_s < t < \infty, -\infty < t < t_u$, respectively. We denote by $q_\varepsilon(t) = (p_\varepsilon(t), t), t \in (0, T)$ the periodic orbit of the extended system (32) with $q_\varepsilon(0) = (p_\varepsilon, 0)$. The invariant manifolds for the periodic orbit $q_\varepsilon(t)$ are denoted $W^{\text{cs}}(q_\varepsilon(t)), W^{\text{u}}(q_\varepsilon(t))$ and*

$$W^j(p_\varepsilon(t_0)) = W^j(q_\varepsilon(t)) \cap (\mathbb{H}^1 \times \{t_0\}), \quad j = \text{cs}, \text{u}$$

4 Persistent homoclinic orbits

In this section, we shall combine the invariant manifold theory with explicit global information from the unperturbed sine-Gordon equation to establish the existence of persistent homoclinic orbit to the singular point O_ε .

4.1 Geometry of Mel'nikov Method

We recall that the unstable manifold $W^{\text{u}}(O_\varepsilon) \subset W^{\text{u}}(\mathcal{M}_\varepsilon)$ is two-dimensional with one direction in the plane of spatial-independent solutions Π and the other direction off the plane. We consider an orbit in $W^{\text{u}}(O_\varepsilon)$ that leaves O_ε and takes off away from the plane, then defines an orbit which lies on the unstable manifold. We will construct a distance function $\mathcal{D}$ between the unstable and stable manifolds, whose zeros correspond to orbits that do not lie on the plane Π and are asymptotic to the saddle-focus point O_ε in backward and forward time.

Let us rewrite equation (1) in the vector form

$$\mathbf{u}_t = J \nabla_{\mathbf{u}} H(\mathbf{u}) + \varepsilon G(t, \mathbf{u}, \mu) \quad (40)$$

where $G(t, \mathbf{u}, \mu) = (0, bg(t, u) - au_t)^\perp$ and $\mu = (a, b)$

We are interested in the dynamics of the solution $u_\varepsilon(t) = (r_s, r_u, r_c)$ of (1) in forward and backward time. For $\varepsilon \leq \varepsilon_0$, we study the dynamics of (1) within some fixed open set

$$Y = \left\{ (r_s, r_u, r_c) \in \mathbb{H}^1 : |r_s| < C_s, |r_u| < C_u, \|r_c\|_{\mathbb{H}^1} < C_c \right\}$$

Figure 3: The stable and unstable boundaries of the set $\mathcal{U}$.

where C_s, C_u, C_c are fixed positive constants. We define a neighborhood $\mathcal{U}$ of $\mathcal{M}_\varepsilon$

$$\mathcal{U} = \{ (r_s, r_u, r_c) \in Y : |r_s| < \delta, |r_u| < \delta, \|r_c\|_{\mathbb{H}^1} < \delta \leq C_{r_c} \}$$

and its boundary $\partial\mathcal{U}$ consists of two parts $\partial\mathcal{U}_s$ stable and $\partial\mathcal{U}_u$ unstable as follows

$$\partial\mathcal{U}_s = \{ (r_s, r_u, r_c) \in \mathcal{U} : |r_s| = \delta \}$$

$$\partial\mathcal{U}_u = \{ (r_s, r_u, r_c) \in \mathcal{U} : |r_u| = \delta \}$$

where the solution curves enter through $\partial\mathcal{U}_s$ and they exit through $\partial\mathcal{U}_u$, cf. Fig. 3.

As was said before, the breather solution $U(x, t)$ is also a solution of the SG and is homoclinic to the point $O = (0, 0) \in \Pi$ which has two unstable directions. The unperturbed system possesses a homoclinic orbit u which evolves along the unstable direction from the plane Π , intersects the unstable boundary $\partial\mathcal{U}_u$ at the point u_0^u and defines a unique orbit $(u^u(t), t)$ which lies on the codimension two center-unstable manifold. After a finite time, this homoclinic orbit u intersects the stable boundary $\partial\mathcal{U}_s$ at the point u_0^s and comes back to the plane Π and is asymptotic to the singular point O , following a unique curve $(u^s(t), t)$ which lies on the center-stable manifold. Thus, for $\varepsilon \neq 0$, there exists a perturbed orbit u_ε^u which evolves from the perturbed saddle point $O_\varepsilon = O + O(\varepsilon) \in \Pi$ and intersects the unstable boundary at the point u_ε^u .

Now, choose a point u_h on the homoclinic orbit for the unperturbed system. Let Σ be a codimension 2 hyperplane at u_h which is transversal to the homoclinic orbit u and which contains the vector

$$\vec{\nu} = \text{grad}\Delta$$

(because the Floquet discriminant Δ Poisson commutes with the integrable Hamiltonian which defines the homoclinic orbit). $W^s(\mathcal{M}_0) \cap \Sigma$ has a codimension 1 in Σ and $\vec{\nu}$ is transversal to $W^s(\mathcal{M}_0) \cap \Sigma$. Let u_s be the intersection of the line through $u_u \in W^u(O_\varepsilon) \subset W^u(\mathcal{M}_\varepsilon)$ and the manifolds $W^s(\mathcal{M}_\varepsilon)$ along to direction $\vec{\nu}$, as shown in Fig. 4. We may thus define the inner product

$$\mathcal{D} = \langle \nabla\Delta, u_u - u_s \rangle, \tag{41}$$

Figure 4: The geometry of Mel'nikov analysis. The function $\mathcal{D}$, as defined in (41), measures the distance between u_u , u_s .

as a measure of the distance between u_u and u_s .

To calculate $\mathcal{D}$, we parametrise the orbits as follows,

$$\text{for } t \leq 0, \quad u = u_h(t + t_*), \quad u^u = u_\varepsilon^u(t + t_*)$$

$$\text{for } t \geq 0, \quad u = u_h(t + t_*), \quad u^s = u_\varepsilon^s(t + t_*)$$

where u^s, u^u be the solutions of the SG on center manifold. Since u_h remains in a δ -neighborhood of S for $t \geq t_*$ the invariant manifold theorem holds in a δ -neighborhood of $S \subset \mathbb{H}^1$. For $t \leq -t_*$ both orbits u, u^u remain in a δ -neighborhood of $(0, 0) \in \Pi \subset \mathbb{H}^1$. From $\|u^u - u_h\|_{\mathbb{H}^1} \leq C_1 \varepsilon$, for $t < 0$ we have,

$$\|u^u - u_h\|_{\mathbb{H}^1} \leq \varepsilon C_2 e^{-\sigma t} \quad (42)$$

Similar, for $t \geq 0$

$$\|u^s - u_h\|_{\mathbb{H}^1} \leq \varepsilon C_2 e^{\sigma t} \quad (43)$$

The above analysis allows to decompose (41) as follows;

$$\mathcal{D} = \langle \nabla \Delta(u_h), u_u - u_h \rangle - \langle \nabla \Delta(u_h), u_s - u_h \rangle.$$

Let

$$\mathcal{D}^s = \langle \nabla \Delta, u_\varepsilon^s(t) - u(t) \rangle, \quad t \geq 0$$

$$\mathcal{D}^u = \langle \nabla \Delta, u_\varepsilon^u(t) - u(t) \rangle, \quad t \leq 0$$

Then

$$\mathcal{D} = \mathcal{D}^u(0) - \mathcal{D}^s(0), \quad i = 1, 2$$

Proposition 4.1 *The distance function (41) are given by*

$$\mathcal{D} = \mathcal{D}^u(0) - \mathcal{D}^s(0) = \varepsilon M + O(\varepsilon^2) \quad (44)$$

where

$$M = \int_{-\infty}^{\infty} \langle \nabla \Delta(u(t)), G(u(t), t) \rangle dt, \quad (45)$$

denote the Mel'nikov integral with $\langle a, b \rangle = \int_0^L a(x)b(x) dx$.

Proof. We start the proof by asserting from (40) that $u = u(t)$, $u_\varepsilon^u = u_\varepsilon^u(t)$, $u_\varepsilon^s = u_\varepsilon^s(t)$ solve the following system of equations

$$\begin{aligned}\partial_t u &= J \nabla H(u) \\ \partial_t u_\varepsilon^u &= J \nabla H(u_\varepsilon^u) + \varepsilon G(u_\varepsilon^u, \mu) \\ \partial_t u_\varepsilon^s &= J \nabla H(u_\varepsilon^s) + \varepsilon G(u_\varepsilon^s, \mu)\end{aligned}\tag{46}$$

Differentiating $\mathcal{D}^u(t)$, we have

$$\begin{aligned}\dot{\mathcal{D}}^u(t) &= \varepsilon \langle \nabla \Delta(u), G(u) \rangle + \langle \nabla^2 \Delta(u) J \nabla H(u), u_\varepsilon^u - u \rangle \\ &\quad + \langle \nabla \Delta(u), \nabla(J \nabla H(u))(u_\varepsilon^u - u) \rangle + \langle \nabla \Delta(u), \mathcal{S}(u_\varepsilon^u, u) \rangle\end{aligned}$$

where

$$J \nabla H(u_\varepsilon^u) = J \nabla H(u) + \nabla(J \nabla H(u))(u_\varepsilon^u - u) + \mathcal{S}(u_\varepsilon^u, u)$$

we leave out the argument t of the functions for the sake of a less cumbersome notation. From (17) we have $\langle J \nabla H, \nabla \Delta \rangle = 0$, differentiating with respect to u gives

$$\nabla(J \nabla H)^\perp \nabla \Delta + \nabla(\nabla \Delta) J \nabla H = 0,\tag{47}$$

where $^\perp$ denotes the adjoint operator. Taking the inner product of (47) with $u_\varepsilon^u - u$ gives

$$\langle \nabla^2 \Delta J \nabla H, u_\varepsilon^u - u \rangle + \langle \nabla \Delta, \nabla(J \nabla H)(u_\varepsilon^u - u) \rangle = 0$$

and thus,

$$\dot{\mathcal{D}}^u(t) = \varepsilon \langle \nabla \Delta(u), G(u) \rangle + \langle \nabla \Delta(u), \mathcal{S}(u_\varepsilon^u, u) \rangle\tag{48}$$

There exists $t_* > 0$ for $t \in (-\infty, t_*)$ such that

$$\|\nabla \Delta(u(t))\|_{\mathbb{H}^1} < C e^{\sigma t}, \quad \sigma := \sigma_1 = \sqrt{1 - (2\pi c/L)^2}\tag{49}$$

since the $\nabla \Delta$ is evaluated on the homoclinic solution which satisfies the condition (19) and (42) entails

$$\|\mathcal{S}\|_{\mathbb{H}^1} \leq \|u^u - u\|_{\mathbb{H}^1}^2 \leq C_1 e^{-2\delta t} \varepsilon^2\tag{50}$$

with $2\delta < \sigma$. Now, we want to find out the limit $\lim_{t \rightarrow -\infty} \mathcal{D}^u(t)$ in order to represent $\mathcal{D}(0)$ by

$$\mathcal{D}^u(-\infty) + \int_{-\infty}^0 \dot{\mathcal{D}}^u(t) dt\tag{51}$$

We calculate $\lim_{t \rightarrow -\infty} \mathcal{D}^u(t)$ and show that $\lim_{t \rightarrow -\infty} \mathcal{D}^u(t) = 0$. We have

$$\lim_{t \rightarrow -\infty} \mathcal{D}^u(t) = \lim_{t \rightarrow -\infty} \langle \nabla \Delta(u(t)), u_\varepsilon^u(t) - u(t) \rangle$$

There are $C_2, C_3 > 0$ such that

$$\begin{aligned}\|u_\varepsilon^u(\tau - t_\varepsilon^u) - F_\varepsilon^\tau(u)\|_{\mathbb{H}^1} &\leq C_2 e^{\lambda_1 \tau} \\ \|u(\tau - t_0^u) - F_\varepsilon^\tau(u)\|_{\mathbb{H}^1} &\leq C_3 e^{\lambda_1 \tau}\end{aligned}\tag{52}$$

for all $\tau \in (-\infty, 0]$, where $\lambda_1 > 0$, $u_\varepsilon^u(-t_\varepsilon^u) = u_\varepsilon^u$, $u(-t_0^u) = u^u$ and

$$F_\varepsilon^\tau(u) \rightarrow O_\varepsilon \quad \text{as } \tau \rightarrow -\infty \quad (53)$$

$$F_0^\tau(u) = O \in S \quad (54)$$

By these relations (52-54), we obtain that as $t \rightarrow -\infty$

$$\|u_\varepsilon^u(t) - u(t)\|_{\mathbb{H}^1} \sim O(t) \quad (55)$$

From relations (49) and (55) we see that

$$\lim_{t \rightarrow -\infty} \mathcal{D}^u(t) = 0$$

All this finally yields upon integration of (48)

$$\mathcal{D}^u(0) = \varepsilon \int_{-\infty}^0 \langle \nabla \Delta(u(t)), G(t, u(t)) \rangle dt + O(\varepsilon^2) \quad (56)$$

Similarly, we have

$$\mathcal{D}^s(0) = -\varepsilon \int_0^\infty \langle \nabla \Delta(u(t)), G(t, u(t)) \rangle dt + O(\varepsilon^2) \quad (57)$$

whence the distance function (44) is given by

$$\mathcal{D} = \mathcal{D}^u(0) - \mathcal{D}^s(0) = \varepsilon M + O(\varepsilon^2) \quad (58)$$

with

$$M = \int_{-\infty}^\infty \langle \nabla \Delta(u(t)), G(t, u(t)) \rangle dt, \quad (59)$$

where $u(t) = U(x, t)$ the homoclinic solution of the unperturbed system. $\square$

Lemma 4.1 *The Mel'nikov integral (59) is convergent.*

Proof. Indeed, the perturbation term $G = bg(t, u) - au_t$ is exponential decreasing expression, since we have assumed that $g(t, u)$ satisfies the conditions $g(t, 0) = 0$, $\partial_u g(t, 0) = 0$ and in the neighborhood of zero

$$\|g(u)\|_{\mathbb{H}^1} \leq \mu_9 \|u\|^2 \leq \mu_9 e^{-2\sigma t} \quad (60)$$

evaluated on the homoclinic orbit, with $\mu_9 > 0$. Also, hold

$$\|u_t\|_{\mathbb{H}^1} \leq \mu_{10} e^{-\sigma t}$$

and (49). If, we consider the integral as the following limit of a sequence in time $\{\tau_n^{s,u}\} = \{\pm 2\pi n/\omega\}$, $n = 1, 2, \dots$

$$M(\vartheta, a, \beta, c) = \lim_{n \rightarrow \infty} \int_{-\frac{2\pi n}{\omega}}^{\frac{2\pi n}{\omega}} \int_0^L \langle \nabla \Delta(u(t)), G(t, u(t)) \rangle dt, \quad \vartheta = 1, 2 \quad (61)$$

we are allowed to extend the integration limits in (61) to infinity and thus obtaining convergent integral M . $\square$

Using the integral of motion, we may derive explicit formula for the Mel'nikov functions

$$M(\vartheta, a, \beta, c, \mu) = \int_{-\infty}^{\infty} \langle \nabla \Delta(U(x, t)), G(t - \vartheta, U(x, t)) \rangle dt \quad (62)$$

evaluated on the homoclinic orbit $U(x, t)$. Thus,

$$M(\vartheta, a, \beta, c, \mu) = b \int_{-\infty}^{\infty} \int_0^L \frac{\delta \Delta}{\delta u_t}(U) g(t - \vartheta, U) dx dt - a \int_{-\infty}^{\infty} \int_0^L \frac{\delta \Delta}{\delta u_t}(U) U_t dx dt \quad (63)$$

with $\beta, \frac{\delta \Delta}{\delta u_t}$ defined in Lemma 2.1.

Setting $M(\vartheta, a, \beta, c, \mu) = 0$ we obtain an algebraic equation for our parameters which describe a surface in the parameter space, for a particular ϑ . The value ϑ must satisfy the second condition on the invariant plane Π .

4.2 Sufficient condition for homoclinic orbits

We are now in position to specify this parameter analyzing the second condition. We write the perturbation g as

$$g(t, u) = g_0(t, u) + \sum_{j \geq 1} g_j(t, u) (\cos 2j\pi x / L) \quad (64)$$

The unperturbed SG equation possesses (on the first mode) a family of homoclinic solution

$$u_{\vartheta}(t) = 4 \tan^{-1} \left(\tanh \beta \operatorname{sech}((t - \vartheta) \tanh \beta) \right) \quad (65)$$

such that $u_{\vartheta}(t) \rightarrow 0$ as $|t| \rightarrow \infty$.

We study the perturbed SG along the eigenmode for $j = 0$ and $j = 1$ (these modes describe the dynamics close to the invariant plane Π). We set

$$u(x, t) = u(t) + v(x, t) = \bar{u}(t) + v(x, t) \quad (66)$$

where

$$\bar{u}(t) = \frac{1}{L} \int_0^L u(x, t) dx, \quad \text{and} \quad v(x, t) = \sum_{j \geq 1} v_j \cos(2\pi j x / L) \in \mathbb{E}_2$$

where $\mathbb{E}_2$ will denote the elements of the form $v(x, t)$ which have zero mean value in the spatial variable x . Substituting

$$u(x, t) = (u_{\vartheta}(t) + w(t)) + v(x, t)$$

in SG equation, we obtain the set of second order equations:

$$\begin{aligned} -\ddot{w} - \varepsilon a \dot{w} + w &= -\frac{1}{L} \int_0^L \sin(u_{\vartheta} + w + v(x, t)) dx + \sin u_{\vartheta} + w(t) - \varepsilon g_0(t, u_{\vartheta} + w + v) + \varepsilon a \dot{u}_{\vartheta} \\ v_{tt} + \varepsilon a v_t - c^2 v_{xx} - v \cos u_{\vartheta} &= \Lambda + \varepsilon g_1(t, u_{\vartheta} + w + v) \end{aligned} \quad (67)$$

where

$$\Lambda = \sin(u_{\vartheta} + w + v(x, t)) + \frac{1}{L} \int_0^L \sin(u_{\vartheta} + w + v(x, t)) dx - v \cos u_{\vartheta}$$

Lemma 4.2 For all $f \in \mathbb{E}_2$ there exists a unique solution v of

$$v_{tt} + \varepsilon a v_t - c^2 v_{xx} - v \cos u_\vartheta = f \quad (68)$$

such that the linear Green operator N_ε satisfy

$$\|N_\varepsilon f\|_{\mathbb{E}_2} \leq \left(\frac{C_3}{a\varepsilon} \right) \|f\|$$

and

$$N_\varepsilon f = v = \sum_{j \geq 1} v_j \cos(2\pi j x / L)$$

Proof. Indeed, we look for a solution of (68) as

$$v = \sum_{j \geq 1} v_j \cos(2\pi j x / L)$$

we obtain that v_j must satisfy

$$\ddot{v}_j + \varepsilon a \dot{v}_j + (c j \pi^2 - \cos u_\vartheta) v_j = f_j \quad (69)$$

Let consider the homogeneous equation

$$\ddot{v}_j + \varepsilon a \dot{v}_j + ((c j \pi)^2 - \cos u_\vartheta) v_j = 0 \quad (70)$$

It is not difficult to prove that solutions of (70) decay exponentially for $t \rightarrow +\infty$, in fact if we consider $v_j(t) = \mathbf{v}_j(t) e^{-(\varepsilon a/2)t}$ in (70) we get

$$\ddot{\mathbf{v}}_j + (\kappa^2 - \cos u_\vartheta) \mathbf{v}_j = 0, \quad \kappa^2 = (c j \pi)^2 - \left(\frac{\varepsilon a}{2}\right)^2 \quad (71)$$

Setting

$$E_j(t) = \frac{1}{2} \left(\dot{\mathbf{v}}_j^2 + \kappa^2 \mathbf{v}_j^2 \right)$$

and using Gronwall's estimates we obtain that solutions of (71) are bounded

$$\|\mathbf{v}_j\|_\infty \leq \frac{\sqrt{2E_j^0(\mathbf{v})}}{\kappa^2} \quad (72)$$

By the variation of constants formula we get that if f_j is bounded then equation (69) has a unique bounded solution which is given in integral form by

$$v_j(t) = - \left[\int_{-\infty}^t \exp\left[\frac{\varepsilon a}{2}(s-t)\right] v_{h,j}^{(2)}(s) f_j(s) ds \right] v_{h,j}^{(1)}(t) + \left[\int_{-\infty}^t \exp\left[\frac{\varepsilon a}{2}(s-t)\right] v_{h,j}^{(1)}(s) f_j(s) ds \right] v_{h,j}^{(2)}(t)$$

where $v_{h,j}^{(1),(2)}$ are solutions of the homogeneous equation. By (72), estimating $E_j^0(v_{h,j}^{(1)})$, $E_j^0(v_{h,j}^{(2)})$, we easily see that there exist $C_1, C_2 > 0$ with

$$\|v_{h,j}^{(1)}\|_\infty \leq C_1 \quad \text{and} \quad \|v_{h,j}^{(2)}\|_\infty \leq C_2$$

Then $\|v_j\| \leq \frac{C_3}{\varepsilon a} \|f_j\|$ where C_3 is a constant. Moreover, if $f_j \rightarrow 0$ as $|t| \rightarrow \infty$ then $v_j(t) \rightarrow 0$ as $|t| \rightarrow \infty$. $\square$

Lemma 4.3 *There exists a unique C^2 solution $w \in C^0(\mathbb{R})$:*

$$-\ddot{w} - \varepsilon a \dot{w} + w = f$$

given by

$$w = K_\varepsilon(f) = \frac{1}{\sqrt{(\varepsilon a)^2 + 4}} \left[\int_t^\infty f(s) e^{\lambda_{0,\varepsilon}^+(t-s)} ds + \int_{-\infty}^t f(s) e^{\lambda_{0,\varepsilon}^-(t-s)} ds \right]$$

where

$$\lambda_{0,\varepsilon}^\pm = -\frac{\varepsilon a}{2} \pm \sqrt{\left(\frac{\varepsilon a}{2}\right)^2 + 1},$$

We can write the system (67) as

$$\mathcal{S}_\varepsilon(w, v) = 0$$

with $\mathcal{S}_\varepsilon : C^0(\mathbb{R}) \times \mathbb{E}_2 \rightarrow C^0(\mathbb{R}) \times \mathbb{E}_2$ defined by

$$\begin{aligned} \mathcal{S}_\varepsilon^{(1)}(w, v) &= w - K_\varepsilon \left[\frac{1}{L} \int_0^L \sin(u_\vartheta + w + v(x, t)) dx + \sin u_\vartheta + w(t) \right. \\ &\quad \left. - \varepsilon g_0(t, u_\vartheta + w + v) + \varepsilon a \dot{u}_\vartheta \right] \\ \mathcal{S}_\varepsilon^{(2)}(w, v) &= v - N_\varepsilon \left[\Lambda(t, w, v) + \varepsilon g_1(t, u_\vartheta + w + v) \right] \end{aligned} \quad (73)$$

where u_ϑ is the reduction of the breather solution on the subspace of x -independent solutions.

Proposition 4.2 *There are $\varepsilon_1, C_4 > 0$ and smooth function $(w(\varepsilon, \vartheta), v(\varepsilon, \vartheta), \eta(\varepsilon, \vartheta)) : (-\varepsilon_1, \varepsilon_1) \times \mathbb{R} \rightarrow C^0(\mathbb{R}) \times \mathbb{E}_2 \times \mathbb{R}$ such that*

a.

$$\mathcal{S}_\varepsilon(u_\vartheta(t) + w_\varepsilon, v_\varepsilon) = (\eta_\varepsilon(t) K_\varepsilon(\dot{u}_\vartheta), 0) \quad (74)$$

b.

$$\langle w_\varepsilon(t), \dot{u}_\vartheta(t) \rangle_{\mathbb{L}^2} = 0 \quad (75)$$

c. Let $0 < \varepsilon < \varepsilon_1$. If $\eta_\varepsilon(\hat{\vartheta}) = 0$ then

$$\mathcal{S}_\varepsilon(u_{\hat{\vartheta}} + w_\varepsilon(\hat{\vartheta}), v_\varepsilon(\hat{\vartheta})) = 0$$

and then $(u_{\hat{\vartheta}} + w_\varepsilon(\hat{\vartheta}), v_\varepsilon(\hat{\vartheta}))$ is an homoclinic orbit to the time-periodic orbit originated from 0Π as $t \rightarrow \infty$.

The function η_ε is given by

$$\eta_\varepsilon(\vartheta) = -\varepsilon \frac{\mathcal{M}_\Pi(\vartheta)}{\mathcal{I}} + \mathcal{O}(\varepsilon^2) \quad (76)$$

and

$$\mathcal{M}_\Pi(\vartheta) = \int_{-\infty}^\infty \dot{u}_\vartheta(t) \left[g_0(t - \vartheta, u_\vartheta(t)) - a \dot{u}_\vartheta(t) \right] dt \quad (77)$$

with

$$\mathcal{I} := \int_{-\infty}^\infty \dot{u}_\vartheta^2 dt$$

Proof. We apply the Lyapunov-Schmidt reduction method those to the invariant plane Π . We introduce the function

$$\Xi : \mathbb{R} \times \mathbb{R} \times C^0(\mathbb{R}) \times \mathbb{R} \times \mathbb{E}_2 \rightarrow C^0(\mathbb{R}) \times \mathbb{R} \times \mathbb{E}_2$$

with components:

$$\Xi_1(\varepsilon, \vartheta, w, \eta, v) \in C^0(\mathbb{R}) \times \mathbb{R}, \quad \Xi_2(\varepsilon, \vartheta, w, \eta, v) \in \mathbb{E}_2$$

where

$$\begin{aligned} \Xi_1 &:= \left(\mathcal{S}_\varepsilon^{(1)}(w, v) - \eta K_\varepsilon(\dot{u}_\vartheta(t)), \langle w(t), \dot{u}_\vartheta(t) \rangle_{\mathbb{L}^2} \right) \\ \Xi_2 &:= \mathcal{S}_\varepsilon^{(2)}(w, v) \end{aligned} \quad (78)$$

and we have to find (w, η, v) such that $\Xi(\varepsilon, \vartheta, w, \eta, v) = 0$. The proof of the statements (a) and (b) is based on the implicit function theorem and the contraction mapping theorem. We refer the reader to Chow *et al.* 1982, Palmer 1984 and Chicone 1999 (Chapters 5, 6).

We prove the statement (c). Since $(u_\vartheta(\vartheta) + w_\varepsilon(\vartheta), v_\varepsilon(\vartheta))$ satisfies the conditions (74, 75), $u_\vartheta(\vartheta) + w_\varepsilon(\vartheta)$ satisfies the equation

$$\begin{aligned} -(u_\vartheta + w_\varepsilon)'' - \varepsilon a(u_\vartheta + w_\varepsilon)' + (u_\vartheta + w_\varepsilon) &= -\frac{1}{L} \int_0^L \sin(u_\vartheta + w_\varepsilon + v_\varepsilon(x, t)) \, dx \\ &+ \sin u_\vartheta + w_\varepsilon - \varepsilon g_0(t, u_\vartheta + w_\varepsilon + v_\varepsilon) + \varepsilon a \dot{u}_\vartheta - \eta_\varepsilon(t) \dot{u}_\vartheta \end{aligned}$$

Multiplying by $\dot{u}_\vartheta$ and integrating by part using the fact that $\dot{u}_\vartheta$ is the solution of the linearized reduced system on Π plane (x -independent solutions), we obtain (76, 77).

The integral $\mathcal{M}_\Pi$ is convergent. We consider the improper integral as the following limit of a sequence in time $\{\tau_n^{s,u}\} = \{\pm 2\pi n/\omega\}$, $n = 1, 2, \dots$

$$\mathcal{M}_\Pi(\vartheta) = \lim_{n \rightarrow \infty} \int_{-\frac{2\pi n}{\omega}}^{\frac{2\pi n}{\omega}} \dot{u}_\vartheta(t) \left[g_0(t - \vartheta, u_\vartheta(t)) - a \dot{u}_\vartheta(t) \right] dt, \quad (79)$$

Since $g(t, u)$ and u_ϑ are rapidly decreasing expressions, we are allowed to extend the integration limits in (79) to infinity and thus obtaining convergent integral. $\square$

The existence topologically simple zeros of the function $\mathcal{M}_\Pi(\vartheta)$ consists the sufficient condition for the persistent homoclinic orbits.

In particular, for $g(t, u) = \Gamma \cos \omega t$, we find

$$\mathcal{M}_\Pi(\vartheta, \Gamma, a, \beta, c) = \mathcal{A} b \Gamma \sin \omega \vartheta - a \mathcal{I}, \quad (80)$$

where

$$\mathcal{A} \sin \omega \vartheta = \int_{-\infty}^{\infty} \left[\dot{u}_\vartheta(t) \cos \omega(t - \vartheta) \right] dt, \quad \mathcal{I} = \int_{-\infty}^{\infty} \dot{u}_\vartheta^2(t) dt$$

Setting $\mathcal{M}_\Pi(\vartheta) = 0$ (cf. (77)), we have an algebraic equation with respect to parameter ϑ .

$$\hat{\vartheta}_\pm = \pm \frac{1}{\omega} \sin^{-1} \left(\frac{a}{b\Gamma} \chi \right) \quad \chi = \frac{\mathcal{I}}{\mathcal{A}} \quad (81)$$

Introducing the simple zero of $\vartheta = \hat{\vartheta}$ into $M = 0$ we obtain an algebraic equation for our parameters which describe a surface in the external parameter space. We denote this surface by $\mathcal{E}_\beta$

$$\mathcal{E}_{\beta,c} : \beta_0 = \mathcal{B}(c_0; b, a),$$

There exists a region $\mathcal{R}$ of the surface $\mathcal{E}_\beta$ such that

$$\frac{\partial}{\partial \beta} M \neq 0 \quad (82)$$

By the implicit function theorem there is a neighborhood $\hat{\mathcal{R}}$ of $(c_0; b, a)$ and a unique C^{k-2} function

$$\beta_0 = \hat{\mathcal{B}}(c_0; b, a, \varepsilon)$$

defined in $\hat{\mathcal{R}}$ such that

$$\hat{\mathcal{B}}(c_0; b, a; 0) = \hat{\beta}_0,$$

and

$$\mathcal{D}(\hat{\mathcal{B}}(b_0, a_0; \varepsilon), c_0; b, a, \varepsilon) = 0$$

Since $\mathcal{D}$ (the distance of stable and unstable manifolds) is C^{k-2} smooth function by (82), we obtain

$$\frac{\partial}{\partial \beta} \mathcal{D}(\hat{\mathcal{B}}(b_0, a_0; \varepsilon), c_0; b, a, \varepsilon) \neq 0$$

for $(c_0; b, a, \varepsilon) \in \hat{\mathcal{R}}$.

Proposition 4.3 *Suppose the function $n \mathcal{M}_\Pi(\vartheta)$ has a topologically simple zero $\hat{\vartheta} \in [0, T]$ and*

$$M(\vartheta, \beta, c; \mu) = 0, \quad \frac{\partial}{\partial \beta} M \neq 0$$

There exists a subregion $\hat{\mathcal{R}}_1$ of external parameter space

$$\hat{\mathcal{R}} = \left\{ (c, \omega, \Gamma, a, \beta) : \frac{1}{4} < c^2 < 1, \omega \in (0, 0.87), \Gamma, a \in (0.1, 1), \beta \in (-\pi/2, \pi/2) \right\}$$

for the parameter b, a, c and a positive constant $\varepsilon_0 > 0$ such that for any $\|\varepsilon\| < \varepsilon_0$ and $(b, a, c) \in \hat{\mathcal{R}}$, there exists a solution $u(x, t)$ of the perturbed sine-Gordon equation (1) homoclinic to the x -independent time periodic orbit $u_\vartheta(t)$ originated from $u = 0$ and $u(x, t) \rightarrow 0$ as $t \rightarrow \pm\infty$.

5 Conclusions

In this paper, we have used a Mel'nikov type analysis to detect the splitting of homoclinic manifolds in a perturbed sine-Gordon equation with even spatial periodic boundary conditions. We extended the two-dimensional phase portrait for the pendulum into the infinite dimensional phase space $\mathbb{H}^1$ of the integrable periodic sine-Gordon equation.

This analysis consists first of a study of homoclinic and chaotic dynamics on perturbed SG systems. The prerequisite for such study is that the unperturbed soliton system, viewed as a Hamiltonian system should have a homoclinic structure. In the process of proving the

existence of invariant manifolds for such a system several mathematical tools from dynamical systems theory are utilized. We established, via Mel'nikov theory the existence of transversal intersection of a codimension 2 stable and unstable manifold, of the time-periodic orbit originated from O_ε and through the dynamics of the reduced system we derived the second condition on the plane Π . From implicit function theorem we proved the existence of a solution $u(x, t)$ of the perturbed sine-Gordon (1) homoclinic to x -independent, time-periodic orbit originated from $u = 0$ and has same asymptotic phase as $t \rightarrow \pm\infty$.

Acknowledgment

This research was supported by the postdoctoral fellowship No. ERBFMBICT983236 of the Commission of the European Communities. The author had stimulating discussions with T.Bountis and G.Haller. He would also like to thank the Nonlinear Center, DAMTP, University of Cambridge for very grateful hospitality.

References

- Bishop, A.R., Forest, M.G., McLaughlin, D.W. & Overman E.A.II, 1986, A quasi-periodic route to chaos in near-integrable PDEs, *Physica D* **23**, 293–328,
- Chow, S.-N., Hale, J. and Mallet-Paret, J., 1982 *An example of bifurcation to homoclinic orbits*, *J. of Diff. Eq.* **37**, 351–373.
- Chicone, C., 1999 *Ordinary Differential Equations with Applications*, TAM **34** Springer-Verlag.
- Ercolani, N.M., Forest, M.G. and McLaughlin, D.W., 1990 Geometry of the modulation instability, Part III: Homoclinic orbits for the periodic sine-Gordon equation, *Physica D* **43**, 349,
- Faddeev, L.D. and Takhtajan, L.A., 1987 *Hamiltonian Methods in the Theory of Solitons*, Springer-Verlag, Berlin.
- Guckenheimer, J. and Holmes, P.J., 1986 *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer-Verlag, Berlin, (2nd ed.)
- Haller, G., 1998, Multi-dimensional Homoclinic Jumping and the Discretized NLS Equation, *Comm. Math. Phys.* **193** (1), 1–46.
- Haller, G., 1999a, Homoclinic Jumping in the Perturbed Nonlinear Schrödinger Equation, *Comm. Pure Appl. Math.* **52** (1), 1–47.
- Haller, G., 1999b, *Chaos Near Resonances*, Springer-Verlag.
- Haller, G., Mennon, G. and Rothos, V. M., 1999c Silnikov manifolds in coupled nonlinear Schrodinger equations *Phys. Lett. A*, **263**, 175–185, 1999.
- Hirsch, M., Pugh, C. and Shub M., 1977, *Invariant Manifolds*, Springer Lectures Notes in Mathematics, **583**.

- Holmes, P.J. and Marsden, J., 1981, A Partial Differential Equation with Infinitely Many Periodic Orbits and Chaotic Oscillators of a Forced Beam, *Archive for Rational Mechanics and Analysis*, **76**, 135–165.
- Kovacic, G. and Wiggins, St., 1992, Orbits Homoclinic to Resonances, with an Application to Chaos in Damped and Forced sine-Gordon Equation *Physica D* **57**, 185–225.
- McLaughlin, D.W. and Shatah, J., 1998, Homoclinic orbits for PDE's in *Recent advances in partial differential equations, Venice 1996*, 281–299, Proc. Sympos. Appl. Math., 54, Amer. Math. Soc., Providence, RI.
- McLaughlin, D.W. and Overman E.A.II, 1995, Whiskered Tori for Integrable PDEs: Chaotic Behavior in Near-Integrable PDEs, *Surveys in Applied Mathematics*, **1**, Plenum NY, 83–203.
- Mel'nikov, V.K., 1963, On the Stability of the Center for Time Periodic Oscillations, *Trans. Moscow Math.*, **12**, 1–57.
- Palmer, K. 1984 Exponential Dichotomies and Transversal Homoclinic points, *J. Diff. Eq.* **55**, 225–256.
- Pedersen, N.F., 1982, *Advances in superconductivity*, Deaver and Ruvolds, eds. NY Plenum.
- Rothos, V.M. and Bountis, T.C., 1998, The Dynamics of Coupled Perturbed Discretized NLS Equations, *Physica D* **113**, 326–331.
- Rothos, V.M., 1999, Mel'nikov Theory of Coupled Perturbed Discretized NLS Equations, *Chaos, Solitons and Fractals* **10**, (7), 1119–1134.
- Rothos, V.M., 2000, Persistent homoclinic orbits in Near-Integrable PDEs, in Proceedings of the conference “Nonlinearity, Integrability and all that twenty years after NEEDS '79”, Eds. M. Boiti *et al*, 402–408 World Scientific.
- Rothos, V.M., 2001, Homoclinic Orbits in Near-Integrable Double Discrete sine-Gordon Equation, *J. Phys. A: Math & Gen* (accepted).
- Temam, R., 1997, *Infinite Dimensional Dynamical Systems in Mechanics and Physics* (2nd ed.), Springer-Verlag, NY.
- Wiggins, St., 1988, *Global Bifurcations and Chaos: Analytical Methods*, Springer-Verlag.

Figure Captions

Figure 1. (a) Homoclinic orbit and (b) the geometry of the unperturbed system.

Figure 2. The spectrum of the linearized operator $\mathcal{L}_0$ in the complex plane.

Figure 3. The stable and unstable boundaries of the set $\mathcal{U}$.

Figure 4. The geometry of Mel'nikov analysis. The function $\mathcal{D}$, as defined in (41), measures the distance between u_u, u_s .

Figure 1a (Rothos)

Figure 1b (Rothos)

Figure 2 (Rothos)

Figure 3 (Rothos)

Figure 4 (Rothos)