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Further Results on the Control Law via the Convex Hull of Ellipsoids

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Abstract

A new Lyapunov function based on the convex hull of ellipsoids was introduced in [8] for the study of uncertain and/or time-varying linear discrete-time systems with/without constraints. The new Lyapunov function has many attractive features such as: i) it provides a necessary and sufficient conditions for robust stability and robust stabilization; ii) the design conditions are formulated as linear matrix inequality constraints. The control law is obtained by solving a convex optimization problem online. This optimization generally does not have a closed-form solution, and hence it is solved by numerical methods. In this paper, we intend to complement the results in [8] by analyzing the solution of the optimization problem as well as the geometric structure of the control law. In particular, we show that the control law is a piecewise linear and continuous function of the state.

I. Introduction

Lyapunov functions play a central role in the study of dynamical systems, and the construction of Lyapunov functions is one of the most fundamental problems in systems theory. The most direct applications is stability analysis, but similar problems appear in performance analysis, and controller design. Consequently, methods for constructing Lyapunov functions is of great theoretical and practical interest.

For uncertain and/or time-varying linear discrete-time systems, the most popular types of Lyapunov functions are the polyhedral functions [2] and the quadratic functions. Polyhedral functions have been involved mostly in control problems with state and input constraints [1], [2], [5], [7]. Their main strengths are: i) the arbitrary approximation of the domain of attraction; ii) various analysis and design problems can be transformed into algebraic problems. The main weakness of polyhedral functions lies in the construction of the corresponding polyhedral sets.

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In general, this is a difficult problem, especially for high dimensional systems. In contrast with the polyhedral functions, the quadratic functions are tractable because of the existence of the linear matrix inequality (LMI) technique. Combining quadratic functions and the LMIs, several analysis and design problems can be converted into a convex optimization problems. However, the results obtained by quadratic functions can be conservative. It is well known [9] that there are stable/stabilizable systems that are not quadratically stable/stabilizable..

In the recent publication [8], the author proposed a novel Lyapunov function which is based on the convex hull of ellipsoids. The new Lyapunov function has several advantages over the standard quadratic and polyhedral Lyapunov functions. Compared to the quadratic one, the new Lyapunov function reduces the conservativeness as it provides a necessary and sufficient condition for stability and stabilization. Compared to the polyhedral Lyapunov function, the design conditions are formulated as LMI constraints. Hence the new Lyapunov function overcomes the main construction challenge of the polyhedral Lyapunov function. It is legitimate to say that the new Lyapunov function goes for the best of both quadratic and polyhedral functions worlds.

The main objective of this paper is to complement the results in [8]. We will analyze the geometric structures of the control law, and of the solution of the optimization problem.

The paper is organized as follows. Section II covers notations and preliminaries. Section III is dedicated to the problem formulation. Section IV is concerned with the question of the uniqueness of the solution. Then in Section V, geometric structures of control law are presented. One simulated example is evaluated in Section VI before drawing the conclusions in Section VII.

II. Notation and Preliminaries

Notation: For a given set $\mathcal{C}$, its boundary is denoted as Fr($\mathcal{C}$). We denote by $\mathbf{0}_n/\mathbf{I}_n$ $n \times n$ zero/identity matrices. A positive definite matrix P is denoted by $P \succ 0$. For symmetric matrices, the symbol $(*)$ denotes each of its symmetric block. We denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^{n\times m}$ the set of real $n \times m$ matrices, and by $\mathbb{S}^n$ the set of positive definite $n \times n$ matrices. For a given $P \in \mathbb{S}^n$, $\mathcal{E}(P)$ represents the following ellipsoid

$$\mathcal{E}(P) = \{x \in \mathbb{R}^n : x^T P^{-1} x \leq 1\} \tag{1}$$

The convex hull of ellipsoids $\mathcal{E}(P_1), \mathcal{E}(P_2), \ldots, \mathcal{E}(P_p)$ is denoted as

$$\mathcal{P} = \text{Co}\,\{\mathcal{E}(P_1), \mathcal{E}(P_2), \ldots, \mathcal{E}(P_p)\} \tag{2}$$

$\mathcal{P}$ is the smallest convex set containing $\mathcal{E}(P_j)$, $\forall j = \overline{1,p}$. For any $x \in \mathcal{P}$, there exist v_j and λ_j, $j = \overline{1,p}$ such that

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_p x_p \tag{3}$$

where $v_j \in \mathcal{E}(P_j)$, $\sum\limits_{j=1}^{p} \lambda_j = 1$, and $\lambda_j \geq 0$, $\forall j = \overline{1,p}$.

Definition 1 (Redundant Ellipsoid): For a given convex hull of ellipsoids $\mathcal{P} = \text{Co}(\mathcal{E}(P_j))$, $j = \overline{1,p}$, the set $\mathcal{P}_{-l}$ is defined by removing the $l-$th ellipsoid $\mathcal{E}(P_l)$ from $\text{Co}\left(\mathcal{E}(P_j)\right)$, i.e.,

$$\mathcal{P}_{-l} = \text{Co}\left(\mathcal{E}(P_j)\right), \forall j = \overline{1,p}, j \neq l \tag{4}$$

The ellipsoid $\mathcal{E}(P_l)$ is redundant if and only if

$$\mathcal{E}(P_l) \subseteq \mathcal{P}_{-l} \tag{5}$$

Definition 2 (Minimal Representation): $\mathcal{P} = \text{Co}(\mathcal{E}(P_j))$ has the minimal representation if and only if the removal of any ellipsoid would change $\mathcal{P}$, i.e., there are no redundant ellipsoids.

Clearly, the minimal representation of $\mathcal{P}$ can be achieved by removing all the redundant ellipsoids.

Definition 3 (Supporting Hyperplane): For a given vector $\beta \in \mathbb{R}^n$, and a given convex set $\mathcal{C}$, the hyperplane $\beta^T x = 1$ is a supporting hyperplane of $\mathcal{C}$ if and only if $\beta^T x \leq 1, \forall x \in \mathcal{C}$, and there exists at least one point $x_0 \in \text{Fr}(\mathcal{C})$ such that $\beta^T x_0 = 1$.

If $\mathcal{C}$ is an ellipsoid, then x_0 is unique [6]. If $\mathcal{C}$ is the convex hull of ellipsoids, i.e., $\mathcal{C} = \mathcal{P}$, then there are several $x_0 \in \text{Fr}(\mathcal{P})$ such that $\beta^T x_0 = 1$. To characterized the set of x_0, the following definition is recalled [11].

Definition 4 (Face): A face of $\mathcal{P}$ is the intersection of $\mathcal{P}$ with a supporting hyperplane of $\mathcal{P}$.

Definition 5 (Extreme Point): A point $v \in \text{Fr}(\mathcal{P})$ is an extreme point of $\mathcal{P}$ if it cannot be represented as a convex combination of other points in $\mathcal{P}$.

III. Problem Formulation

In this section, we first summarize the results in [8]. We then formulate the problems that need to be solved.

Consider the following uncertain and/or time-varying linear discrete-time systems

$$x(k+1) = A(k)x(k) + B(k)u(k) \tag{6}$$

where $x(k) \in \mathbb{R}^n$ is the measurable state, $u(k) \in \mathbb{R}^m$ is the control input. The matrices $A(k), B(k)$ satisfy

$$A(k) = \sum_{i=1}^{s} \alpha_i(k) A_i, B(k) = \sum_{i=1}^{s} \alpha_i(k) B_i \tag{7}$$

where $A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times m}, \forall i = \overline{1,s}$ are known matrices. $\alpha(k) = [\alpha_1(k)\ \alpha_2(k)\ \ldots\ \alpha_s(k)]^T$ is a vector of unknown and time-varying parameters, with

$$\sum_{i=1}^{s} \alpha_i(k) = 1, \alpha_i(k) \geq 0 \tag{8}$$

One of the most simple and well-known ways to control the system (6) is to employ a linear state feedback control law $u(k) = Kx(k)$ and an associated quadratic Lyapunov function $V(x) = x(k)^T P^{-1} x(k)$. In this case it is well known [3] that the problem of finding K and P can be converted into a convex semidefinite program (SDP). However, requiring the existence of a linear control law and a quadratic function can be quite restrictive. This is because the same control gain and the same Lyapunov matrix must verify for all vertices of the uncertain domain (7).

In [8], to overcome the conservative weakness of the quadratic Lyapunov function and the linear control law, the convex hull of quadratic functions $\mathcal{P} = \mathrm{Co}\,(\mathcal{E}(P_j))$ and the associated matrix gains $K_j \in \mathbb{R}^{m\times n}$, $j = \overline{1,p}$ are employed. It was shown that the new Lyapunov function is universal in the sense that (6) is robustly stabilizable if and only there exist a Lyapunov function based on the convex hull of ellipsoids. From this point on using the results in [8], it is assumed that $P_j \in \mathbb{S}^n$, as well as $K_j, \forall j = \overline{1,p}$ are known.

Remark 1: As shown in [8], although the set $\mathcal{P} = \mathrm{Co}\,(\mathcal{E}(P_j))$ is robustly invariant, the ellipsoids $\mathcal{E}(P_j)$, $\forall j = \overline{1,p}$ are generally not. The gains K_j are also generally not robustly stabilizing $\forall j = \overline{1,p}$. □

At time instant k, for a given state $x(k) \in \mathcal{P}$, the control action is computed as

$$u(k) = \sum_{j=1}^{p} \lambda_j^*(k) K_j v_j^*(k) \tag{9}$$

where $\lambda_j^*(k)$ and $v_j^*(k)$ are a solution of the following optimization problem

$$\begin{aligned} &\min_{\lambda_j, v_j} \left\{ \sum_{j=1}^{p} \lambda_j \right\}, \\ &\text{s.t.} \begin{cases} \sum_{j=1}^{p} \lambda_j v_j = x, \\ v_j^T P_j^{-1} v_j \leq 1, \forall j = \overline{1,p}, \\ \lambda_j \geq 0, \forall j = \overline{1,p} \end{cases} \end{aligned} \tag{10}$$

The optimization problem (10) is a nonlinear and non-convex due to the multiplication of λ_j, v_j. Using a change of variables, (10) can be converted into a convex optimization problem, for which there exists an efficient solver [8]. It was shown that the closed-loop system is robustly asymptotically stable under the control law (9), (10).

Problem (10) might have multiple solutions because the cost function (10) is linear. Multiple solutions are undesirable, as they might lead to a fast switching between the different control actions when (10) is solved on-line.

In this paper, we aim to answer the following three questions

Q1: What conditions need to be hold for (10) to have a unique solution?

Q2: The implementation of the control law (9) is based on solving online the optimization problem (10), which generally does not have an analytical solution. Therefore $u(k)$ is an implicit function of $x(k)$, i.e., $u(k) = f(x(k))$. What is the form of the function $f(x)$?

Q3: Is the control law $u(k) = f(x(k))$ a continuous function of the state?

Remark 2: The results in [8] were obtained with/without state and input constraints. Because the aim of this paper is to answer Q1, Q2, Q3, these constraints are not considered here for simplicity. □

IV. Geometrical Properties of the Solution

In this section we aim to reveal the geometrical properties of the solution of (10) to answer the question Q1. For this purpose, we will first propose a procedure to eliminate redundant ellipsoids in the convex hull of ellipsoids. We will then study the geometrical properties of the solution.

A. Removing Redundant Ellipsoid

The objective of this section is to propose a procedure to remove redundant ellipsoids. This redundancy elimination is with two purposes

- To reduce the online computational burden of the optimization problem (10). Obviously, if the number of constraints in (10) is smaller, then the computational burden is lower.
- It will be shown that if there is no redundant ellipsoid in the constraints (10), then the solution is unique.

For the given set of ellipsoids $\mathcal{E}(P_j), j=\overline{1,p}$, consider the following optimization problem

$$\begin{aligned} &\min_{\gamma_j} \left\{ \sum_{j=1,j\neq l}^{p} \gamma_j \right\} \\ &\text{s.t.} \begin{cases} P_l \preceq \sum\limits_{j=1,j\neq l}^{p} \gamma_j P_j, \\ \gamma_j \geq 0, \forall j=\overline{1,p}, j\neq l \end{cases} \end{aligned} \tag{11}$$

Denote γ_j^*, $\forall j=\overline{1,p}, j\neq l$ as an optimal solution of (11). The following theorem holds

Theorem 1: The ellipsoid $\mathcal{E}(P_l)$ is redundant in $\text{Co}(\mathcal{E}(P_j))$ if and only if $\sum\limits_{j=1,j\neq l}^{p} \gamma_j^* \leq 1$.

Proof: Consider the set $\mathcal{P}_{-l}$ in (4). Using the proof of Theorem 1 in [8], it follows that $\mathcal{P}_{-l}$ can be parameterized as $\mathcal{E}\left(\sum\limits_{j=1,j\neq l}^{p} \gamma_j P_j\right)$ with

$$\sum_{j=1,j\neq l}^{p} \gamma_j \leq 1, \gamma_j \geq 0 \tag{12}$$

In other words, x belongs to $\mathcal{P}_{-l}$ if and only if there exist γ_j satisfying (12) such that

$$x^T \left(\sum_{j=1,j\neq l}^{p} \gamma_j P_j\right)^{-1} x \leq 1 \tag{13}$$

Using (5), (13), it follows that $\mathcal{E}(P_l)$ is redundant if and only if $\exists \gamma_j$ satisfying (12), $j=\overline{1,p}, j\neq l$, such that

$$P_l^{-1} \succeq \left(\sum_{j=1,j\neq l}^{p} \gamma_j P_j\right)^{-1}$$

or equivalently $P_l \preceq \sum\limits_{j=1,j\neq l}^{p} \gamma_j P_j$. This completes the proof. □

Remark 3: Theorem 1 is the first one that provides a convex condition to verify if a given ellipsoid is redundant in the convex hull of ellipsoids. To the best of the author's knowledge, there does not exist any condition in the literature. □

Using Theorem 1, the following procedure can be used for removing the redundant ellipsoids in $\mathcal{P}=\text{Co}(\mathcal{E}(P_j))$, $j=\overline{1,p}$.

Algorithm 1: Redundant Ellipsoids Elimination

1: Set $l \leftarrow 1$, $p_m \leftarrow 0$;

2: Obtain $\gamma_j^*, \forall j=\overline{1,p}, j\neq l$ by solving (11);

3: If $\sum\limits_{j=1,j\neq l}^{p} \gamma_j^* > 1$, then set $p_m \leftarrow p_m+1$, $S_{p_m}=P_l$;

4: If $l<p$, then set $l \leftarrow l+1$ and go to step 2, else terminate;

5: The minimal representation of $\mathcal{P}$ is given by $\mathcal{P}=\text{Co}\left(\mathcal{E}(S_j)\right), j=\overline{1,p_m}$.

B. Uniqueness of Solutions

The main aim of this section is to derive a condition to guarantee the uniqueness of the solution of (10). To this aim, it is assumed that $\mathcal{P} = \text{Co}(\mathcal{E}(P_j)), j = \overline{1,p}$ has the minimal representation.

The following theorem concerns a geometrical property of the optimal solution.

Theorem 2: For a given state $x(k)$, (λ_j^*, v_j^*) is an optimal solution of (10) if and only if

- $\sum_{j=1}^{p} \lambda_j^* = g^*$ where $g^* \geq 0$ is the scalar such that $\frac{x(k)}{g^*} \in \text{Fr}(\mathcal{P})$ if $x(k) \neq \mathbf{0}$, and $g^* = 0$ if $x(k) = \mathbf{0}$.
- Either $(v_j^*)^T P_j^{-1} v_j^* = 1$ or $v_j^* = \mathbf{0}, \forall j = \overline{1,p}$.

Proof: If $x(k) \in \text{Fr}(\mathcal{P})$, then it is clear that $\sum_{j=1}^{p} \lambda_j^* = 1$. In this case, v_j^* is a solution of (10) if and only if either $(v_j^*)^T P_j^{-1} v_j^* = 1$ or $v_j^* = \mathbf{0}, \forall j = \overline{1,p}$.

Consider now the case when $x(k)$ is strictly inside $\mathcal{P}$. If $x(k) = \mathbf{0}$, then $v_j^* = \mathbf{0}$, $\forall j = \overline{1,p}$. Otherwise there exists $0 < g^* < 1$ such that $x_f(k) \in \text{Fr}(\mathcal{P})$, where $x_f(k) = \frac{1}{g^*} x(k)$, see Fig. 1. Define $\lambda_{f,j} = \frac{1}{g^*} \lambda_j, \forall j = \overline{1,p}$.

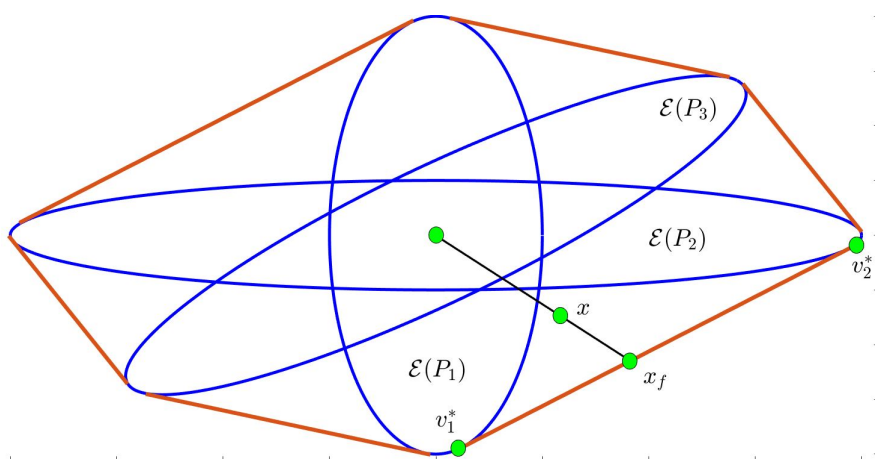

Fig. 1: Geometrical interpretation for the proof of Theorem 2.

Rewrite the problem (10) as

$$\min_{\lambda_j, v_j} \left\{ g^* \sum_{j=1}^{p} \lambda_{f,j} \right\},$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^{p} \lambda_{f,j} v_j = x_f, \\ v_j^T P_j^{-1} v_j \leq 1, \forall j = \overline{1,p}, \\ \lambda_{f,j} \geq 0, \forall j = \overline{1,p} \end{cases} \tag{14}$$

Since $x_f \in \text{Fr}(\mathcal{P})$, one has $\sum_{j=1}^{p} \lambda_{f,j}^* = 1$. It follows that $\sum_{j=1}^{p} \lambda_j^* = g^* \sum_{j=1}^{p} \lambda_{f,j}^* = g$. One also has either $(v_j^*)^T P_j^{-1} v_j^* = 1$ or $v_j^* = \mathbf{0}$. The proof is complete. □

Remark 4: Using the proof of Theorem 2, three observations can be made

1) Note that $x(k) = g^* x_f(k), 0 < g < 1$. Hence $x(k)$ lies on the line segment joining $x_f(k)$ and the origin.
2) The level set of the optimal value function is given by scaling the boundary of $\mathcal{P}$.
3) For any $j = \overline{1,p}$, if $v_j^* = \mathbf{0}$ then $\lambda_j^* = 0$. If $(v_j^*)^T P_j^{-1} v_j^* = 1$ then $\lambda_j^* > 0$. In other words, for any $j = \overline{1,p}$ if the constraint $(v_j)^T P_j^{-1} v_j \leq 1$ is active then the constraint $\lambda_j \geq 0$ is inactive and vice versa. □

For a given $x(k)$, if there is only one active ellipsoidal constraint, i.e., there exists only one index $1 \leq l \leq p$ such that $v_l^* P_l^{-1} v_l^* = 1$ and $v_j^* = \mathbf{0}, \forall j = \overline{1,p}, j \neq l$. Then one obtains in this case, $\lambda_l^* = g^*, \lambda_j^* = 0, \forall j = \overline{1,p}, j \neq l$. If in addition $x(k) \in \text{Fr}(\mathcal{P})$, then $\lambda_l^* = g^* = 1$ and $v_l^* = x(k)$. In this case, $x(k)$ is an extreme point of $\mathcal{P}$, as it cannot be represented as the convex combination of other points in $\mathcal{P}$.

Consider now the case where we have more than one active ellipsoidal constraints. Without loss of generality, it is assumed that the first p_a ellipsoidal constraints are active for a given $x(k)$, $2 \leq p_a \leq p$, i.e.,

$$v_j^T P_j^{-1} v_j = 1, \forall j = \overline{1,p_a}$$

The following result holds.

Theorem 3: The optimal solution v_j^*, $\forall j = \overline{1,p_a}$ and $\frac{x(k)}{g^*}$ belong to the same supporting hyperplane of $\mathcal{P}$, i.e.,

$$\beta^T v_j^* = \beta^T \frac{x(k)}{g^*} = 1 \tag{15}$$

where $0 < g^* \leq 1$ is a scalar such that $\frac{x(k)}{g^*} \in \text{Fr}(\mathcal{P})$. The normal vector $\beta \in \mathbb{R}^n$ satisfies the following set of equations

$$\beta^T P_j \beta = 1, \forall j = \overline{1,p_a} \tag{16}$$

Proof: Following the proof of Theorem 2, if $v_j^T P_j^{-1} v_j = 1, \forall j = \overline{1,p_a}$, one gets $\lambda_j > 0, \forall j = \overline{1,p_a}$, and $v_j = \mathbf{0}$ $\lambda_j = 0, \forall j = \overline{p_a+1,p}$.

Define $g = \sum_{j=1}^{p_a} \lambda_j, \eta_j = \frac{\lambda_j}{g}$, and $e_j = g v_j$, $\forall j = \overline{1,p_a}$. One has

$$\sum_{j=1}^{p_a} \eta_j = \frac{1}{g} \sum_{j=1}^{p_a} \lambda_j = 1$$

Rewrite the problem (10) as

$$\min_{g,\eta_j,e_j} \{g\}, \quad \text{s.t.} \begin{cases} \sum_{j=1}^{p_a} \eta_j e_j = x, \\ \sum_{j=1}^{p_a} \eta_j = 1, \\ e_j^T P_j^{-1} e_j = g^2, \forall j = \overline{1, p_a} \end{cases} \tag{17}$$

Consider the Lagrange function

$$\mathcal{L}(g, \beta, \eta_j, e_j, \rho, \mu_j) = g + \beta^T (x - \sum_{j=1}^{p_a} \eta_j e_j) + \rho(\sum_{j=1}^{p_a} \eta_j - 1) + \frac{1}{2} \sum_{j=1}^{p_a} \mu_j (e_j^T P_j^{-1} e_j - g^2) \tag{18}$$

The factor $\frac{1}{2}$ introduced in the Lagrange function is for the purpose of scaling. An optimal solution $(g^*, \beta^*, \eta_j^*, e_j^*, \rho^*, \mu_j^*)$ satisfies the following conditions

$$g^* \sum_{j=1}^{p_a} \mu_j^* = 1, \left(\text{from } \frac{\partial \mathcal{L}}{\partial g} = 0\right) \tag{19}$$

$$(\beta^*)^T e_j^* = \rho^*, \left(\text{from } \frac{\partial \mathcal{L}}{\partial \mu_j} = 0\right), \tag{20}$$

$$\beta^* = \frac{\mu_j^*}{\eta_j^*} P_j^{-1} e_j^*, \left(\text{from } \frac{\partial \mathcal{L}}{\partial e_j} = 0\right) \tag{21}$$

Using (20), one gets

$$(\beta^*)^T x = (\beta^*)^T \left(\sum_{j=1}^{q} \eta_j^* e_j^*\right) = \sum_{j=1}^{q} \eta_j^* (\beta^*)^T e_j^* = \rho^* \sum_{j=1}^{q} \eta_j^*$$

It follows that

$$(\beta^*)^T x = \rho^* \tag{22}$$

Substituting (21) to (20), one obtains

$$\frac{\mu_j^*}{\eta_j^*} (e_j^*)^T P_j^{-1} e_j^* = \rho^*, \forall j = \overline{1, p_a} \tag{23}$$

Recall $(e_j^*)^T P_j^{-1} e_j^* = (g^*)^2, \forall j = \overline{1, p_a}$. It follows that

$$\frac{\mu_1^*}{\eta_1^*} = \frac{\mu_2^*}{\eta_2^*} = \ldots = \frac{\mu_{p_a}^*}{\eta_{p_a}^*}$$

thus, $\forall j = \overline{1, p_a}$

$$\frac{\mu_j^*}{\eta_j^*} = \frac{\sum_{j=1}^{p_a} \mu_j^*}{\sum_{j=1}^{p_a} \eta_j^*} = \frac{1}{g^*}$$

Hence $\eta_j^* = g^*\mu_j^*, \forall j = \overline{1, p_a}$. Using (23), one has $\rho^* = g^*$. As a consequence, using (20), (22)

$$(\beta^*)^T e_j^* = (\beta^*)^T x = g^*, \forall j = \overline{1, p_a}$$

or equivalently

$$(\beta^*)^T v_j^* = (\beta^*)^T \frac{x}{g^*} = 1 \tag{24}$$

Hence $v_j^*, \forall j = \overline{1, p_a}$ and $\frac{x}{g^*}$ belong to the same supporting hyperplane.

Using (21) and since $\eta_j^* = g^*\mu_j^*$, one gets $e_j^* = g^* P_j \beta^*$. Using the fact that $(e_j^*)^T P_j^{-1} e_j^* = (g^*)^2$, one obtains

$$(\beta^*)^T P_j \beta^* = 1, \forall j = \overline{1, p_a}$$

The proof is complete. □

Remark 5: For β given in (16), on has $\beta^T x \leq 1, \forall x \in \mathcal{E}(P_j)$. The hyperplane $\beta^T x = 1$ touches the ellipsoid $\mathcal{E}(P_j)$ at the extreme point $v_j = P_j\beta, \forall j = \overline{1, p_a}$. Hence $\beta^T x = 1$ is a supporting hyperplane of $\mathcal{E}(P_j), \forall j = \overline{1, p_a}$. Because there is no redundant ellipsoid in $\mathcal{P}$, it follows that $v_j, \forall j = \overline{1, p_a}$ are also extreme points of $\mathcal{P}$. □

We are now ready to state the main theorem of this section.

Theorem 4: If $\mathcal{P} = \mathrm{Co}\left(\mathcal{E}(P_j)\right), j = \overline{1, p}$ has the minimal representation, then the solution of (10) is unique.

Proof: Using Theorem 3, the normal vector β of the supporting hyperplane for given $x(k)$ can be found by solving the set of equations (16). Obviously, one needs at most n equations in (16) to obtain β, since $\beta \in \mathbb{R}^n$. It follows that the number of points $v_j = P_j\beta$ is at most equal to n. Combining with the fact that $v_j \in \mathbb{R}^n$ are extreme points of $\mathcal{P}$, i.e., they cannot be represented as the convex combination of other points in $\mathcal{P}$, one concludes that v_j are linearly independent.

Now suppose on the contrary that $x(k)$ can be decomposed as

$$x = \lambda_1 v_1 + \ldots + \lambda_q v_q = \zeta_1 v_1 + \ldots + \zeta_q v_q \tag{25}$$

where $q \leq n$, and $\sum\limits_{j=1}^{q} \lambda_j = \sum\limits_{j=1}^{q} \zeta_j$. Using (25), one obtains

$$(\lambda_1 - \zeta_1)v_1 + \ldots + (\lambda_q - \zeta_q)v_q = \mathbf{0} \tag{26}$$

Because v_j are linearly independent, (26) holds if and only if $\lambda_j = \zeta_j, \forall j = \overline{1, q}$. In other words, the solution of (10) is unique. □

Remark 6: The number of active extreme points q in (25) can be different to the number of active ellipsoidal constraints p_a in (17). Indeed one always has $q \leq p_a$. This is because two or many non-redundant ellipsoids can share the same extreme points. For example, consider the following matrices P_1, P_2

$$P_1 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q_1 \end{bmatrix}, \; P_2 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & Q_2 \end{bmatrix}$$

where $Q_1 \succ 0, Q_2 \succ 0$. It is clear that $v = [1 \; \mathbf{0}]^T$ is as extreme point of both P_1 and P_2. Hence if $x(k) = v$, then both ellipsoidal constraints are active, i.e., $p_a = 2$. However there is only one active extreme point v. □

V. Geometrical Properties of the Control Law

In this section we aim to study the geometrical properties of the control law to answer the questions Q2 and Q3.

A. Continuous Piecewise Linear Control Law

Definition 6 (Dimension of Face): Suppose that a given supporting hyperplane has q extreme points of $\mathcal{P}$, $q \geq 1$. A face of dimension $q-1$ is the convex hull of all the q extreme points.

Using the proof of Theorem 4, one has $q \leq n$. The boundary of $\mathcal{P}$ is the union of faces of dimension $0, 1, \ldots, n-1$. For example in $\mathbb{R}^2$, the boundary of $\mathcal{P}$ is composed of elliptical arcs and line segments. The elliptical arcs have dimension 0, and the line segments have dimension 1. In $\mathbb{R}^3$, the boundary of $\mathcal{P}$ consists of three different kinds of faces

- Elliptical faces, which are parts of the ellipsoids. The dimension of the elliptical faces is 0.
- Conical faces, which are the convex hull of two extreme points. The dimension of the conical faces is 1.
- Planar faces, which are the convex hull of three extreme points, i.e., triangles. The dimension of the planar faces is 2.

The following definition is borrowed from [10].

Definition 7 (Critical Region): A critical region (CR) is the set of all states x which have the same set of active extreme points.

For example the origin is a CR, because in this case, the set of active extreme points is empty. Otherwise for any $x \neq \mathbf{0}$, there is always at least one active extreme point. Consider the case

of an extreme point, such that a corresponding supporting hyperplane contains only this point. Using remark 4 - point 1, it is clear that the half-open line segment connecting the extreme point and the not-included origin is a CR.

Without loss of generality, consider now the case where the first q extreme points $v_1, \ldots, v_q$ are active, $2 \leq q \leq n$. With a slight abuse of notation, $\text{Co}(\mathbf{0}, v_1, \ldots, v_q)$ is used to denote the convex hull of the origin and of $v_1, v_2, \ldots, v_q$. We also denote by $\text{Co}_{-\mathbf{0}}(\mathbf{0}, v_1, \ldots, v_q)$ the set $\text{Co}(\mathbf{0}, v_1, \ldots, v_q)$ where the origin is excluded from the set. Note that $\text{Co}(\mathbf{0}, v_1, \ldots, v_q)$ is a closed set, while $\text{Co}_{-\mathbf{0}}(\mathbf{0}, v_1, \ldots, v_q)$ is neither open nor closed.

The following result holds

Theorem 5: $\text{Co}_{-\mathbf{0}}(\mathbf{0}, v_1, \ldots, v_q)$ is a CR.

Proof: Note that $\text{Co}(v_1, \ldots, v_q)$ is a face of $\mathcal{P}$. For any $x \in \text{Co}(v_1, \ldots, v_q)$, it is clear that $v_1, \ldots, v_q$ are active extreme points.

Now consider the case x is strictly inside $\text{Co}_{-\mathbf{0}}(\mathbf{0}, v_1, \ldots, v_q)$. Define x_f as the intersection between the line connecting the origin and x and the face $\text{Co}(v_1, \ldots, v_q)$. One has

$$x = gx_f + (1-g)\mathbf{0} = gx_f \tag{27}$$

where $0 < g < 1$. Because $x_f \in \text{Co}(v_1, \ldots, v_q)$, one has

$$x_f = \lambda_1 v_1 + \ldots + \lambda_q v_q$$

where $\sum_{j=1}^{q} \lambda_j = 1, \lambda_j \geq 0, \forall j = \overline{1,q}$. Using (27), one has

$$x = g\lambda_1 v_1 + \ldots + g\lambda_q v_q$$

with $g\lambda_1 + \ldots + g\lambda_q = g\left(\sum_{i=1}^{q} \lambda_j\right) = q$. Therefore $v_1, \ldots, v_q$ are active extreme points $\forall x \in \text{Co}_{-\mathbf{0}}(\mathbf{0}, v_1, \ldots, v_q)$. □

Our next step is to reveal the form of the control law in a CR. Clearly, if $x(k) = \mathbf{0}$, then $u(k) = 0$.

Consider the case where $x(k) \in \text{Co}_{-\mathbf{0}}(\mathbf{0}, v_j)$ with $v_j \in \mathcal{E}(P_j)$ being an extreme point of $\mathcal{P}, j = \overline{1,p}$. In this case $x(k)$ is rewritten as $x(k) = \lambda_j(k) v_j$, where $0 < \lambda_j(k) \leq 1$. The control action is computed as

$$u(k) = \lambda_j(k) K_j v_j = K_j \lambda_j(k) v_j = K_j x(k) \tag{28}$$

If $x(k) = \mathbf{0}$, then using the control law (28), one has $u(k) = \mathbf{0}$. Hence, $\forall x \in \text{Co}(\mathbf{0}, v_j)$, the control law is (28).

Without loss of generality, consider now the case $x(k) \in \text{Co}(\mathbf{0}, v_1, \ldots, v_q)$, where $v_j \in \mathcal{E}(P_j)$ are extreme points of $\mathcal{P}$, and $2 \leq q \leq n$. One has

$$x(k) = \lambda_1 v_1 + \ldots + \lambda_q v_q \tag{29}$$

where $\lambda_j \geq 0, j = \overline{1, p}$. Rewrite (29) in a compact vector form as

$$x(k) = V\Lambda \tag{30}$$

where $\Lambda = [\lambda_1 \ \lambda_2 \ \ldots \ \lambda_q]^T \in \mathbb{R}^q$ and

$$V = \begin{bmatrix} v_1 & v_2 & \ldots & v_q \end{bmatrix} \tag{31}$$

Since $v_1, v_2, \ldots, v_q$ are linearly independent, one has rank$(V) = q$. Using the singular decomposition (SVD), rewrite the matrix $V \in \mathbb{R}^{n \times q}$ as

$$V = U_v S_v V_v^T \tag{32}$$

where $U_v \in \mathbb{R}^{n \times q}, V_v \in \mathbb{R}^{q \times q}$ with $U_v^T U_v = \mathbf{I}_q, V_v^T V_v = \mathbf{I}_q$, and $S_v \in \mathbb{R}^{q \times q}$ is a diagonal matrix. Since rank$(V) = q$, it follows that the diagonal elements of S_v are positive. Using (30), (32), one obtains

$$\Lambda = V_v S_v^{-1} U_v^T x(k) \tag{33}$$

The control action for the given $x(k)$ is computed as

$$u(k) = \lambda_1 K_1 v_1 + \ldots + \lambda_q K_q v_q$$

Thus, with $u_j = K_j v_j, \forall j = \overline{1, q}$

$$u(k) = [u_1 \ u_2 \ \ldots \ u_q] \, \Lambda$$

Combining with (33), one obtains

$$u(k) = [u_1 \ u_2 \ \ldots \ u_q] \, V_v S_v^{-1} U_v^T x(k) = F_v x(k) \tag{34}$$

where $F_v = [u_1 \ u_2 \ \ldots \ u_q] \, V_v S_v^{-1} U_v^T$.

If $x(k) = \mathbf{0}$ then using (34), one gets $u(k) = \mathbf{0}$. Hence $\forall x \in \text{Co}(\mathbf{0}, v_1, v_2, \ldots, v_q)$, the control law is (34).

With a slight abuse of notation, a partition of dimension q is the convex hull of the origin and of a face of dimension $q - 1$, $1 \leq q \leq n$. The following result holds.

Theorem 6: The control law (9), (10) is a piecewise linear function of the state over a partition of dimension $1, 2, \ldots, n$ of the state space.

Proof: The proof comes directly by using (28), (34), and by the fact that the set $\mathcal{P}$ is the union of the partitions of dimension $1, 2, \ldots, n$. $\square$

Remark 7: We separate two cases with one, and with more than one active extreme points only for clarity. The SVD technique (32) works also with one active extreme point. $\square$

Theorem 7: The control law (9), (10) is a continuous function of the state.

Proof: The proof comes from two facts that: i) the partitions are closed sets; ii) the control law is continuous in any partition. $\square$

B. Particular Case: $n = 2$

The aim of this section is to illustrate graphically the discussions in Section V-A for the case $n = 2$.

Consider the convex hull of ellipsoids $\mathcal{P} = \text{Co}(P_1, \ldots, P_p)$, $P_j \in \mathbb{S}^2$, and the associated matrix gains $K_j \in \mathbb{R}^{m\times 2}, \forall j = \overline{1, p}$. It is clear that in $\mathbb{R}^2$, our main problem is to construct the partitions of dimension 2 as well as the control gains in these partitions. Our first step is to calculate all possible normal vectors β of all faces of dimension 1. This can be done by solving the following set of equation, $\forall j_1 = \overline{1, p-1}, \forall j_2 = \overline{j_1 + 1, p}$

$$\begin{cases} \beta^T P_{j_1} \beta = 1, \\ \beta^T P_{j_2} \beta = 1 \end{cases} \tag{35}$$

Once the normal vector β is computed, the extreme points v_{j_1}, v_{j_2} are given as

$$v_{j_1} = P_{j_1}\beta, \; v_{j_2} = P_{j_2}\beta \tag{36}$$

Since v_{j_1}, v_{j_2} are linearly independent, and $[v_{j_1} \; v_{j_2}]$ is a square matrix, it follows that $[v_{j_1} \; v_{j_2}]$ is invertible. In this case, one does not need to perform the SVD technique to factorize $[v_{j_1} \; v_{j_2}]$. The control law for $x \in \text{Co}(\mathbf{0}, v_{j_1}, v_{j_2})$ is given as

$$u(k) = K_{j_1 j_2} x(k) \tag{37}$$

where the control gain $K_{j_1 j_2}$ is computed as

$$F_{j_1 j_2} = [K_{j_1} v_{j_1} \; K_{j_2} v_{j_2}][v_{j_1} \; v_{j_2}]^{-1} \tag{38}$$

Remark 8: Except the partitions of dimension n in $\mathbb{R}^n$, the other partitions are degenerate. Consider now the partitions of dimension 1. Recall that these partitions are the convex hull of the origin and of faces of dimension 0. If these faces belong to the same elliptical arc, then

the control gains for these partitions are the same. Hence the partitions with faces of the same elliptical arc can be merged to create a new full dimensional partition. One can expect the same behavior for the partitions of dimension $2, \ldots, n-1$, as the boundary of $\mathcal{P}$ is smooth. However this is beyond the scope of this paper. $\square$

VI. Example

In this section, we will demonstrate the obtained results via an example taken from [8]. Consider the system (6) with

$$A_1 = \begin{bmatrix} 1.0 & -1.4 \\ -1.0 & -0.8 \end{bmatrix} \; A_2 = \begin{bmatrix} 1.0 & 1.4 \\ -1.0 & -0.8 \end{bmatrix}, \; B_1 = \begin{bmatrix} 5.9 \\ 2.8 \end{bmatrix}, \; B_2 = \begin{bmatrix} 3.1 \\ -2.8 \end{bmatrix} \tag{39}$$

In [8], input constraints were considered: $-1 \leq u(k) \leq 1$. The goal is to design a robust stabilizing controller.

As written in [8], it can be verified that (39) is not quadratically stabilizable. LMI conditions for designing a linear feedback gain and its associated quadratic Lyapunov function are not feasible. Also we were not able to construct a robustly controlled invariant polyhedral set using procedures in [2].

Using [8] one obtains the matrices P_1, P_2, K_1, K_2 as

$$\begin{aligned} P_1 &= \begin{bmatrix} 22.9061 & -19.9925 \\ -19.9925 & 18.0114 \end{bmatrix}, \; P_2 = \begin{bmatrix} 21.2384 & -10.5083 \\ -10.5083 & 8.9691 \end{bmatrix}, \\ K_1 &= \begin{bmatrix} -0.0949 & 0.1296 \end{bmatrix}, \; K_2 = \begin{bmatrix} -0.0095 & 0.2060 \end{bmatrix} \end{aligned} \tag{40}$$

Note that the closed-loop system with the linear control law $u(k) = K_1 x(k)$ or with $u(k) = K_2 x(k)$ is are unstable. For example, the eigenvalues of $A_1 + B_1 K_2$ are 1.0887 and -0.3676.

By solving (35), one obtains the set of normal vectors $\beta = [\beta_1 \; \beta_2 \; -\beta_1 \; -\beta_2]$ with

$$\beta_1 = \begin{bmatrix} 0.2271 & 0.0209 \end{bmatrix}, \; \beta_2 = \begin{bmatrix} 0.2568 & 0.5150 \end{bmatrix} \tag{41}$$

Fig. 2 presents the sets $\mathcal{E}(P_1), \mathcal{E}(P_2)$ and the supporting hyperplanes of dimension 1.

Using (36), one obtains the corresponding extreme points $V = [V_h \; -V_h]$ with

$$V_h = \begin{bmatrix} 4.7854 & -4.4150 & 4.6047 & 0.0414 \\ -4.1650 & 4.1428 & -2.1996 & 1.9210 \end{bmatrix} \tag{42}$$

Fig. 3 shows the state space partition. Note that we can merge those partitions, whose faces of dimension 0 belong to the same elliptical arc. We have in total 8 partitions.

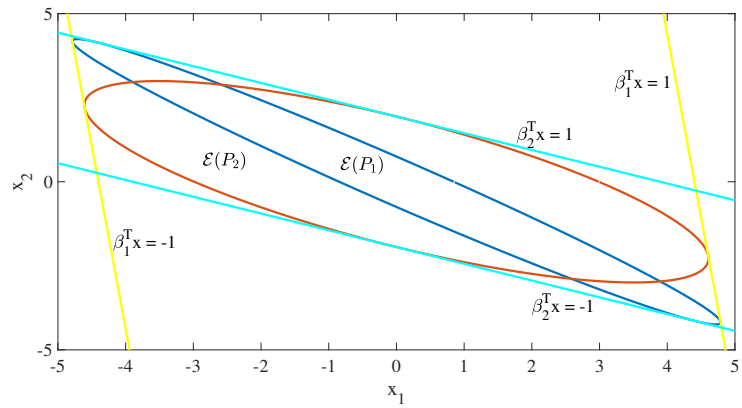

Fig. 2: Convex hull of ellipsoids.

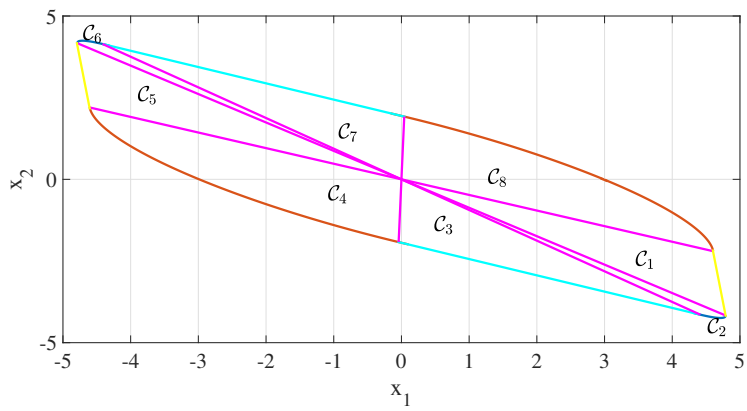

Fig. 3: State Space Partition.

Using (37), (38), the control law over the state space partition is

$$u(k) = \begin{cases} [0.0135 \quad 0.2541]x \text{ if } x \in \mathcal{C}_1 \bigcup \mathcal{C}_5, \\ [-0.0949 \quad 0.1296]x \text{ if } x \in \mathcal{C}_2 \bigcup \mathcal{C}_6, \\ [-0.0229 \quad 0.2063]x \text{ if } x \in \mathcal{C}_3 \bigcup \mathcal{C}_7, \\ [-0.0095 \quad 0.2060]x \text{ if } x \in \mathcal{C}_4 \bigcup \mathcal{C}_8 \end{cases} \tag{43}$$

Using the control law (43), for the initial condition $x(0) = [-4.75 \;\; 4.2]^T$, Fig. 4 presents the state trajectories of the closed-loop system as functions of time. Fig. 5 shows the input trajectory and the realization of α as functions of time.

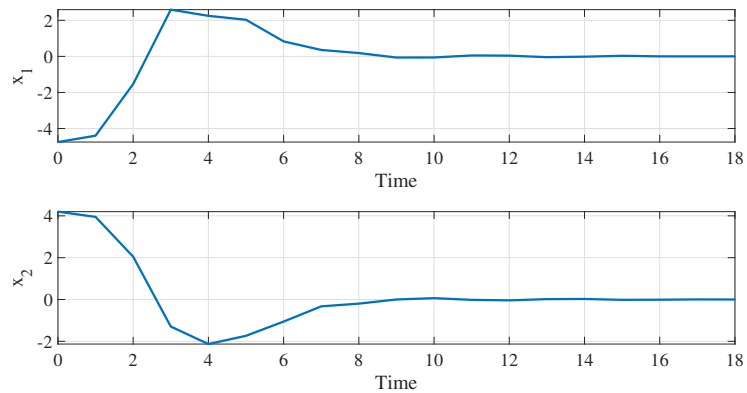

Fig. 4: State trajectories as functions of time.

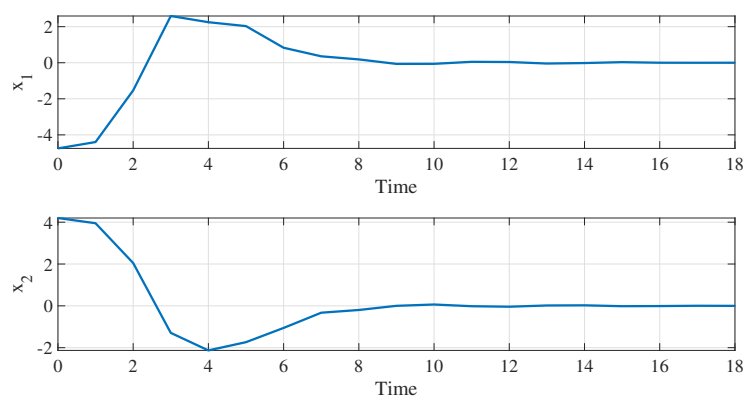

Fig. 5: Input trajectory and α realization as functions of time.

VII. Conclusion

In this paper we complement the recent results in [8] by studying geometric structures of the solution of the optimization problem, and of the control law. We propose a procedure to remove redundant ellipsoids in the convex hull of ellipsoids. We prove that if the convex hull of ellipsoid has the minimal representation, then the solution of the optimization problem is unique. We also show that the control law is a continuous piecewise linear function of the state. An unstable uncertain time-varying second order system example is used to validate the theoretical results.

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