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Viscoelastic bearings with fractional constitutive law for Fractional Tuned Mass Dampers

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Abstract

The paper aims at studying the effects of the inherent fractional constitutive law of viscoelastic bearings used as devices for tuned mass dampers. First, the proper constitutive law of the viscoelastic supports is determined by the local constitutive law. Then, the characteristic force-displacement relationship at the top of the bearing is found. Taking advantage of the whole bearing constitutive laws, the tuning of the mass damper is proposed by defining the damped fractional frequency, which is the analogous of the classical damped frequency. The effectiveness of the optimal tuning procedure is validated by a numerical application on a system subjected to a Gaussian white noise.

keywords: fractional calculus, Tuned Mass Dampers, elastomeric bearings, damped fractional frequency

Introduction

Tuned Mass Damper (TMD) devices for the reduction of structural vibrations have been widely studied and applied in both fields of mechanical and civil engineering [1]-[4]. Several applications of TMDs have been realized worldwide for reducing the wind-induced motions of tall buildings such as the Centerpoint Tower in Sidney, the Citicorp Center in New York, the John Hancock Tower in Boston, the Funade Bridge Tower in Osaka, among others [5]-[7].

A TMD is a device consisting of mass, spring and damper attached to the main structure in order to reduce its dynamic response. The frequency of the damper is tuned to a selected structural frequency so that when the corresponding mode shape is excited, the damper will resonate out of phase with the main structural motion [1]. The mass of the damper transmits its inertia force to the main structure in a direction opposite to the motion of the structure itself, thereby reducing its oscillations.

The initial theory of TMD, as developed by Den Hartog [1], was limited to undamped single degree of freedom (SDOF) systems subjected to sinusoidal force excitations. Extension to damped systems has been investigated by numerous researchers [2]-[11].

Several devices commonly used to enhance damping in mass dampers, such as laminated elastomeric rubber bearings or fluid viscous dampers, have viscoelastic properties and their behaviour has been studied by means of fractional derivative constitutive laws. Theoretical basis for the application of fractional calculus to model viscoelastic materials can be found in [12]-[15] while some applications of viscoelastic dampers and elastomeric rubber bearings modeled by means of fractional integral and derivatives are reported in [16]-[22]. In [23] TMDs with viscoelastic damping elements applied to SDOF systems excited by white noise are considered. The viscoelastic damping is modelled as a force proportional to the fractional derivative of the relative displacement and placed in parallel with a linear spring between the structure and the secondary mass. Numerical analyses are conducted to obtain optimal parameters of the damper and it has been shown that TMDs with fractional derivative damping elements have effectiveness comparable with optimal conventional TMDs, if they are tuned correctly.

In this paper, Fractional Tuned Mass Damper (FTMD) devices, constituted by an oscillating mass connected to the main structure through a viscoelastic link with fractional constitutive law, are studied in detail. Since the most commonly used viscoelastic devices for base isolation and tuned mass dampers are laminated elastomeric rubber bearings, the fractional viscoelastic constitutive law of the damper device is initially determined, starting from the constitutive law of the elastomer. It is worth to stress that such fractional model only needs two parameters for capturing both relaxation and creep behaviours, while more traditional methods involving the combination of simple models, as Maxwell and/or Kelvin models, requires several parameters. The proposed formulation accounts for the presence of the multi-layered elastomeric structure typical of the devices available by manufacturers.

In order to properly analyse the system composed by the main structure and the FTMD, the latter is, at first, studied as a single oscillator. A novel function, denominated Damped Fractional Frequency, is defined for the FTMD, studying the transfer function of the fractional oscillator. It is shown as, using the fractional constitutive law, both the Elasto-Viscous and Visco-Elastic behaviours are modelled at once [24]. A critical value for the order of the fractional derivative, defining the limit between the two behaviours, is calculated based on the defined Damped Fractional Frequency. Then, the latter is utilized for the tuning of the FTMD, adopting classical tuning procedures proposed in literature for TMDs. Moreover, an innovative design formula for the fractional damping coefficient is presented.

Practical application of the FTMD concept is illustrated by numerical examples involving a SDOF system excited by a zero mean Gaussian white noise. The response of the structure is evaluated by using classical tools of stochastic analysis in the complex domain, showing the easiness of the mathematical formulation. Various optimal tuning ratios are tested, based on classic

TMD theory, and the corresponding performance indexes are compared, showing the robustness of the proposed method with respect to the selected tuning criteria for the design of the FTMD.

1. Fractional mechanical properties of elastomeric rubber bearings

Laminated elastomeric rubber bearings have been suggested as support for a damper mass in place of conventional springs [16] to realize innovative TMD devices. Elastomeric bearings consist of alternate layers of rubber and steel shims so that the composite unit exhibits large vertical stiffness and low horizontal stiffness (Figure 1). The rubber layers are introduced because of their high elastic deformations, large elongation at break and their virtual incompressibility. The large vertical stiffness prevents undesirable rocking responses of isolated structures, reduces shear strain and creep deformations in the rubber, while increases the capacity of the bearings to carry axial loads at large displacements. A complete study on the characterization, design and construction of elastomeric rubber bearings is reported in [25], where the effects of axial, shear, torsional and bending loadings on the bearing are studied.

Figure 1. Typical section of an Elastomeric Bearing.

In this section only the effects of the shear forces are considered in order to define a relationship among the geometrical properties of the bearing and the damping coefficient.

The viscoelastic behaviour of the elastomeric bearing can be defined starting from the relaxation test. The latter consists in applying a constant (unitary) deformation and measuring the correspondent loss of the stress in time. Such a stress history is the so-called *relaxation function*, labelled $G(t)$. It has been shown in [26] that the experimental data of the relaxation test are well suited by a power law in the form [12]

$$G(t) = \frac{\bar{C}_\beta}{\Gamma(1-\beta)} t^{-\beta}; \quad 0 \leq \beta \leq 1 \quad (1)$$

where $\Gamma(\cdot)$ is the Euler Gamma function, while $\bar{C}_\beta$ and β are two parameters that characterize the viscoelastic behaviour of the material, as it will be shown later. $\bar{C}_\beta$ and β can be obtained by fitting the experimental data with Eq. (1). The Boltzmann superposition principle asserts that the

stress history $\tau(t)$ for an assigned strain history $\gamma(t)$ is simply given as

$$\tau(t) = \int_0^t G(t-\bar{t}) \dot{\gamma}(\bar{t}) d\bar{t} \quad (2)$$

By inserting the power law (1) in Eq. (2) the following expression is obtained

$$\tau(t) = \bar{C}_\beta \left[\frac{1}{\Gamma(1-\beta)} \int_0^t (t-\bar{t})^{-\beta} \dot{\gamma}(\bar{t}) d\bar{t} \right] \quad (3)$$

The term in the square brackets in Eq. (3) is the so called *Caputo's fractional derivative* of the deformation $\gamma(t)$ [27], that is

$$({}^c D_{0^+}^\beta \gamma)(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\bar{t})^{-\beta} \dot{\gamma}(\bar{t}) d\bar{t} \quad (4)$$

Hence, the local constitutive law is given in the form

$$\tau(t) = \bar{C}_\beta \times ({}^c D_{0^+}^\beta \gamma)(t); \quad 0 \leq \beta \leq 1 \quad (5)$$

It has to be emphasized that, for $\beta = 0$, it follows $\tau(t) = \bar{C}_0 \gamma(t)$, that is a purely elastic behavior is observed, and $\bar{C}_0$ represents the shear modulus. Conversely, for $\beta = 1$ it follows $\tau(t) = \bar{C}_1 \dot{\gamma}(t)$, that is the constitutive law of a purely viscous fluid, and $\bar{C}_1$ is the dissipation coefficient. All other values of β return a variety of viscoelastic behaviors, characteristic of the material at hands.

Once the local constitutive law has been derived, the global relationship of an interlayer can be properly obtained by taking into account for the thickness of each polymeric interlayer, the shape, and the transversal area of the elastomeric device. It is assumed that the elastomeric bearing has a circular cross section. The exact solution in terms of shear stress distribution of a purely elastic beam with circular cross section is given as

$$\begin{aligned} \tau_{zx} &= -\frac{1+2\nu}{4(1+\nu)} \frac{T}{I_x} xy \\ \tau_{zy} &= \frac{3+2\nu}{8(1+\nu)} \frac{T}{I_x} \left(R^2 - y^2 - \frac{1-2\nu}{3+2\nu} x^2 \right) \end{aligned} \quad (6)$$

From Eqs. (6) it is apparent that the shear distribution, and therefore the strain distribution along the transverse section, are different from point to point. However, Eqs. (6) are evaluated under the De Saint Venant hypotheses that in our case are not fulfilled. In fact, the interlayer elastomer is not a long element and it is constrained by steel plates that do not allow displacements in the z direction, so that the shear strains are $\gamma_{zx} = \partial u_x / \partial z$, $\gamma_{zy} = \partial u_y / \partial z$. By assuming that the shear T is in the x direction, it follows that $u_y = 0$. Hence, the only strain is γ_{zx} that, because of the vulcanization procedure, has been assumed to be uniform along the cross section rather than distributed according to Eqs. (6). Moreover, the stress in each interlayer should be uniform as well, and it is simply given by the ratio $\tau = T/A$, being A the cross section area of the rubber core. By inserting the latter expression into Eq. (3), and by denoting h the interlayer thickness and n the total number of elastomeric platelets, the displacement $s(t)$ at the top of the elastomeric device is obtained, and the

applied shear force T is expressed in the form

$$T(t) = \frac{A\bar{C}_\beta}{nh\Gamma(1-\beta)} \int_0^t (t-\bar{t})^{-\beta} \dot{s}(\bar{t}) d\bar{t} = C_\beta \times ({}^c D_{0^+}^\beta s)(t) \quad (7)$$

where $C_\beta = A\bar{C}_\beta/nh$ is the coefficient that can be easily evaluated once the relaxation test on the rubber is performed and the quantities A , n , h are defined.

Summing up, the first step is to perform the relaxation test (possibly a shear test) on a specimen of the rubber that will be used for the FTMD. Then, the relevant parameters $\bar{C}_\beta$ and β that characterize the rubber are determined fitting the relaxation data by Eq. (1). Finally, by means of Eq. (7), the fractional constitutive law of the whole device is obtained. Alternatively, if an appropriate test machine is available, the unitary displacement at the top of the elastomeric bearing can be imposed and the loss of the total force necessary to maintain such displacement can be measured leading to the identification of C_β and β , directly. However, the latter procedure has to be repeated for each elastomeric bearing having different cross sectional area or number of steel platelets or different interlayers.

2. Behaviour of FTMDs

In this section the behavior of the FTMD alone is analyzed in order to prepare the way for the subsequent analyses of the FTMD inserted in a structure, that is the system depicted in Figure 2 is studied. The formulation proposed is limited to the hypothesis of linear behaviour of the system, and therefore small displacements are considered for the mass of the system. In fact, in the case of elastomeric rubber bearings, non-linear hysteretic behaviour should be expected in case of high level of displacements.

Figure 2. Fractional SDOF.

Let m be the mass of the damper device, and C_β and β the characteristic parameters of the viscoelastic device. Let $u^{(a)}(t)$ be the absolute displacement of the support and $s^{(a)}(t)$ the absolute displacement of the mass m . The equilibrium equation of the mass m leads to

$$m\ddot{s}^{(a)}(t) + C_\beta \times ({}^c D_{0^+}^\beta s)(t) = 0 \quad (8)$$

where $s(t) = s^{(a)}(t) - u^{(a)}(t)$ is the relative displacement between the mass m and the support. The Eq. (8) may be rewritten in the form

$$m\ddot{s} + C_\beta \times ({}^c D_{0^+}^\beta s) = -m\ddot{u}^{(a)} \quad (9)$$

By dividing this equation by m it follows

$$\ddot{s} + \eta \left({}^C D_{0^+}^\beta s \right) = -\ddot{u}^{(a)} \quad (10)$$

where $\eta = C_\beta / m$. Eq. (10) is a fractional differential equation in which neither a frequency not a viscous term can be recognized, since the two phases are not well distinguished each other. This is a serious limitation for the design of the parameters η and C_β (once the constituent material of the FTMD is assigned), since the previous studies of the TMD can not be used.

In order to overcome this difficulty, the definition of a Damped Fractional Frequency (DFF) for Eq. (10) is here proposed, analogously to the damped frequency in the classical single oscillator. The DFF is defined as the frequency ω_s such that the square absolute modulus of the transfer function $H_s(\omega)$ of system (10) attains a relative maximum for $\omega > 0$. The Fourier Transform of the fractional term $\eta \left({}^C D_{0^+}^\beta s \right)(t)$ is given as

$$\mathcal{F} \left\{ \eta \left({}^C D_{0^+}^\beta s \right)(t); \omega \right\} = \eta (i\omega)^\beta \hat{S}(\omega) \quad (11)$$

where $\hat{S}(\omega)$ is the Fourier Transform of $s(t)$. It follows that the transfer function $H_s(\omega)$ is given as

$$H_s(\omega) = \frac{1}{\eta (i\omega)^\beta - \omega^2} \quad (12)$$

Taking into account that

$$(i\omega)^\beta = |\omega|^\beta \left[\cos(\pi\beta/2) + i \operatorname{sgn}(\omega) \sin(\pi\beta/2) \right] \quad (13)$$

the transfer function $H_s(\omega)$ is readily found as

$$H_s(\omega) = - \frac{(\eta \omega^\beta \cos(\pi\beta/2) - \omega^2) - i \eta \omega^\beta \sin(\pi\beta/2)}{[\eta \omega^\beta \cos(\pi\beta/2) - \omega^2]^2 + [\eta \omega^\beta \sin(\pi\beta/2)]^2}; \quad \omega > 0 \quad (14)$$

and $H_s(-\omega) = H_s^*(\omega)$, where the star means complex conjugate. The derivative of $|H_s(\omega)|^2$ with respect to ω is given as

$$\frac{\partial |H_s(\omega)|^2}{\partial \omega} = - \frac{2[2\omega^4 + \eta^2 \beta \omega^{2\beta} - \eta(2 + \beta) \omega^{2+\beta} \cos(\pi\beta/2)]}{\omega [\omega^4 + \eta^2 \omega^{2\beta} - 2\eta \omega^{2+\beta} \cos(\pi\beta/2)]^2}; \quad \omega > 0 \quad (15)$$

Hence, the DFF can be found by imposing $\partial |H_s(\omega)|^2 / \partial \omega = 0$, that is by solving the following equation

$$2\omega^4 + \eta^2 \beta \omega^{2\beta} - \eta(2 + \beta) \omega^{2+\beta} \cos(\pi\beta/2) = 0; \quad \omega > 0 \quad (16)$$

The solutions of Eq. (16) are

$$\omega^{\min} = \left(\frac{2\eta\beta}{a + \sqrt{a^2 - 8\beta}} \right)^{\frac{1}{2-\beta}}; \quad \omega^{\max} = \omega_s = \left(\frac{2\eta\beta}{a - \sqrt{a^2 - 8\beta}} \right)^{\frac{1}{2-\beta}} \quad (17)$$

where $a = (2 + \beta) \cos(\pi\beta/2)$, $\omega^{\min}$ corresponds to the relative minimum and $\omega^{\max} = \omega_s$ corresponds to the relative maximum, that is the DFF. In Figure 3 the plot of $|H_s(\omega)|^2$ is reported for different values of β and η versus ω .

Figure 3. Plot of $|H_s(\omega)|^2$ with respect to the frequency ω , for different values of η and β .

From these figures it can be concluded that, for any value of η , the various curves exhibit a relative minimum and a relative maximum (excluding the value $\omega = 0$ in which all the curves attain an infinite value with the exception of the case $\beta = 0$) for values of β lesser than a certain critical value, labeled as β_c and analogous to the critical damping for the classic oscillator. The critical value of β is such that $\omega^{\min} \equiv \omega^{\max}$ and, therefore, it can be determined by the following equation

$$(2 + \beta)^2 \cos(\pi\beta/2)^2 - 8\beta = 0 \quad (18)$$

whose solution, independent of η , is $\beta_c \cong 0.44102$. In Figure 4a the positions of $\omega^{\min}$ and $\omega^{\max}$ versus β are reported in non-dimensional form by considering the non-dimensional values $\eta^{1/(\beta-2)} \omega^{\min}$ and $\eta^{1/(\beta-2)} \omega^{\max}$.

Figure 4. Plot of $\eta^{1/(\beta-2)} \omega^{\min}$ and $\eta^{1/(\beta-2)} \omega^{\max}$ versus β for different values of η .

It has to be stressed that in [24] two different behaviours of the fractional term $({}^c D_{0+}^\beta s)(t)$ are proposed, that is for $0 \leq \beta \leq 0.5$ the system has been termed Elasto-Viscous (EV) in the sense that the elastic phase prevails, while for $0.5 \leq \beta \leq 1$ the system has been termed Visco-Elastic (VE) in the sense that the viscous phase prevails. The critical value that distinguishes the two different behaviours was obtained for the case $m = 0$ (quasi static case) since two different mechanical models are necessary to exactly return the fractional constitutive law expressed in Eq. (5). The

critical value β_c here obtained takes into account the dynamic behaviour. If $\beta < \beta_c$ then the single fractional oscillator shows a resonant behaviour (as for the case of the critical damping $\zeta < 1$), while for $\beta > \beta_c$ the term $|H_s(\omega)|^2$ decreases monotonically (as for the case $\zeta > 1$ in the classical oscillator). These aspects are evidenced in Figure 3 and confirmed in Figure 4 in which for $\beta > \beta_c$ no relative maximum can be found in the real domain $\omega > 0$. In Table 1 the non-dimensional values of $\eta^{1/(\beta-2)}\omega^{\min}$ and $\eta^{1/(\beta-2)}\omega^{\max}$ are reported for different values of β . For design purposes, when $\beta \leq \beta_c$, the DFF $\omega_s = \omega^{\max}$ can be selected with respect to the optimal tuning ratio, as shown in the numerical application sections. While the β value is dependent on the material of the damper device, and therefore it has to be fixed *a priori*, the value of the parameter η can be determined by inverse formula from the Eq. (17):

$$\eta = \frac{a - \sqrt{a^2 - 8\beta}}{2\beta} \omega_s^{2-\beta} \quad (19)$$

allowing the geometric design of the damper device as described in section 1.

Table 1. Non-dimensional values $\eta^{\frac{1}{\beta-2}}\omega^{\min}$ and $\eta^{\frac{1}{\beta-2}}\omega^{\max}$ for different values of β

β	$\eta^{\frac{1}{\beta-2}}\omega^{\min}$	$\eta^{\frac{1}{\beta-2}}\omega^{\max}$
0.00	0.000	1.000
0.05	0.151	0.998
0.10	0.208	0.993
0.15	0.251	0.982
0.20	0.288	0.966
0.25	0.324	0.942
0.30	0.361	0.907
0.35	0.405	0.858
0.40	0.468	0.782
β_c	0.616	0.616

3. Stochastic dynamics of a SDOF system equipped with FTMD

Let us consider a SDOF system excited at the base and equipped with a FTMD as depicted in Figure 5. The equation of motion of this system can be written as

$$\begin{cases} M\ddot{u} + C\dot{u} + Ku - T = -M\ddot{z}_g \\ m\ddot{u} + m\ddot{s} + T = -m\ddot{z}_g \end{cases} \quad (20)$$

where $\ddot{z}_g(t)$ is the base acceleration; $u^{(a)}(t)$ and $u(t) = u^{(a)}(t) - z_g(t)$ are the absolute and the relative displacements with respect to the ground of the primary mass, respectively; M , C and K are the mass, damping coefficient and stiffness of the primary system; $s^{(a)}(t)$ is the damper absolute displacement and $s(t) = s^{(a)}(t) - u^{(a)}(t)$ is the relative displacement of the damper mass m with respect to the primary mass. Dots stand for integer time derivatives. $T(t)$ is the force that the damper mass exerts on the primary mass at each time instant t as defined in Eq. (7).

Figure 5. SDOF Structural system with FTMD.

Eq. (20) can be written in canonical form as:

$$\begin{cases} \ddot{u} + 2\zeta_0\omega_0\dot{u} + \omega_0^2u - \mu\eta(D_{0+}^\beta s) = -\ddot{z}_g \\ \ddot{u} + \ddot{s} + \eta(D_{0+}^\beta s) = -\ddot{z}_g \end{cases} \quad (21)$$

where $\omega_0 = \sqrt{K/M}$ is the natural frequency and $\zeta_0 = C/2\sqrt{MK}$ the damping ratio of the main system, $\mu = m/M$ is the damper/structure mass ratio and $\eta = C_\beta/m$ as defined in the previous section.

Since the excitation is a stochastic process, the response statistics of the linear system in Eq. (21) can be computed by means of the standard tools of stochastic calculus. In the following, capital letters for displacements have been introduced, as customary, since the response to a stochastic process is, in turn, a stochastic process. Then, by applying the Fourier transform to Eq. (21) and rearranging in matrix form, it is possible to obtain:

$$\mathbf{A}(\omega)\mathbf{V}(\omega) = -\mathbf{I}\ddot{Z}_g(\omega) \quad (22)$$

where

$$\mathbf{A}(\omega) = \begin{bmatrix} H_U^{-1}(\omega) & -\mu\eta(i\omega)^\beta \\ -\omega^2 & H_S^{-1}(\omega) \end{bmatrix}; \quad \mathbf{V}(\omega) = \begin{bmatrix} U(\omega) \\ S(\omega) \end{bmatrix}; \quad \mathbf{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (23)$$

and

$$H_U(\omega) = \frac{1}{2\zeta_0\omega_0(i\omega) + \omega_0^2 - \omega^2}; \quad H_S(\omega) = \frac{1}{\eta(i\omega)^\beta - \omega^2} \quad (24)$$

The one-sided Power Spectral Density function (PSDF) of the response $\mathbf{V}(\omega)$ can be computed as:

$$\mathbf{G}_V(\omega) = G_{\ddot{Z}_g}(\omega)\mathbf{A}^{-1}(\omega)\mathbf{1}\mathbf{1}^T\mathbf{A}^{-T*}(\omega) \quad (25)$$

where

$$\mathbf{G}_V(\omega) = \begin{bmatrix} G_{UU}(\omega) & G_{US}(\omega) \\ G_{US}^*(\omega) & G_{SS}(\omega) \end{bmatrix} \quad (26)$$

and $G_{\ddot{Z}_g}(\omega)$ is the PSDF of the base acceleration process. Since, in the following, the base acceleration is modeled as a zero-mean Gaussian white noise, the effectiveness of the damping device can be estimated by comparing the variance of the primary system:

$$\sigma_U^2 = \int_0^\infty G_{UU}(\omega)d\omega \quad (27)$$

with the variance of the uncontrolled structure:

$$\sigma_{U_0}^2 = \int_0^\infty |H_U(\omega)|^2 G_{\ddot{Z}_g}(\omega) d\omega \quad (28)$$

In the following sections, the tuning procedure for the FTMD devices will be applied and the effectiveness of the control strategy will be assessed by means of numerical applications on a SDOF system.

4. Applications

Numerical applications are reported in this section to validate the FTMD effectiveness measured in terms of reduction of the main structure response variance. The proposed formulation has been implemented for a SDOF system using the tuning criterion proposed by Gosh and Basu [28] for the case of damped structures excited by broad-band signals:

$$\nu_{opt} = \sqrt{\frac{1 - 4\zeta_0^2 - \mu(2\zeta_0^2 - 1)}{(1 + \mu)^3}} \quad (29)$$

In Eq. (29), ν_{opt} is the optimal tuning ratio, defined as the ratio between the frequency of the TMD and of the primary structure. Of course, in this case, the DFF is used. In this way, once the mass ratio μ is chosen, it is possible to compute the value of the damper frequency $\omega_s = \nu_{opt}\omega_0$ and the FTMD damping coefficient by using Eq. (19). Of course, a different tuning criteria can be used if the excitation on the structure is of different nature ([1], [4]). Also, for the considered case, being the structure excited by an external white noise, the H2 tuning criterion proposed in [29] can be used in lieu of Gosh and Basu's criterion.

It is worth to be noted that the optimal tuning ratio reported in Eq. (29) has been derived for a TMD model in which the zero and first order derivatives appear. However, since we have assumed that the DFF is the frequency at which $|H_s(\omega)|^2$ exhibits a maximum, as soon as we impose $\omega_s = \nu_{opt}\omega_0$ the implicit dependence on the order of the fractional derivative is taken into account. For this reason, a second tuning procedure to find the values of C_β that numerically minimize the main structure response variance has been implemented. Results of the two tuning criteria have been compared in terms of statistics of the structural response.

Let us consider a system described by Eq. (21), where the primary system parameters have been selected as $\omega_0 = \pi$ rad/s and $\zeta_0 = 0.01$, while the FTMD mass ratio is $\mu = 0.03$. Numerical simulations have been performed for two different values of the fractional derivative order $\beta = 0.15$ and $\beta = 0.30$.

The base excitation is modelled as a zero-mean white noise, i.e. $\ddot{Z}_g(t) = W(t)$, with two-sided power spectral density S_w defined as:

$$E[W(t)W(t+\tau)] = 2\pi S_w \delta(\tau) \quad (30)$$

where $E[\cdot]$ is the expectation operator and $\delta(\cdot)$ is the Dirac delta function. The one-sided PSDF of

the input can be computed as $G_{\ddot{z}_g}(\omega) = 2S_w$ and the matrix $\mathbf{G}_v(\omega)$ takes the form:

$$\mathbf{G}_v(\omega) = 2S_w \mathbf{A}^{-1}(\omega) \mathbf{I} \mathbf{I}^T \mathbf{A}^{-T*}(\omega) \quad (31)$$

The equation of motion has been solved by means of stochastic analysis for an input white noise having unitary strength: $G_{\ddot{z}_g}(\omega) = 1 \text{ (m/s}^2\text{)}^2 / (\text{rad/s})$. The displacement variance of the uncontrolled system has been determined by means of Eq. (28) and according to [30] $(\sigma_{U_0}^2 = \pi G_{\ddot{z}_g} / 4\zeta_0 \omega_0^3 = 2.537 \text{ m}^2/\text{s}^4)$. For the controlled system, the variance of the response has to be calculated for each value of the FTMD parameters, using Eq. (27). Hence, the following performance index has been defined:

$$\varepsilon_s = \frac{\sigma_{U_0} - \sigma_U}{\sigma_{U_0}} \quad (32)$$

Table 2 summarizes results evaluated with the two adopted tuning criteria and considering $\beta = 0.15$ and $\beta = 0.30$. Each column reports results in terms of the DFF, optimal frequency ratio, η , displacements variance of the controlled system and performance index.

The PSDF of the uncontrolled system displacements has been evaluated as:

$$G_{U_0} = |H_U(\omega)|^2 G_{\ddot{z}_g}(\omega) = 2S_w |H_U(\omega)|^2 \quad (33)$$

and it is depicted in Figure 6-a and Figure 6-b together with the PSDFs of the controlled system displacements, computed as described in Eq. (25) for $\beta = 0.15$ and $\beta = 0.30$, respectively.

Table 2. Displacement variances and performance indices for $\beta = 0.15$ and $\beta = 0.30$.

	$\beta = 0.15$		$\beta = 0.30$	
	ν_{opt} eq. (29)	Min Variance	ν_{opt} eq. (29)	Min Variance
ω_s [rad/s]	3.048	2.989	3.048	2.749
ν_{opt}	0.971	0.952	0.971	0.876
η [s $^{\beta-2}$]	8.121	7.832	7.844	6.583
σ_U^2 [m 2 /s 4]	1.063	1.051	1.540	1.418
ε_s	0.542	0.545	0.449	0.471

Results confirm that the FTMD is able to reduce the structural response under broad band random loads. In fact, the data obtained by using Eq. (29) and by means of numerical minimization of the primary mass displacement variance show that the damper efficiency is slightly influenced by the used tuning criterion, as reported in Table 2, thus assessing the correctness of DFF as the maximum of $|H_s(\omega)|^2$.

Between the two reported numerical examples, the damping effect on the primary mass is enhanced when $\beta = 0.15$. In this case, two distinct peaks can be observed in the PSDF of the controlled system (Figure 6-a), analogously to the typical behaviour of the TMDs. Conversely, when $\beta = 0.30$ a single peak is shown (Figure 6-b). This depends on the fact that the Elasto-Viscous behaviour for $\beta = 0.15$ is more pronounced than the case $\beta = 0.30$.

Figure 6. Primary mass displacement PSDFs. Uncontrolled vs. controlled system for: a) $\beta = 0.15$; b) $\beta = 0.30$.

Sensitivity analysis of the performance index with respect to the mass ratio, the β coefficient, and damping on the primary system, has been performed. First, for two different primary systems ($\zeta = 0.5\%$ and $\zeta = 3.0\%$), the FTMD efficiency has been analysed varying μ and β ; for each case, the tuning criterion in Eq. (29) has been used. The β values have been considered in the range $0 < \beta < \beta_c$. In fact, for larger values of β , the concepts of DFF and optimal tuning make no sense. Results are illustrated in Figure 7a-b. As expected, increasing the mass ratio leads to a larger reduction of the vibrations of the primary system, but the improvement in the FTMD efficiency is gradually reduced for increasing mass ratios.

Figure 7. Sensitivity analysis of the performance index with respect to the mass ratio: a) $\zeta_0 = 0.5\%$; b) $\zeta_0 = 3.0\%$.

Conversely, Figure 8a-b depict similar curves considering two selected mass ratios and various values of damping in the primary system. It is possible to observe that, for each selected configuration, the FTMD becomes less efficient for increasing damping in the primary structure. Finally, although a closed-form expression for the variance of the controlled system and for the FTMD efficiency has not yet been determined, it can be observed that the optimal β value remains the same for all values of the damping in the primary structure, once the mass ratio is fixed. Therefore, it can be deduced that the β optimum value only depends on the two masses of the primary structure and FTMD.

Figure 8. Sensitivity analysis of the performance index with respect to the primary system damping ratio: a) $\mu = 1.0\%$; b) $\mu = 3.0\%$.

Conclusions

A novel formulation for Fractional Tuned Mass Dampers has been proposed, considering the case of elastomeric rubber bearings linking the damper mass to the main structure and using a fractional derivative constitutive model. A relation between the geometry of the bearing itself and the value of the damping coefficient has been presented, thus allowing an easy design of the damping device for the practical case of elastic rubber bearings. In fact, the fractional constitutive law only requires two parameters for modeling both relaxation and creep behaviors, in opposition to more traditional models involving several parameters. Numerical applications have been implemented for a SDOF system subjected to a Gaussian white noise. The displacement variances for both uncontrolled and controlled systems have been determined by stochastic calculus and the efficiency of the FTMD has been analyzed for various values of the damper parameters. Based on the proposed formulation and results, the following main concepts should be highlighted.

- A specific design frequency can be identified for the FTMD, the Damped Fractional Frequency. The closed-form expression of the DFF shows its dependency on both the parameters of the fractional oscillator.
- A critical value has been identified for the β parameter, distinguishing the dynamic Elasto/Viscous behaviour and the Visco/Elastic behaviour. This critical value can be determined starting from the definition of the DFF, and it has been calculated numerically as approximately 0.44.
- The efficiency of the FTMD for reducing vibrations in the case of structures subjected to white noise has been tested. Thanks to the definition of the DFF, existing tuning criteria defined for TMD can be easily applied for the FTMD case, as well. A design formula for the FTMD has been proposed, linking the damping coefficient with the DFF and the fractional order derivative of the constitutive law.
- The use of a performance index, defined in terms of reduction of the main structure displacement variances, has given the possibility of comparing the efficiency of the FTMD in several cases. The FTMD has shown to have a very small sensitivity to small variations

of the tuning frequency. Hence, the presence of uncertainties in the knowledge of the primary structure parameters has very small influence on the FTMD efficiency.

- The FTMD efficiency, in terms of performance index, increases when incrementing the mass ratio between the primary structure and the damping device. However, the gain in performance becomes less and less significant increasing the mass ratio itself. Conversely, the FTMD efficiency decreases when incrementing the damping in the primary structure.
- For each studied configuration of damping in the primary system and mass ratio between the two oscillators, and optimal β value can be identified. Although a closed-form expression for the optimal value has still not been found. Numerical tests suggest that such optimal value should be independent on the damping in the primary structure.

Further analytical and experimental aspects regarding the effects of the bearing geometry (e.g., cross section area, rubber layer thickness) and material on the fractional parameters as well as optimal tuning of the device are currently under investigations.

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