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Extending a Theorem of Bergweiler and Langley Concerning Nonvanishing Derivatives

E. F. Clifford

Abstract

We consider the differential operator Λ_k defined by

$$\Lambda_k(y) = \Psi_k(y) + a_{k-1}\Psi_{k-1}(y) + \dots + a_1\Psi_1(y) + a_0,$$

where $a_0, \dots, a_{k-1}$ are analytic functions of restricted growth and $\Psi_k(y)$ is a differential operator defined by $\Psi_1(y) = y$ and $\Psi_{k+1}(y) = y\Psi_k(y) + (\Psi_k(y))'$ for $k \in \mathbb{N}$. We suppose that $k \geq 3$, that F is a meromorphic function on an annulus $\mathcal{A}(r_0)$, and that $\Lambda_k(F)$ has all its zeros on a set E such that E has no limit point in $\mathcal{A}(r_0)$. We suppose also that all simple poles a of F in $\mathcal{A}(r_0) \setminus E$ have $\text{Res}(F, a) \notin \{1, \dots, k-1\}$. We then deduce that F is a function of restricted growth in the Nevanlinna sense. This extends a theorem of Bergweiler and Langley [1]. We show also that this result does not hold for $a_0, \dots, a_{k-1}$ meromorphic functions.¹

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1 Introduction

In [1], Bergweiler and Langley define a differential operator $\Psi_k(F)$ for $k \in \mathbb{N}$ by

$$\Psi_1(F) = F, \quad \Psi_{k+1}(F) = F\Psi_k(F) + (\Psi_k(F))', \quad (1)$$

for which we have the following lemma.

Lemma 1.1 ([1]) *Let f be meromorphic on a domain Ω and let $F = f'/f$. Then for each $k \in \mathbb{N}$ we have $\Psi_k(F) = f^{(k)}/f$.*

Bergweiler and Langley then prove the following theorem. For background material regarding Nevanlinna theory, the reader is referred to [5].

Theorem A ([1]) *Let $k \geq 3$ be an integer, and let F be a meromorphic and non-constant function in the plane that satisfies both of the following conditions:*

- (i) $\Psi_k(F)$ has no zeros.
- (ii) if a is a simple pole of F then $\text{Res}(F, a) \notin \{1, \dots, k-1\}$.

Then F has the form

$$F(z) = \frac{(k-1)z + \alpha}{z^2 + \beta z + \gamma}, \quad (2)$$

or

$$F(z) = \frac{1}{\alpha z + \beta}. \quad (3)$$

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Here $\alpha, \beta, \gamma \in \mathbb{C}$ with $\alpha \neq 0$ in (3).

Conversely, if F has the form (2) or (3), and if (ii) holds, then $\Psi_k(F)$ has no zeros. If F has the form (2) or (3), but (ii) does not hold, then $\Psi_k(F) \equiv 0$.

We note that the conclusion of this theorem can be summarised as saying that F is a rational function, of a special form, and hence,

$$T(r, F) = O(\log r) \quad \text{as } r \rightarrow \infty.$$

Defining an annulus $\mathcal{A}(r_0)$ by

$$\mathcal{A}(r_0) = \{z : r_0 \leq |z| < \infty\} \quad (4)$$

we extend Theorem 1 in two ways. First, we let F be meromorphic and non-constant on $\mathcal{A}(r_0)$, by which we mean that F is meromorphic in a domain containing $\mathcal{A}(r_0)$. Second, we weaken condition (i) as follows. We let $a_0, \dots, a_{k-1}$ be analytic functions of restricted growth as $z \rightarrow \infty$, and define $\Lambda_k(F)$ by

$$\Lambda_k(F) = \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \dots + a_1\Psi_1(F) + a_0. \quad (5)$$

We then assume that $\Lambda_k(F) = 0$ only on a set E such that E has no limit point in the annulus $\mathcal{A}(r_0)$. This implies that $\Lambda_k(F) = 0$ only on a countable set E . The new conclusion is that F is a function of restricted growth in the Nevanlinna sense. We state the extended theorem as follows, denoting by $S(r, F)$ any quantity satisfying

$$S(r, F) = O(\log r + \log^+ T(r, F)),$$

as $r \rightarrow \infty$ outside a set of finite measure, not necessarily the same set at each occurrence.

Theorem 1.2 *Let $k \geq 3$ be an integer and let F be meromorphic and non-constant in an annulus $\mathcal{A}(r_0)$, as defined by (4). Suppose $a_0, \dots, a_{k-1}$ are analytic functions on $\mathcal{A}(r_0)$, with*

$$a_j(z) = O(|z|^{(\lambda-1)(k-j)}) \quad \text{as } z \rightarrow \infty, \quad (6)$$

for some fixed $\lambda \geq 0$. Let $f_1, \dots, f_k$ be solutions of $L(w) = 0$ where L is defined by

$$L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \dots + a_0f, \quad (7)$$

in $\mathcal{A}(r_0) \setminus \mathbb{R}^-$. Let $\Lambda_k(F)$ be defined as in (5) by

$$\Lambda_k(F) = \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \dots + a_1\Psi_1(F) + a_0.$$

Suppose there exists a set E , such that E has no limit point in $\mathcal{A}(r_0)$, and such that $\Lambda_k(F)$ has all its zeros in E . Suppose further that all simple poles a of F in $\mathcal{A}(r_0) \setminus E$ have $\text{Res}(F, a) \notin \{1, \dots, k-1\}$. Set

$$N_E(r) = \int_{r_0}^r \frac{n_E(t)}{t} dt$$

where $n_E(t)$ is the number of points in $E \cap \{z : r_0 \leq |z| \leq t\}$.

Then either:

- (i) $T(r, F) \leq cN_E(r) + S(r, F)$, as $r \rightarrow \infty$,
where c is a constant depending only on k ,

or

- (ii) F is a rational function of the f_j and their derivatives, in which case

$$T(r, F) = O(r^\lambda + \log r), \quad \text{as } r \rightarrow \infty.$$

We note that when $\lambda = 0$, it follows from (ii) that $\lim_{z \rightarrow \infty} F(z)$ exists.

The following corollaries are deduced from Theorem 1.2.

Corollary 1.3 *Let $k \geq 3$ be an integer and let F be meromorphic and non-constant in an annulus $\mathcal{A}(r_0)$, as defined by (4). Suppose there exists a set E , such that E has no limit point in $\mathcal{A}(r_0)$, and such that $\Psi_k(F)$ has all its zeros in E . Suppose further that all simple poles a of F , such that $a \notin E$, have $\text{Res}(F, a) \notin \{1, \dots, k-1\}$. Then either:*

- (i) $T(r, F) \leq cN_E(r) + O(\log r + \log^+ T(r, F))$ (n.e.),
where c is a constant depending only on k ,

or

- (ii) $\lim_{z \rightarrow \infty} F(z)$ exists.

Corollary 1.4 *Let $k \geq 3$ be an integer, and let F be meromorphic and non-constant in an annulus $\mathcal{A}(r_0)$, as defined by (4). Suppose F satisfies both of the following conditions:*

- (i) $\Psi_k(F)$ has no zeros.
- (ii) if a is a simple pole of F then $\text{Res}(F, a) \notin \{1, \dots, k-1\}$.

Then $\lim_{z \rightarrow \infty} F(z)$ exists.

Corollary 1.5 *Let F be meromorphic on $\mathbb{C}$ and satisfy both of the following conditions:*

- (i) $\Psi_k(F)$ has finitely many zeros.
- (ii) for all but finitely many simple poles of F we have $\text{Res}(F, a) \notin \{1, \dots, k-1\}$.

Then F is a rational function.

The following example shows that we cannot extend Theorem 1.2 to the case where $a_0, \dots, a_{k-1}$ are meromorphic functions.

Example Let $k \geq 3$ and let $F = f'/f$ where f is a meromorphic function which is nonvanishing, that is, has no zeros. Define $a_0, \dots, a_{k-1}$ by

$$a_j = \begin{cases} -\frac{f^{(k)}}{f} & \text{if } j = 0, \\ \frac{f}{f^{(j)}} & \text{if } j = 1, \dots, k-1. \end{cases}$$

Then $a_0, \dots, a_{k-1}$ are meromorphic functions and we have, using Lemma 1.1, that

$$\begin{aligned} \Lambda_k(F) &= \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \dots + a_1\Psi_1(F) + a_0 \\ &= \frac{f^{(k)}}{f} + \frac{f}{f^{(k-1)}} \frac{f^{(k-1)}}{f} + \dots + \frac{f}{f'} \frac{f'}{f} - \frac{f^{(k)}}{f} \\ &= k-1 \end{aligned}$$

Thus $\Lambda_k(F)$ has no zeros, and $E = \emptyset$. And so, all the hypotheses of Theorem 1.2 are satisfied except for the analyticity and growth of the a_j . However, since f may be any nonvanishing meromorphic function, no conclusions may be drawn about the growth of F .

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Some preliminaries for the proof of Theorem 1.2 are in § 2, and the main part of the proof is in § 3.

2 Proof of Theorem 1.2 (Preliminaries)

First we include some background material about Wronskians, referring the reader to [7] for further material. Let $f_1, \dots, f_k$ be meromorphic functions. We define the *Wronskian* $W(f_1, \dots, f_k)$ as follows

$$W(f_1, \dots, f_k) = \begin{vmatrix} f_1 & \cdots & f_k \\ f_1' & \cdots & f_k' \\ \vdots & & \vdots \\ f_1^{(k-1)} & \cdots & f_k^{(k-1)} \end{vmatrix}.$$

Following the notation in [7], we denote by $W_q(f_1, \dots, f_k)$, for $q = 0, \dots, k-1$, the determinant which is obtained from $W(f_1, \dots, f_k)$ by replacing the row $(f_1^{(q)}, \dots, f_k^{(q)})$ by $(f_1^{(k)}, \dots, f_k^{(k)})$. We include two useful lemmas regarding Wronskians, the second of which combines results from [4] and [7].

Lemma 2.1 ([7]) *Let $f_1, \dots, f_k$ be meromorphic functions in a domain Ω . Then $W(f_1, \dots, f_k)$ vanishes identically on Ω , if and only if $f_1, \dots, f_k$ are linearly dependent on Ω .*

Lemma 2.2 ([4], [7]) *Let $k \geq 1$ and let $f_1, \dots, f_k$ be linearly independent meromorphic functions in a domain Ω , that satisfy the homogeneous linear differential equation $L(w) = 0$, where L is defined by (7), and in which the a_q are meromorphic on Ω . Then the a_q can be written in the form*

$$a_q = -\frac{W_q(f_1, \dots, f_k)}{W(f_1, \dots, f_k)} \quad (8)$$

for $q = 0, \dots, k-1$ and, in particular,

$$a_{k-1} = -W(f_1, \dots, f_k)' / W(f_1, \dots, f_k). \quad (9)$$

The poles of a_q in Ω have multiplicity $\leq k-q$ and can only arise among the poles of $f_1, \dots, f_k$ and the zeros of $W(f_1, \dots, f_k)$. Furthermore, if f is meromorphic on Ω then

$$W(f_1, \dots, f_k, f) = W(f_1, \dots, f_k)L(f). \quad (10)$$

Several lemmas are needed for the proof of Theorem 1.2. We state them here, providing proofs where necessary. The first assertion in the following lemma is proved in [1], and the second is an extension which follows immediately.

Lemma 2.3 ([1]) *Let $k \geq 2$ be an integer. Let y be meromorphic on a domain Ω , such that if a is a simple pole of y then $\text{Res}(y, a) \notin \{1, \dots, k-1\}$. Let $n \in \mathbb{N}$ be such that $n \leq k$. If y has a pole at a of multiplicity m then $\Psi_n(y)$ has a pole at a of multiplicity nm , and $\Lambda_n(y)$ has a pole at a of multiplicity nm , where $\Lambda_n(y)$ is defined as in (5) by*

$$\Lambda_n(y) = \Psi_n(y) + a_{n-1}\Psi_{n-1}(y) + \dots + a_1\Psi_1(y) + a_0$$

where $a_0, \dots, a_{n-1}$ are analytic functions on Ω .

The following summarises some results from Nevanlinna theory which are used in Lemma 2.8. For background material about Nevanlinna theory in an annulus, we refer the reader to [2] or [4].

Lemma 2.4 *Let $p \in \mathbb{N}$ and let $\mathcal{B}$ be a simply connected domain in the annulus $\mathcal{A}(r_0)$, as defined by (4). Suppose that w is meromorphic on $\mathcal{B}$ such that $W = w^p$ is meromorphic on $\mathcal{A}(r_0)$. Then $w^{(k)}/w$ is meromorphic on $\mathcal{A}(r_0)$, for each $k \in \mathbb{N}$, with poles of multiplicity at most k . Further, we have*

$$T(r, w^{(k)}/w) = O(T(r, W)), \quad (\text{n.e.}),$$

and

$$m(r, w^{(k)}/w) = S(r, W), \quad (\text{n.e.}),$$

as $r \rightarrow \infty$.

Proof For $k = 1$ we note that $\frac{w'}{w} = \frac{1}{p} \frac{W'}{W}$, and the result follows. For $k \geq 2$, the result is easily proved by induction.

Next we state a lemma from [4].

Lemma 2.5 ([4]) *Suppose that $f_1, \dots, f_k$ each admit unrestricted analytic continuation in an annulus $\mathcal{A}(r_0)$, as defined by (4), and satisfy $\log^+ \log^+ |f_j(z)| = O(\log |z|)$ for z in $\mathcal{S} = \{z : |z| > r_0, |\arg z| < 2\pi\}$. Suppose that F is meromorphic in $\mathcal{A}(r_0)$. Suppose further that, for some non-negative integer Q , each of the functions $h_1, \dots, h_k$ on $\mathcal{S}$ is a polynomial in the $f_j^{(m)}$, $F^{(m)}$, $1 \leq j \leq k$, $0 \leq m \leq Q$. Suppose finally that $h_1, \dots, h_k$ are linearly independent solutions in $\mathcal{S}$ of an equation*

$$w^{(k)} + \sum_{j=0}^{k-1} B_j w^{(j)} = 0$$

in which the B_j are meromorphic in $\mathcal{A}(r_0)$. Then we have, for $j = 0, \dots, k-1$,

$$m(r, B_j) = S(r, F).$$

The following lemma is proved in [4] for $a_0, \dots, a_{k-1}$ rational functions, and F meromorphic in the plane. The proof extends without modification to the case where $a_0, \dots, a_{k-1}$ are analytic functions and F is meromorphic in $\mathcal{A}(r_0)$. This gives the following lemma.

Lemma 2.6 *Let $\lambda \geq 0$ and $k \geq 2$, and let $a_0, \dots, a_{k-1}$ be analytic functions satisfying $a_j(z) = O(|z|^{(\lambda-1)(k-j)})$ as $z \rightarrow \infty$. Suppose that F is meromorphic in the annulus $\mathcal{A}(r_0)$, as defined by (4), and has in some domain $\mathcal{B}$ a representation as a rational function in solutions f_j of the equation $L(w) = 0$ and their derivatives, where L is defined by (7). If $\lambda > 0$ then $T(r, F) = O(r^\lambda)$ as $r \rightarrow \infty$. If $\lambda = 0$ then $\lim_{z \rightarrow \infty} F(z)$ exists.*

The following lemma is found in [8].

Lemma 2.7 ([8]) *Suppose that $k \geq 1$ and that $a_0, \dots, a_{k-1}$ are analytic in an annulus $\mathcal{A}(r_0)$, as defined by (4), such that, for some $\lambda \geq 0$,*

$$a_j(z) = O(|z|^{(\lambda-1)(k-j)}), \quad \text{as } z \rightarrow \infty. \quad (11)$$

Let $f_j(z)$ be a solution of $L(w) = 0$, where L is defined by (7), in a sectorial region

$$\mathcal{S} = \{z : |z| > r_0, \alpha < \arg z < \alpha + 2\pi\},$$

where α is real. Then as $z \rightarrow \infty$ in $\mathcal{S}$,

$$\log^+ |f_j(z)| = O(|z|^\lambda + \log |z|). \quad (12)$$

Suppose now that k , F , $\mathcal{A}(r_0)$ and $a_0, \dots, a_{k-1}$ are as in the statement of Theorem 1.2. In particular, the a_j satisfy (11). We may define linearly independent analytic solutions $f_1, \dots, f_k$ of (7) in $\mathcal{A}(r_0) \setminus \mathbb{R}^-$. These are analytic in $\mathcal{A}(r_0) \setminus \mathbb{R}^-$ and since the a_j are analytic in $\mathcal{A}(r_0)$, the f_j admit unrestricted analytic continuation in $\mathcal{A}(r_0)$. By (12), $\log^+ |f_j(z)| = O(|z|^\lambda + \log |z|)$ for $j = 1, \dots, k$ and thus the continuations satisfy $\log^+ \log^+ |f_j(z)| = O(\log |z|)$ for $|z| > r_0$, $|\arg z| < 2\pi$. Then $f_1, \dots, f_k$ satisfy the conditions of Lemma 2.5.

Choose a simply connected domain $\mathcal{B} \subseteq \mathcal{A}(r_0)$, on which F has no poles and $\Lambda_k(F)$ has no zeros. Define functions f , g , and h in $\mathcal{B}$ by

$$f'/f = F, \quad \Lambda_k(F) = g^{-k}, \quad h = -Fg. \quad (13)$$

Then f , g and h are analytic on $\mathcal{B}$.

We note that by Lemma 1.1, we have in $\mathcal{B}$ that

$$L(f)/f = f^{(k)}/f + \sum_{j=0}^{k-1} a_j f^{(j)}/f = \Psi_k(F) + \sum_{j=1}^{k-1} a_j \Psi_j(F) + a_0 = \Lambda_k(F). \quad (14)$$

The following lemma is fundamental to our proof of Theorem 1.2 and uses ideas such as analytic continuation and Wronskians. Background material on these topics can be found in [3] and [7].

Lemma 2.8 Define on $\mathcal{B}$ functions w_j and h_j , $j = 1, \dots, k$, by

$$w_j(z) = f'_j(z)g(z) + f_j(z)h(z), \quad h_j(z) = -f'_j(z) + f_j(z)F(z). \quad (15)$$

Then the w_j form a fundamental solution set on $\mathcal{B}$ of the differential equation

$$w^{(k)} + \sum_{j=0}^{k-1} A_j w^{(j)} = 0, \quad (16)$$

in which the A_j are meromorphic functions on $\mathcal{A}(r_0)$ with

$$T(r, A_q) \leq cN_E(r) + S(r, F), \quad r \rightarrow \infty, \quad (17)$$

for $j = 0, \dots, k-1$, where c is a constant depending only on k .

Proof We divide this proof into a number of steps.

(i) The w_j are linearly independent solutions of an equation (16) on $\mathcal{B}$.

We note first that $w_j = fg(f_j/f)'$ on $\mathcal{B}$ by (13), and by (10) and the properties of Wronskians we have

$$W(w_1, \dots, w_k) = f^{-1}g^k(-1)^k W(f_1, \dots, f_k, f) = f^{-1}g^k(-1)^k L(f)W(f_1, \dots, f_k). \quad (18)$$

Then since $L(f)/f = \Lambda_k(F)$ by (14), and $g^k = (\Lambda_k(F))^{-1}$ by (13), we have from (18) that,

$$W(w_1, \dots, w_k) = (-1)^k W(f_1, \dots, f_k). \quad (19)$$

By Lemma 2.1, the right-hand side is not identically zero, since the f_j form a linearly independent solution set of (7). Thus $w_1, \dots, w_k$ form a linearly independent solution set for a differential equation (16). Also, by (9), we have

$$A_{k-1} = -\frac{W(w_1, \dots, w_k)'}{W(w_1, \dots, w_k)} = -\frac{(-1)^k W(f_1, \dots, f_k)'}{(-1)^k W(f_1, \dots, f_k)} = a_{k-1}. \quad (20)$$

(ii) The h_j are linearly independent solutions of a differential equation,

$$w^{(k)} + \sum_{j=0}^{k-1} B_j w^{(j)} = 0, \quad (21)$$

with coefficients B_j that are meromorphic on $\mathcal{A}(r_0) \setminus \mathbb{R}^-$.

We have by (13) and (15) that $w_j = -h_j g$, and so

$$W(w_1, \dots, w_k) = (-1)^k g^k W(h_1, \dots, h_k) = (-1)^k \Lambda_k(F)^{-1} W(h_1, \dots, h_k).$$

Then by (19), we have

$$W(h_1, \dots, h_k) = \Lambda_k(F) W(f_1, \dots, f_k).$$

The right-hand side is not identically zero on $\mathcal{B}$ since $\Lambda_k(F) \neq 0$ on $\mathcal{B}$ and since $f_1, \dots, f_k$ form a linearly independent solution set to (7) on $\mathcal{A}(r_0) \setminus \mathbb{R}^-$. Thus, $h_1, \dots, h_k$ form a linearly independent solution set to a differential equation (21) on $\mathcal{B}$. By (15), we have $h_j = -f'_j + f_j F$ and thus the $h_1, \dots, h_k$ are meromorphic on $\mathcal{A}(r_0) \setminus \mathbb{R}^-$ since the f_j are analytic there, and F is meromorphic on $\mathcal{A}(r_0)$. Hence the coefficients B_j are meromorphic on $\mathcal{A}(r_0) \setminus \mathbb{R}^-$.

(iii) The B_j extend to be meromorphic on $\mathcal{A}(r_0)$ and have poles of multiplicity $\leq k-j$ on $\mathcal{A}(r_0)$.

By analytic continuation of the f_j , the B_j extend to be meromorphic on $\mathcal{A}(r_0)$. Furthermore, (21) has k linearly independent solutions on a neighbourhood of each point of $\mathcal{A}(r_0)$, namely $h_1, \dots, h_k$. Hence, by Lemma 2.2, the B_j have poles of multiplicity $\leq k-j$ on $\mathcal{A}(r_0)$.

(iv) Estimate for $m(r, A_j)$.

First we recall by (15) that $h_j = -f'_j + f_j F$, and note that all the conditions of Lemma 2.5 are satisfied, and thus we have that

$$m(r, B_j) = S(r, F) \quad (22)$$

for $j = 0, \dots, k-1$. Next expressing A_j in terms of B_j and $g^{(p)}/g$, we have for $j = 0, \dots, k-1$,

$$A_j = B_j - \binom{k}{j} \frac{g^{(k-j)}}{g} - \sum_{q=j+1}^{k-1} A_q \binom{q}{j} \frac{g^{(q-j)}}{g}, \quad (23)$$

which is initialised by

$$A_{k-1} = B_{k-1} - \binom{k}{k-1} \frac{g'}{g}.$$

We now note by Lemma 2.4 that $g^{(p)}/g$ is meromorphic in $\mathcal{A}(r_0)$ for all $p \in \mathbb{N}$, and that $m(r, g^{(p)}/g) = S(r, F)$ for $p = 1, \dots, k$. Thus we have by (22) and by induction on (23), that

$$m(r, A_j) = S(r, F)$$

for $j = 0, \dots, k-1$.

(v) Estimate for $N(r, A_j)$.

We show first that the poles of A_j can only arise on E , the set containing all points where $\Lambda_k(F) = 0$.

We know by Lemma 2.2, that the poles of A_j can only arise among the zeros of the continuation of $W(w_1, \dots, w_k)$ and the poles of the continuations of $w_1, \dots, w_k$. By (20), $a_{k-1} = -W(w_1, \dots, w_k)' / W(w_1, \dots, w_k)$ and since a_{k-1} is analytic on $\mathcal{A}(r_0)$, we have that $W(w_1, \dots, w_k)$ continues without zeros. We recall from (13) and 15) that $w_j = (f'_j - f_j F) / (\Lambda_k(F))^{1/k}$. Thus the poles of A_j only arise at poles of F and zeros of $\Lambda_k(F)$.

Now let $z_0 \in \mathcal{A}(r_0) \setminus E$ and suppose that a pole of A_j arises at z_0 . Since $\Lambda_k(F) \neq 0$ on $\mathcal{A}(r_0) \setminus E$ we must have that z_0 is a pole of F , of multiplicity m say, and if $m = 1$ then $\text{Res}(F, z_0) \notin \{1, \dots, k-1\}$ since F satisfies the hypotheses of Theorem 1.2. Then by Lemma 2.3, $\Lambda_k(F)$ has a pole at z_0 of multiplicity mk , and so g can be analytically continued to a neighbourhood of z_0 and has a zero of multiplicity m there. Thus $h = -Fg$ can be analytically continued to z_0 and since the f_j can be continued analytically in $\mathcal{A}(r_0)$, we have that $w_j = f'_j g + f_j h$ can be analytically continued to z_0 . We therefore deduce that the A_j are analytic at z_0 . This contradicts our hypothesis, and so the poles of A_j can only arise in E .

We recall from (iv) that the poles of B_j have multiplicity $\leq k-j$ on $\mathcal{A}(r_0)$. We note also by Lemma 2.4, that $g^{(p)}/g$ have poles of multiplicity $\leq p$ there. Thus we have by (23) that the poles of A_j must have multiplicity $\leq c$ where c is a constant depending only on k . Therefore we have, for $j = 0, \dots, k-1$,

$$N(r, A_j) \leq cN_E(r).$$

(vi) Conclusion.

We have since $T(r, A_j) = N(r, A_j) + m(r, A_j)$ that

$$T(r, A_j) \leq cN_E(r) + S(r, F), \quad \text{as } r \rightarrow \infty,$$

for $j = 0, \dots, k-1$, where c is a constant depending only on k . This completes the proof of Lemma 2.8.

The following lemma appears in [4] and is the final lemma needed for the main part of the proof of Theorem 1.2.

Lemma 2.9 ([4]) *Let $k \geq 1$ be an integer, and let $f_1, \dots, f_k, G, H$ and $a_0, \dots, a_{k-1}$ and $A_0, \dots, A_{k-1}$ be meromorphic in a domain Ω . Suppose that $f_1, \dots, f_k$ are linearly independent solutions in Ω of $L(w) = 0$, where L is defined as in (7). Then the functions $f'_1 g + f_1 h, \dots, f'_k g + f_k h$ are solutions in Ω of the equation (16) if and only if, setting $A_k = 1$ and $A_{-1} = a_{-1} = 0$ and, for $0 \leq q \leq k$,*

$$M_{k,q}(w) = \sum_{m=q}^k \binom{m}{q} A_m w^{(m-q)}, \quad M_{k,-1}(w) = 0,$$

we have, for $0 \leq q \leq k-1$,

$$M_{k,q}(h) - a_q h = -M_{k,q-1}(g) + a_q M_{k,k-1}(g) - (a_q a_{k-1} - a'_q - a_{q-1})g. \quad (24)$$

3 Proof of Theorem 1.2 (Main Part)

We are now in a position to complete the proof of Theorem 1.2. We note that we use methods found in [4, Theorem 3].

Proof of Theorem 1.2 We apply Lemma 2.9 to equation (16) and to g and h in $\mathcal{B}$. The k equations (24) can be written in the form

$$T_q(g) = S_q(h) = \sum_{j=0}^{k-q} c_{j,q} h^{(j)}, \quad 0 \leq q \leq k-1, \quad (25)$$

in which T_q and S_q are homogeneous linear differential operators with coefficients λ_ν which are rational functions in the a_j , A_j and their derivatives. Then by (17) we have

$$T(r, \lambda_\nu) \leq cN_E(r) + S(r, F), \quad \text{as } r \rightarrow \infty, \quad (26)$$

where c is a constant depending only on k .

We have in particular that $q = k-1$ gives

$$M_{k,k-1}(h) - a_{k-1}h = -M_{k,k-2}(g) + a_{k-1}M_{k,k-1}(g) - (a_{k-1}a_{k-1} - a'_{k-1} - a_{k-2})g.$$

Then since

$$\begin{aligned} M_{k,k-1}(h) &= A_{k-1}h + kh', \\ M_{k,k-1}(g) &= A_{k-1}g + kg', \\ M_{k,k-2}(g) &= A_{k-2}g + (k-1)A_{k-1}g' + k(k-1)g''/2, \end{aligned}$$

we now have that

$$\begin{aligned} A_{k-1}h + kh' - a_{k-1}h &= -A_{k-2}g - (k-1)A_{k-1}g' - k(k-1)g''/2 + a_{k-1}A_{k-1}g + \\ &\quad + ka_{k-1}g' - (a_{k-1}a_{k-1} - a'_{k-1} - a_{k-2})g \end{aligned}$$

which gives

$$h' = U(g) = -(k-1)g''/2 + a_{k-1}g'/k + (a'_{k-1} + a_{k-2} - A_{k-2})g/k \quad (27)$$

since $a_{k-1} = A_{k-1}$ on $\mathcal{B}$ by (20). We note that we can then write (25) in the form

$$T_q(g) = c_{0,q}h + \sum_{j=1}^{k-q} c_{j,q} \frac{d^{j-1}}{dz^{j-1}}(U(g)). \quad (28)$$

We distinguish two cases here.

Case 1. We assume that the coefficient of h in at least one S_q in (25) is not identically zero.

Let ν be the largest integer, $0 \leq \nu \leq k-1$, such that $c_{0,\nu} \neq 0$. Then since $h = -Fg$ by (13), equations (25) and (28) give

$$h = -Fg = (c_{0,\nu})^{-1} \left(T_\nu(g) - \sum_{j=1}^{k-\nu} c_{j,\nu} \frac{d^{j-1}}{dz^{j-1}}(U(g)) \right) = V(g). \quad (29)$$

Then by (25), (27) and (29) we know that g solves the system of equations

$$U(g) = \frac{d}{dz}(V(g)), \quad S_q(V(g)) = T_q(g), \quad 0 \leq q \leq k-2. \quad (30)$$

Here we distinguish two sub-cases.

Case 1A. We assume that the solution space of (30) has dimension 1. That is, every common solution of the equations (30) is a constant multiple of g .

Then (26) and a standard reduction procedure, see [6, p.126], give a first order equation

$$p_1 g' + p_0 g = 0, \quad p_1 \not\equiv 0,$$

where the p_j are rational functions in the λ_ν and their derivatives. It follows by (26) that

$$T(r, g'/g) \leq cN_E(r) + S(r, F), \quad \text{as } r \rightarrow \infty,$$

where c is a constant depending only on k . Hence, since $F = -h/g$ and using (26) and (29),

$$T(r, F) \leq cN_E(r) + S(r, F), \quad \text{as } r \rightarrow \infty.$$

Hence we have conclusion (i) of the theorem.

Case 1B. We assume that there is a solution G for the system (30) such that G/g is non-constant. In particular, we note that this will be the case if the system (30) is trivial.

Define H by $H = V(G)$. Then, by (30),

$$H' = U(G), \quad S_q(H) = T_q(G), \quad 0 \leq q \leq k-2.$$

In particular, the equations (25) hold with g and h replaced by G and H respectively. And so, by Lemma 2.9, the functions $f_j H + f'_j G$ are solutions of (16) and so are linear combinations of $w_1, \dots, w_k$. Hence, there are solutions g_j of $L(f) = 0$ where L is defined by (7) such that

$$f_j H + f'_j G - g_j h - g'_j g = 0, \quad 1 \leq j \leq k. \quad (31)$$

We regard the equations in (31) as a system of k equations in H, G, h, g , over the field $\mathbb{F}$ of functions meromorphic in $\mathcal{B}$, with coefficients f_j, f'_j, g_j, g'_j .

Next, we note that the rank of the coefficient matrix is ≤ 3 , since there is a non-trivial solution for the system. Using the method found in [4, Theorem 3], we have that the rank of the system (31) is precisely 3. We can then solve for $-F = h/g$ as a quotient of determinants in f_j, f'_j, g_j, g'_j . Thus F is a rational function of the f_j and their derivatives. By Lemma 2.6

$$T(r, F) = O(r^\lambda + \log r), \quad \text{as } r \rightarrow \infty,$$

and so we have conclusion (ii) of the theorem.

Case 2. We assume that $c_{0,q} \equiv 0$ for $0 \leq q \leq k-1$ in (25).

We then have that the equations (25) are satisfied when g and h are replaced by 0 and 1 respectively and thus so are the equations (24). Then, by Lemma 2.9, the f_j are solutions of (16). Thus the equations $L(f) = 0$ and (16) are the same, where L is defined by (7), and for $1 \leq q \leq k$ we may write

$$f_j h + f'_j g = g_j, \quad (32)$$

in which each g_j is a solution of $L(f) = 0$. Then since f_1 and f_2 are linearly independent, we have $f_1 f'_2 - f'_1 f_2 \not\equiv 0$ and so

$$F = -h/g = (f'_1 g_2 - f'_2 g_1)/(f_1 g_2 - f_2 g_1),$$

which gives F as a quotient of determinants in f_j, f'_j, g_j, g'_j . Then by Lemma 2.6 we have that

$$T(r, F) = O(r^\lambda + \log r), \quad \text{as } r \rightarrow \infty,$$

and so we have conclusion (ii) of the theorem.

This completes the proof of Theorem 1.2.

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