This item was submitted to [Loughborough's Research Repository](#) by the author.
Items in Figshare are protected by copyright, with all rights reserved, unless otherwise indicated.

Two-dimensional development of the dynamic coupled consolidation scaled boundary finite-element method for fully saturated soils

PLEASE CITE THE PUBLISHED VERSION

PUBLISHER

© Elsevier

VERSION

AM (Accepted Manuscript)

LICENCE

CC BY-NC-ND 4.0

REPOSITORY RECORD

El-Hamalawi, Ashraf, and Mahmoud Hassanen. 2019. "Two-dimensional Development of the Dynamic Coupled Consolidation Scaled Boundary Finite-element Method for Fully Saturated Soils". figshare. <https://hdl.handle.net/2134/5159>.

This item was submitted to Loughborough's Institutional Repository (<https://dspace.lboro.ac.uk/>) by the author and is made available under the following Creative Commons Licence conditions.

For the full text of this licence, please go to:
<http://creativecommons.org/licenses/by-nc-nd/2.5/>

2D DEVELOPMENT OF THE DYNAMIC COUPLED CONSOLIDATION SCALED BOUNDARY FINITE ELEMENT METHOD FOR FULLY SATURATED SOILS

A. El-Hamalawi^{a, *} and M. Hassanen^b

^a *Civil & Building Engineering Department, Loughborough University, Loughborough
LE11 3TU, United Kingdom*

^b *Department of Civil Engineering, Strathclyde University, 107 Rottenrow, Glasgow G4
0NG, UK*

Abstract

The scaled boundary finite element method is a powerful tool used to analyse far-field boundary soil-structure interaction problems. In this paper, the method is extended to include Biot's coupled consolidation in order to deal with fully saturated soil as a two-phase medium. The advantages of this method are explained in this paper. The detailed formulation considers the general 2D analysis case, accounting for body forces and surface tractions in both bounded and unbounded media.

Keywords : scaled boundary finite element method, coupled consolidation, soil mechanics, fully saturated.

1. Introduction

For a number of practical engineering applications such as very stiff and massive structures, the numerical study of transient soil-structure interaction under dynamic conditions is important. Most engineers in industry currently attempt to idealise these

* Corresponding author. Address : *Civil & Building Engineering Department, Loughborough University, Loughborough LE11 3TU, United Kingdom. Tel: +44-1509 223206; fax: +44-1509 223981*

E-mail address: cvae4@lboro.ac.uk

problems by performing a quasi-static analysis, and assuming that the soil in question is a single-phase material. Saturated soil is however a multi-phase material and its behaviour is governed by its main constituents: solid particles and pore fluids. As the pore pressures build up with time, such as in an earthquake during liquefaction, the structural response is affected significantly. To therefore produce a more realistic representation of saturated soil under dynamic loads, a two-phase coupled solid-fluid formulation is required. Several numerical methods exist to model dynamic soil-structure interaction, but modelling the soil as a half-space has many shortcomings, as described below.

Fig. 1. Layout of a typical dynamic soil-structure interaction problem

Fig. 1 shows an embedded structure overlying soil, which extends to infinity. The problem constitutes the determination of the dynamic response of the structure interacting with the soil, upon applying a time-varying load such as seismic waves in an earthquake or vibrations due to machinery. The soil can be split into two parts: a bounded near-field medium, which could comprise several layers, and an unbounded far-field medium that extends to infinity. As the dynamic load takes effect, waves are emitted and propagate throughout the soil in all directions either towards infinity or towards the structure depending on the initial input waves' source. The most important question when idealising the soil domain is how to best model the infinite boundary. Numerically, this has usually been achieved by truncating the remote boundaries and then imposing free or fixed boundary conditions. If input waves originate at the structure, and propagate through the soil towards infinity, the artificial far-field domain boundary would reflect waves back into the near-field and structure, thus producing erroneous results. Proper modelling of the far-field boundary is also very important at

high frequencies above the cut-off frequency, in order to provide a better representation of the radiation damping that occurs. A proper quantification of the level of radiation damping that exists would help engineers make use of this condition when designing. Despite much progress in finite element technology, the requirement for a computationally efficient technique that is able to capture accurately the 3-dimensional stress wave transmission in anisotropic soils still poses an important challenge within engineering mechanics.

One solution to this problem when using the "industry-standard" finite element method (FEM) in analysing the problem would be to use an extended FE mesh with a large number of elements to model the region where the stress waves may propagate during the time scale of the analysis. This might provide a solution for short duration transient problems, but with practical earthquake analyses, where the waves will travel a great distance during earthquake excitation (say up to 30 seconds), an excessive and impractical computational cost would be incurred. In 3D, an extended mesh would require prohibitively high computer resources by today's standards, and would thus not be viable. In order to avoid this, different boundaries have been developed to minimise wave reflections and absorb the wave energy, such as viscous energy-absorbing boundaries [1], [2], or Smith boundaries [3]. These however have not been entirely effective at simulating the far-field. Boundaries by Smith [3] were restricted to applications having simple geometries, while the viscous energy-absorbing boundary was ineffective for body waves with angles of incidence of less than 30° and for surface waves. Bettess and Zienkiewicz [4] introduced infinite elements to simulate the far-field boundary for wave propagation problems in single-phase porous elastic media, followed by other researchers such as Chow and Smith [5], and Zhao and Valliappan [6]. The element shape functions used are decay

functions representing the wave propagation towards infinity. Simoni and Schrefler [7], and Khalili et al. [8] used infinite elements for wave propagation in 2-phase saturated porous media. These approaches improved the modelling, but still do not provide an accurate representation, as infinite elements are not exact in the finite-element sense, so are less accurate than the remaining finite element near-field domain.

One method that overcomes the problems associated with modelling the far-field boundary is the boundary element method (BEM). The fundamental solution required for the BEM formulation would satisfy the radiation damping condition at the far-field boundary exactly. Cheng and Liggett [9], and Cheng and Detournay [10], developed a BEM formulation to model 2D static linear elastic consolidation formulated in the Laplace transform space, using Cleary's fundamental solution [11]. Dargush and Banerjee [12] extended this formulation to 3D. By using an analogy between poroelasticity and thermoelasticity, different fundamental solutions were presented by different researchers [13], [14], and used by Cheng et al. [10] and Dominguez [14] to produce dynamic BEM formulations for poroelasticity. However, an understanding of the boundary element method requires a rigorous theoretical mathematics background not generally available to civil engineers. In addition, one of the major drawbacks of the method is the necessity of a fundamental solution to exist. Such an analytical solution is difficult to derive, and in some cases exhibits singularities, while in other cases a fundamental solution does not exist. The latter makes it very complicated and sometimes impossible to apply the BEM to problems involving plasticity, except in the simplest of cases.

The finite-boundary element method (FBEM, e.g. [15]), is one method that combines the advantages of both the FEM and BEM. The near-field is discretised into finite elements, which allows any elastic or plastic constitutive model to represent the soil without the need of a fundamental solution, while the far-field is modeled using the

BEM, which improves the modelling of the far-field boundary and radiation condition. The application of the method however is not easy, and problems occur along the interface separating the finite and boundary elements, and with layered soils.

The scaled boundary finite element method (SBFEM), developed by Wolf and Song [16], overcomes many of these shortcomings when modelling the far-field boundary in a dynamic soil-structure interaction situation. A scaling centre is chosen outside the soil area as shown in Fig. 2, with the soil-structure interface split into 1D finite elements. A typical finite element 1-2 is shown in Fig. 2 in relation to the scaling centre O. A scaled transformation in coordinates is performed from the Cartesian to the radial dimensionless coordinate ξ such that the unbounded far-field soil corresponds to the domain $1 \leq \xi < \infty$ and the bounded near-field area corresponds to the domain $0 \leq \xi \leq 1$. In the frequency domain, the governing differential equations of elastodynamics are transformed into a system of linear second-order ordinary differential equations, either by using the weighted-residual method [17], or by using the concept of similarity [16]. After defining the dynamic stiffness, a system of non-linear first-order ordinary differential equations results, which is solved for the dynamic stiffness matrix as a function of frequency. A similar derivation process yields the unit-impulse response matrix in the time domain. At high frequencies above the cut-off frequency, the use of an asymptotic expansion to represent the dynamic-stiffness matrix yields a boundary condition satisfying the far-field radiation condition exactly. The computed solutions are exact in a radial direction (from the soil-structure boundary pointing towards infinity), while converge to the exact solution in the finite element sense in the circumferential direction parallel to the soil-structure boundary interface. From a computational point of view, the SBFEM results in symmetric dynamic-stiffness and unit-impulse response matrices, which would not be the case if the BEM or FBEM were used, making it much more efficient time and storage-wise.

A major advantage for both the academic and engineer in industry is during the initial discretisation of the problem to be analysed, where similar to the BEM, only the boundary rather than the domain needs discretisation. This means that for 3D problems, 2D surface finite elements are used to discretise the boundary, which result in reducing the spatial dimension by one, and in turn the number of degrees of freedom during the analysis. It also allows very complicated 3D meshes to be discretised easily (overcoming the problems inherent in approximating 3D problems into 2D) and removes the significant time burden associated with generating a 3D mesh. Although 2D modelling of a 3D model is convenient, it has many problems such as the contact area for 2D problems being larger (e.g. if plane strain is assumed) than in 3D, which will overestimate radiation damping for frequencies above the cut-off frequency. As with the BEM, by not discretising the domain, internal displacements and stresses are computed more accurately than with the FEM. However, in contrast to the BEM, the behaviour of the bounded near-field soil can be represented in the SBFEM using any advanced constitutive model (for example, based on plasticity). This is due to the SBFEM not requiring a fundamental solution, unlike the BEM/FEBEM, which also eliminates the need to evaluate singular integrals occurring in most fundamental solutions. Anisotropy can also be modelled using the SBFEM without an increase in computational effort. The advantages of the FEM and BEM are thus combined. The method currently exists for single-phase media. A more accurate representation of the far-field boundary during a dynamic soil-structure interaction analysis would be achieved by extending it to account for two-phase media. The aim of this paper is to develop a numerical formulation capable of correctly modelling the dynamic far-field boundary condition for two-phase media, and study its effect on the time-dependent interaction between a structure, and the underlying local soil.

2. Governing Equations

The scaled boundary finite element method (SBFEM) equations derived in the past have been based on the equilibrium equations (e.g. [18]). In this section, the formulation will be re-derived to incorporate the process of consolidation in the soil. This is especially important in geotechnical engineering as it accounts for the changing pore water pressures that occur in between the drained and undrained states. The standard SBFEM formulation would only allow the modelling of soils in the long-term (drained) case, or short-term (undrained) with some modification to the formulation to take into consideration the pore water pressures, which are a function of volumetric strain during the undrained case. This however would not be satisfactory for most soils undergoing the crucial intermediate consolidating stage.

The coupled consolidation equations [19] comprise a system of simultaneous differential equations which satisfy; (a) the dynamic equilibrium equations of motion and (b) the fluid continuity equations. The following 2-dimensional differential equations of motion (expressed in terms of displacement amplitudes) are exact in the frequency domain.

$$\frac{\partial(\sigma'_x + p)}{\partial \hat{x}} + \frac{\partial(\tau_{xy} + p)}{\partial y'} - \rho \ddot{u}_x + \rho b_x = 0 \quad (1)$$

$$\frac{\partial(\sigma'_y + p)}{\partial y'} + \frac{\partial(\tau_{xy} + p)}{\partial \hat{x}} - \rho \ddot{u}_y + \rho b_y = 0$$

where σ' is the effective stress, p is the fluid pressure (in which the total stress, σ , relates to the fluid pressure as $\{\sigma\} = \{\sigma'\} + \{m\}p$), u is the displacement, ω and ρ are the frequency and density respectively, and $\{b\}$ is a vector representing the body forces per unit volume.

The fluid continuity equation is shown below :

$$-k\left(\frac{\partial^2 p}{\partial \hat{x}^2} + \frac{\partial^2 p}{\partial \hat{y}^2}\right) + \omega^2 k \rho_f \left(\frac{\partial u_x}{\partial \hat{x}} + \frac{\partial u_y}{\partial \hat{y}}\right) + k \rho_f \left(\frac{\partial b_x}{\partial \hat{x}} + \frac{\partial b_y}{\partial \hat{y}}\right) + i\omega \left(\frac{\partial u_x}{\partial \hat{x}} + \frac{\partial u_y}{\partial \hat{y}}\right) + i\omega \frac{n}{K_f} p = 0 \quad (2)$$

where p is the fluid pressure , u , ω , k , n , K_f and ρ are the displacement, frequency, soil permeability and porosity, fluid bulk modulus and density respectively, and $\{b\}$ is the body forces per unit volume vector.

3. Scaled Boundary Finite Element Transformation of Geometry

Fig. 2. Scaled boundary transformation of geometry of line finite element

Consider a finite line element 1-2 forming the base of the triangle in Fig. 2 with the opposite apex of the triangle coinciding with the scaling centre O and the origin of the Cartesian coordinates system. Any point A on the boundary line element 1-2, with local coordinates (x, y) , can be represented by its position vector $\mathbf{r}$,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} \quad (3)$$

If the origin, O , of the Cartesian coordinates $(\hat{x}, \hat{y})$ coincides with the apex of the triangle, then a point within the domain may be described in Cartesian coordinates by its global position vector $\hat{\mathbf{r}}$ as

$$\hat{\mathbf{r}} = \hat{x}\mathbf{i} + \hat{y}\mathbf{j} \quad (4)$$

To transfer from the Cartesian to the curvilinear co-ordinate system (ξ, η) , any point within the domain (in which $\xi = 1$ at the boundary and $\xi = 0$ at the scaling centre) may be described by scaling using the position vector of the corresponding boundary point; $\hat{r} = \xi \mathbf{r}$. Fig. 2 shows the geometry of the line boundary with the tangential vector (slope) in the η direction, represented by the derivative of the point A's position vector on the line as shown in equation (5) :

$$\mathbf{r}_{,\eta} = x_{,\eta} \mathbf{i} + y_{,\eta} \mathbf{j} \quad (5)$$

Mathematically, the geometry of the boundary of the finite element shown in Fig. 2 may be represented by interpolating its nodal coordinates $\{x\}$ and $\{y\}$ using the local coordinates η at the boundary as follows :

$$x(\eta) = [N(\eta)]\{x\} = [N]\{x\} \quad \text{and} \quad y(\eta) = [N(\eta)]\{y\} = [N]\{y\} \quad (6)$$

where $[N] = [N(\eta, \zeta)] = [N_1(\eta, \zeta) \quad N_2(\eta, \zeta) \quad \dots \quad N_n(\eta, \zeta)] = [N_1 \quad N_2 \quad \dots \quad N_n]$ and n is the number of element nodes. Using the scaling relationship $\hat{r} = \xi \mathbf{r}$ to describe the position of any point within the domain leads to

$$\hat{x}(\xi, \eta) = \xi x(\eta) \quad \text{and} \quad \hat{y}(\xi, \eta) = \xi y(\eta) \quad (7)$$

Equation (6) is used to transfer the differential operators in the $(\hat{x}, \hat{y})$ co-ordinate system to those corresponding to the (ξ, η) co-ordinate system.

$$\begin{Bmatrix} \partial/\partial \hat{x} \\ \partial/\partial \hat{y} \end{Bmatrix} = [\hat{\mathbf{J}}(\xi, \eta)]^{-1} \begin{Bmatrix} \partial/\partial \xi \\ \partial/\partial \eta \end{Bmatrix} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} y_{,\eta} & -x_{,\eta} \\ -y & x \end{bmatrix} \begin{Bmatrix} \partial/\partial \xi \\ 1/\xi \partial/\partial \eta \end{Bmatrix} \quad (8)$$

where the Jacobian matrix is $[\hat{J}(\xi, \eta)] = \begin{bmatrix} \hat{x}_{,\xi} & \hat{y}_{,\xi} \\ \hat{x}_{,\eta} & \hat{y}_{,\eta} \end{bmatrix}$ and the determinant is $|J| = xy_{,\eta} - yx_{,\eta}$.

At $\xi=1$, on the curved boundary, the unit outward normal vector is defined by equation (9).

$$\mathbf{n}^\xi = \frac{\mathbf{g}^\xi}{g^\xi} = n_x^\xi \mathbf{i} + n_y^\xi \mathbf{j} = \frac{1}{g^\xi} [y_{,\eta} \mathbf{i} - x_{,\eta} \mathbf{j}] \quad \text{where } \mathbf{g}^\xi = \mathbf{r}_{,\eta} = y_{,\eta} \mathbf{i} - x_{,\eta} \mathbf{j} \quad (9)$$

Similarly, the unit outward normal vector to the line (ξ) is :

$$\mathbf{n}^\eta = \frac{\mathbf{g}^\eta}{g^\eta} = n_x^\eta \mathbf{i} + n_y^\eta \mathbf{j} = \frac{1}{g^\eta} [-y \mathbf{i} + x \mathbf{j}] \quad \text{where } \mathbf{g}^\eta = \mathbf{r} = -y \mathbf{i} + x \mathbf{j} \quad (10)$$

where $g^\xi = |\mathbf{g}^\xi| = \sqrt{(y_{,\eta})^2 + (-x_{,\eta})^2}$ and $g^\eta = |\mathbf{g}^\eta| = \sqrt{(-y)^2 + (x)^2}$. Substituting the derivative relationships results in :

$$\begin{Bmatrix} \frac{\partial}{\partial \hat{x}} \\ \frac{\partial}{\partial \hat{y}} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} g^\xi n_x^\xi & g^\eta n_x^\eta \\ g^\xi n_y^\xi & g^\eta n_y^\eta \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{1}{\xi} \frac{\partial}{\partial \eta} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} g^\xi n_x^\xi \frac{\partial}{\partial \xi} & \frac{1}{\xi} g^\eta n_x^\eta \frac{\partial}{\partial \eta} \\ g^\xi n_y^\xi \frac{\partial}{\partial \xi} & \frac{1}{\xi} g^\eta n_y^\eta \frac{\partial}{\partial \eta} \end{bmatrix} \quad (11)$$

3.1 The Equilibrium Equations of Motion

For harmonic motion, the velocity and acceleration may be represented by

$$\dot{u} = \frac{\partial u}{\partial t} = i\omega u \quad \text{and} \quad \ddot{u} = \frac{\partial^2 u}{\partial t^2} = -\omega^2 u \quad \text{respectively.}$$

The equilibrium equations (1) may thus be rewritten as

$$\begin{aligned}
\frac{\partial(\sigma'_x + p)}{\partial \hat{x}} + \frac{\partial(\tau_{xy} + p)}{\partial y'} + \omega^2 \rho u_x + \rho b_x &= 0 \\
\frac{\partial(\sigma'_y + p)}{\partial y'} + \frac{\partial(\tau_{xy} + p)}{\partial \hat{x}} + \omega^2 \rho u_y + \rho b_y &= 0
\end{aligned} \tag{12}$$

In vector notation;

$$[L]^T (\{\sigma'\} + \{m\}p) + \omega^2 \rho \{u\} + \rho \{b\} = 0 \tag{13}$$

where for the 2-dimensional case;

$$\{m\} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ total normal stress} = \{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}, \text{ strain} = \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}, \{u\} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \{b\} = \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

$$\text{and } [L] = \begin{bmatrix} \partial/\partial \hat{x} & 0 \\ 0 & \partial/\partial \hat{y} \\ \partial/\partial \hat{y} & \partial/\partial \hat{x} \end{bmatrix}$$

Applying the differential transformation equations (11) to the differential operator [L] results in equation (14) :

$$[L] = \frac{1}{|J|} \begin{bmatrix} g^\xi n_x^\xi \frac{\partial}{\partial \xi} + \frac{1}{\xi} g^\eta n_x^\eta \frac{\partial}{\partial \eta} & 0 \\ 0 & g^\xi n_y^\xi \frac{\partial}{\partial \xi} + \frac{1}{\xi} g^\eta n_y^\eta \frac{\partial}{\partial \eta} \\ g^\xi n_y^\xi \frac{\partial}{\partial \xi} + \frac{1}{\xi} g^\eta n_y^\eta \frac{\partial}{\partial \eta} & g^\xi n_x^\xi \frac{\partial}{\partial \xi} + \frac{1}{\xi} g^\eta n_x^\eta \frac{\partial}{\partial \eta} \end{bmatrix} = \frac{g^\xi}{|J|} \begin{bmatrix} n_x^\xi & 0 \\ 0 & n_y^\xi \\ n_y^\xi & n_x^\xi \end{bmatrix} \frac{\partial}{\partial \xi} + \frac{g^\eta}{\xi |J|} \begin{bmatrix} n_x^\eta & 0 \\ 0 & n_y^\eta \\ n_y^\eta & n_x^\eta \end{bmatrix} \frac{\partial}{\partial \eta} = [b^1] \frac{\partial}{\partial \xi} + \frac{1}{\xi} [b^2] \frac{\partial}{\partial \eta} \tag{14}$$

$$\text{where } [b^1] = \frac{g^\xi}{|J|} \begin{bmatrix} n_x^\xi & 0 \\ 0 & n_y^\xi \\ n_y^\xi & n_x^\xi \end{bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_{,\eta} & 0 \\ 0 & -x_{,\eta} \\ -x_{,\eta} & y_{,\eta} \end{bmatrix}, [b^2] = \frac{g^\eta}{|J|} \begin{bmatrix} n_x^\eta & 0 \\ 0 & n_y^\eta \\ n_y^\eta & n_x^\eta \end{bmatrix} = \frac{1}{|J|} \begin{bmatrix} -y & 0 \\ 0 & x \\ x & -y \end{bmatrix}.$$

Note that the transpose of the differential operator $[L]^T = [b^1]^T \frac{\partial}{\partial \xi} + \frac{1}{\xi} [b^2]^T \frac{\partial}{\partial \eta}$.

Applying the transformation in equation (14) to the differential equation of motion yields

$$[b^1]^T \{\sigma'_{,\xi}\} + \frac{1}{\xi} [b^2]^T \{\sigma'_{,\eta}\} + [b^1]^T \{m\} p_{,\xi} + \frac{1}{\xi} [b^2]^T \{m\} p_{,\eta} + \omega^2 \rho \{u\} + \rho \{b\} = 0 \quad (15)$$

3.2 The Fluid Continuity Equation

The fluid continuity equation may be represented in vector form as :

$$-k \{m\}^T [L][L]^T \{m\} p + \omega^2 k \rho_f \{m\}^T [L] \{u\} + k \rho_f \{m\}^T [L] \{b\} + i \omega \{m\}^T [L] \{u\} + i \omega \frac{n}{K_f} p = 0 \quad (16)$$

where p is the fluid pressure , u , ω , k , n , K_f and ρ are the displacement, frequency, soil permeability and porosity, fluid bulk modulus and density respectively, and $\{b\}$ is the body forces per unit volume vector.

Applying the differential transformation in the form of equation (14) to the continuity equations yields

$$\begin{aligned} & -k \{m\}^T ([b^1] \frac{\partial}{\partial \xi} ([b^1]^T \frac{\partial}{\partial \xi} + \frac{1}{\xi} [b^2]^T \frac{\partial}{\partial \eta}) + \frac{1}{\xi} [b^2] \frac{\partial}{\partial \eta} ([b^1]^T \frac{\partial}{\partial \xi} + \frac{1}{\xi} [b^2]^T \frac{\partial}{\partial \eta})) \{m\} p \\ & + \omega^2 k \rho_f \{m\}^T ([b^1] \{u_{,\xi}\} + \frac{1}{\xi} [b^2] \{u_{,\eta}\}) + k \rho_f \{m\}^T ([b^1] \{b_{,\xi}\} + \frac{1}{\xi} [b^2] \{b_{,\eta}\}) \\ & + i \omega \{m\}^T ([b^1] \{u_{,\xi}\} + \frac{1}{\xi} [b^2] \{u_{,\eta}\}) + i \omega \frac{n}{K_f} p = 0 \end{aligned} \quad (17)$$

4. Displacements and pore pressure shape functions

Shape functions similar to the mapping interpolation functions may be used to interpolate the finite element displacements for all lines with constant ξ in Fig. 2.

Using displacement shape functions $[N^u(\eta)]$ and displacements functions in the radial direction $\{u(\xi)\}$, the finite element displacement function may be represented as

$$\{u\} = \{u(\xi, \eta, \zeta)\} = [N^u(\eta, \zeta)] \{u(\xi)\} \quad (18)$$

The displacements derivatives are thus :

$$\{u_{,\xi}\} = [N^u] \{u(\xi)\}_{,\xi}, \quad \{u_{,\eta}\} = [N^u]_{,\eta} \{u(\xi)\} \quad \text{and} \quad \{u_{,\zeta}\} = [N^u]_{,\zeta} \{u(\xi)\} \quad (19)$$

Similarly, the pore pressures are represented using the shape functions $[N^p(\eta, \zeta)]$ and pressure functions in the radial direction $\{p(\xi)\}$. Hence the finite element pore pressure function may be represented as

$$p = \{p(\xi, \eta)\} [N^p(\eta)] \{p(\xi)\} \quad (20)$$

The derivatives of p are:

$$p_{,\xi} = [N^p] \{p(\xi)\}_{,\xi} \quad (21)$$

$$p_{,\eta} = [N^p]_{,\eta} \{p(\xi)\}, \quad p_{,\xi\xi} = [N^p] \{p(\xi)\}_{,\xi\xi}, \quad p_{,\eta\eta} = [N^p]_{,\eta\eta} \{p(\xi)\} \quad \text{and} \quad p_{,\xi\eta} = p_{,\eta\xi} = [N^p]_{,\eta} \{p(\xi)\}_{,\xi}$$

The effective stresses $\{\sigma'\}$, strains $\{\varepsilon\}$ and displacements $\{u\}$ are related by

$$\{\sigma'\} = [D] \{\varepsilon\} = [D][L] \{u\} = [D][L][N^u] \{u(\xi)\} \quad (22)$$

where $[D]$ is the constitutive matrix. Substituting equation (14) into (22) yields:

$$\{\sigma'\} = [D] \left([b^1] \{u_{,\xi}\} + \frac{1}{\xi} [b^2] \{u_{,\eta}\} + \frac{1}{\xi} [b^3] \{u_{,\zeta}\} \right) = [D] \left([b^1] [N^u] \{u(\xi)\}_{,\xi} + \frac{1}{\xi} \left([b^2] [N^u]_{,\eta} + [b^3] [N^u]_{,\zeta} \right) \right) \quad (23)$$

which can be expressed as

$$\{\sigma'\} = [D] \left([B_u^1] \{u(\xi)\}_{,\xi} + \frac{1}{\xi} [B_u^2] \{u(\xi)\} \right) \quad (24)$$

where $[B_u^1] = [b^1] [N^u(\eta)] = [b^1] [N^u]$ and $[B_u^2] = [b^2] [N^u(\eta)]_{,\eta} = [b^2] [N^u]_{,\eta}$. Note that $[B_u^1]$ and $[B_u^2]$ are not dependent on ξ , and hence differentiating $\{\sigma'\}$ leads to

$$\{\sigma'_{,\xi}\} = [D] \left([B^1] \{u(\xi)\}_{,\xi\xi} + \frac{1}{\xi} [B^2] \{u(\xi)\}_{,\xi} - \frac{1}{\xi^2} [B^2] \{u(\xi)\} \right) \quad (25)$$

5. Weighted Residual Finite Element Approximation

To derive the finite element approximation, the Galerkin's weighted-residual method is applied to the transformed differential equations of motion and continuity (1) and (2) respectively by multiplying them with a weighting function $\{w\}^T = \{w(\xi, \eta)\}^T$ and then integrating over the domain V. The domain V is represented by the triangle shown in Fig. 2, with the line boundary as its base and its apex coincides with the origin of the global Cartesian coordinates. The integration limits are from -1 to $+1$ for the boundary variables η while in the ξ direction is from 0 to $+1$ for bounded elements and from $+1$ to ∞ for unbounded elements. The final equations are shown in section 6.

5.1 Weighted Residual Method Applied to the Equilibrium Equation

The weighting function $\{w^u\}^T$ multiplied by the differential equation of motion can be represented by the same displacement function as $\{w^u\} = \{w^u(\xi, \eta)\} = [N^u(\eta)]\{w^u(\xi)\}$.

$$\int_V \left(\{w\}^T [b^1]^T \{\sigma'_{,\xi}\} + \frac{1}{\xi} \{w\}^T [b^2]^T \{\sigma'_{,\eta}\} + \{w\}^T [b^1]^T \{m\} p_{,\xi} + \frac{1}{\xi} \{w\}^T [b^2]^T \{m\} p_{,\eta} + \{w\}^T \omega^2 \rho \{u\} + \{w\}^T \rho \{b\} \right) dV = 0 \quad (26)$$

For the bounded/unbounded element, substituting $dV = \xi |J| d\xi d\eta$ leads to

$$\int_V \int_{-1}^{+1} \left(\{w\}^T [b^1]^T \{\sigma'_{,\xi}\} + \frac{1}{\xi} \{w\}^T [b^2]^T \{\sigma'_{,\eta}\} + \{w\}^T [b^1]^T \{m\} p_{,\xi} + \frac{1}{\xi} \{w\}^T [b^2]^T \{m\} p_{,\eta} + \{w\}^T \omega^2 \rho \{u\} + \{w\}^T \rho \{b\} \right) \xi |J| d\eta d\xi = 0 \quad (27)$$

Applying Green's theorem to the surface integral leads to

$$\begin{aligned} & \int_{\xi} \left(\xi \int_{-1}^{+1} \{w\}^T [b^1]^T \{\sigma'_{,\xi}\} |J| d\eta - \int_{-1}^{+1} (\{w\}^T |J| [b^2]^T)_{,\eta} \{\sigma'\} d\eta + \xi \int_{-1}^{+1} \{w\}^T [b^1]^T \{m\} p_{,\xi} |J| d\eta \right. \\ & - \int_{-1}^{+1} (\{w\}^T |J| [b^2]^T)_{,\eta} \{m\} p d\eta + \left. \left(\{w\}^T |J| [b^2]^T \{\sigma'\} \right)_{-1}^{+1} + \left(\{w\}^T |J| [b^2]^T \{m\} p \right)_{-1}^{+1} \right. \\ & \left. + \xi \int_{-1}^{+1} \{w\}^T \omega^2 \rho \{u\} |J| d\eta + \xi \int_{-1}^{+1} \{w\}^T \rho \{b\} |J| d\eta \right) d\xi = 0 \end{aligned} \quad (28)$$

which may be rewritten as

$$\begin{aligned} & \int_{\xi} \left(\xi \int_{-1}^{+1} \{w\}^T [b^1]^T \{\sigma'_{,\xi}\} |J| d\eta - \int_{-1}^{+1} (\{w\}^T |J| [b^2]^T)_{,\eta} \{\sigma'\} d\eta + \xi \int_{-1}^{+1} \{w\}^T [b^1]^T \{m\} p_{,\xi} |J| d\eta \right. \\ & - \int_{-1}^{+1} (\{w\}^T |J| [b^2]^T)_{,\eta} \{m\} p d\eta + \left. \left(\{w\}^T |J| [b^2]^T (\{\sigma'\} + \{m\} p) \right)_{-1}^{+1} + \xi \int_{-1}^{+1} \{w\}^T \omega^2 \rho \{u\} |J| d\eta \right. \\ & \left. + \xi \int_{-1}^{+1} \{w\}^T \rho \{b\} |J| d\eta \right) d\xi = 0 \end{aligned} \quad (29)$$

Substituting the total stress σ in terms of fluid pressure $\{\sigma\} = \{\sigma'\} + \{m\}p$;

$$\begin{aligned} & \int_{\xi} \left(\xi \int_{-1}^{+1} \{w\}^T [b^1]^T \{\sigma'_{,\xi}\} |J| d\eta - \int_{-1}^{+1} (\{w\}^T |J| [b^2]^T)_{,\eta} \{\sigma'\} d\eta + \xi \int_{-1}^{+1} \{w\}^T [b^1]^T \{m\} p_{,\xi} |J| d\eta \right. \\ & - \int_{-1}^{+1} (\{w\}^T |J| [b^2]^T)_{,\eta} \{m\} p d\eta + \left. \left(\{w\}^T |J| [b^2]^T \{\sigma\} \right)_{-1}^{+1} + \xi \int_{-1}^{+1} \{w\}^T \omega^2 \rho \{u\} |J| d\eta \right. \\ & \left. + \xi \int_{-1}^{+1} \{w\}^T \rho \{b\} |J| d\eta \right) d\xi = 0 \end{aligned} \quad (30)$$

To simplify the above differential equation, some variable manipulations are carried out. The amplitudes of surface tractions on any boundary in the Cartesian coordinates are

$$\{\tau\} = \begin{Bmatrix} \tau_x \\ \tau_y \end{Bmatrix} = \{n\} \{\sigma\} = \begin{Bmatrix} n_x \sigma_x + n_y \tau_{xy} \\ n_y \sigma_y + n_x \tau_{xy} \end{Bmatrix} = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad (31)$$

Accordingly, the traction amplitudes on the η line (where ξ is constant), τ^ξ , may be expressed as shown below in terms of unit normal vectors.

$$\{\tau^\xi\} = \begin{bmatrix} n_x^\xi & 0 & n_y^\xi \\ 0 & n_y^\xi & n_x^\xi \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{|J|}{g^\xi} [b^1]^T \{\sigma\} \Rightarrow |J| [b^1]^T \{\sigma\} = \{\tau^\xi\} g^\xi \quad (32)$$

Similarly, the traction amplitude on the ξ line (where η is constant), τ^η , may be expressed as

$$\{\tau^\eta\} = \frac{|J|}{g^\eta} [b^2]^T \{\sigma\} \Rightarrow |J| [b^2]^T \{\sigma\} = \{\tau^\eta\} g^\eta \quad (33)$$

Substituting the above simplification into the main differential equation of motion yields

$$\begin{aligned}
& \int_{\xi} \left(\xi \int_{-1}^{+1} \{w\}^T [b^1]^T \{\sigma'_{,\xi}\} J |d\eta + \xi \int_{-1}^{+1} \{w\}^T [b^1]^T \{m\} p_{,\xi} J |d\eta \right. \\
& - \int_{-1}^{+1} (\{w\}^T (J [b^2]^T)_{,\eta} + \{w_{,\eta}\}^T J [b^2]^T) \{\sigma'\} d\eta - \int_{-1}^{+1} (\{w\}^T (J [b^2]^T)_{,\eta} + \{w_{,\eta}\}^T J [b^2]^T) \{m\} p d\eta \quad (34) \\
& \left. + (\{w\}^T \{\tau^\eta\} g^\eta)_{-1}^{+1} + \xi \int_{-1}^{+1} \{w\}^T \omega^2 \rho \{u\} J |d\eta + \xi \int_{-1}^{+1} \{w\}^T \rho \{b\} J |d\eta \right) d\xi = 0
\end{aligned}$$

Furthermore, the derivative $(J [b^2])_{,\eta}$ is defined as

$$(J [b^2])_{,\eta} = - \begin{bmatrix} y_{,\eta} & 0 \\ 0 & -x_{,\eta} \\ -x_{,\eta} & y_{,\eta} \end{bmatrix} \quad (35)$$

Comparing the above derivative with the $(J [b^1])$ results in $(J [b^2])_{,\eta} = -(J [b^1])$. Setting the integrand of the integral over ξ to zero satisfies the above equation and effectively enforces the main differential equation of motion exactly in the radial direction. Furthermore, substituting the above equation (35) into equation (34) leads to

$$\begin{aligned}
& \xi \int_{-1}^{+1} \{w\}^T [b^1]^T \{\sigma'_{,\xi}\} J |d\eta d\xi + \xi \int_{-1}^{+1} \{w\}^T [b^1]^T \{m\} p_{,\xi} J |d\eta \\
& + \int_{-1}^{+1} (\{w\}^T [b^1]^T - \{w_{,\eta}\}^T [b^2]^T) \{\sigma'\} J |d\eta + \int_{-1}^{+1} (\{w\}^T [b^1]^T - \{w_{,\eta}\}^T [b^2]^T) \{m\} p J |d\eta \\
& + (\{w\}^T \{\tau^\eta\} g^\eta)_{-1}^{+1} + \xi \int_{-1}^{+1} \{w\}^T \omega^2 \rho \{u\} J |d\eta + \xi \int_{-1}^{+1} \{w\}^T \rho \{b\} J |d\eta = 0 \quad (36)
\end{aligned}$$

Substituting $\{w\}^T = \{w\}^T [N^u]^T$ into the differential equation of motion (15) together with the finite element displacements and pressure functions results equation (37).

$$\begin{aligned}
& \{\mathfrak{w}\}^T \xi \int_{-1}^{+1} [N^u]^T [b^1]^T \{\sigma'_{,\xi}\} J |d\eta + \{\mathfrak{w}\}^T \xi \int_{-1}^{+1} [N^u]^T [b^1]^T \{m\} [N^p] \{p(\xi)\}_{,\xi} J |d\eta \\
& + \{\mathfrak{w}\}^T \int_{-1}^{+1} ([N^u]^T [b^1]^T - [N^u]_{,\eta}^T [b^2]^T) \{\sigma'\} J |d\eta \\
& + \{\mathfrak{w}\}^T \int_{-1}^{+1} ([N^u]^T [b^1]^T - [N^u]_{,\eta}^T [b^2]^T) \{m\} [N^p] \{p(\xi)\} J |d\eta + \left(\{\mathfrak{w}\}^T [N^u]^T \{\tau^\eta\} g^\eta \right)_{-1}^{+1} \\
& + \{\mathfrak{w}\}^T \omega^2 \xi \int_{-1}^{+1} [N^u]^T \rho [N^u] \{u(\xi)\} J |d\eta + \{\mathfrak{w}\}^T \xi \int_{-1}^{+1} [N^u]^T \rho \{b\} J |d\eta = 0
\end{aligned} \tag{37}$$

Substituting the above expressions with $[B_u^1]$ and $[B_u^2]$, defined below, and multiplying

both sides of the equation by $[\{\mathfrak{w}\}^T]^{-1}$ yields :

$$\begin{aligned}
& \xi \int_{-1}^{+1} [B_u^1]^T \{\sigma'_{,\xi}\} J |d\eta + \xi \int_{-1}^{+1} [B_u^1]^T \{m\} [N^p] \{p(\xi)\}_{,\xi} J |d\eta + \int_{-1}^{+1} ([B_u^1]^T - [B_u^2]^T) \{\sigma'\} J |d\eta \\
& + \int_{-1}^{+1} ([B_u^1]^T - [B_u^2]^T) \{m\} [N^p] \{p(\xi)\} J |d\eta + \left([N^u]^T \{\tau^\eta\} g^\eta \right)_{-1}^{+1} \\
& + \omega^2 \xi \int_{-1}^{+1} [N^u]^T \rho [N^u] \{u(\xi)\} J |d\eta + \xi \int_{-1}^{+1} [N^u]^T \rho \{b\} J |d\eta = 0
\end{aligned} \tag{38}$$

Substituting the expressions for $\{\sigma'\}$ and $\{\sigma'_{,\xi}\}$ (equations 22 and 25 respectively)

yields

$$\begin{aligned}
& \xi \int_{-1}^{+1} [B_u^1]^T [D] \left([B^1] \{u(\xi)\}_{,\xi\xi} + \frac{1}{\xi} [B_u^2] \{u(\xi)\}_{,\xi} - \frac{1}{\xi^2} [B_u^2] \{u(\xi)\} \right) J |d\eta + \xi \int_{-1}^{+1} [B_u^1]^T \{m\} [N^p] \{p(\xi)\}_{,\xi} J |d\eta \\
& + \int_{-1}^{+1} ([B_u^1]^T - [B_u^2]^T) \{m\} [N^p] \{p(\xi)\} J |d\eta + \int_{-1}^{+1} ([B_u^1]^T - [B_u^2]^T) [D] \left([B_u^1] \{u(\xi)\}_{,\xi} + \frac{1}{\xi} [B_u^2] \{u(\xi)\} \right) J |d\eta \\
& + \left([N^u]^T \{\tau^\eta\} g^\eta \right)_{-1}^{+1} + \omega^2 \xi \int_{-1}^{+1} [N^u]^T \rho [N^u] \{u(\xi)\} J |d\eta + \xi \int_{-1}^{+1} [N^u]^T \rho \{b\} J |d\eta = 0
\end{aligned} \tag{39}$$

Further simplification and rearranging results in;

$$\begin{aligned}
& \xi^2 \int_{-1}^{+1} [B_u^1]^T [D][B_u^1] J |d\eta\{u(\xi)\}_{,\xi\xi} + \xi^2 \int_{-1}^{+1} [B_u^1]^T \{m\} [N^p] J |d\eta\{p(\xi)\}_{,\xi} + \xi \int_{-1}^{+1} [B_u^1]^T [D][B_u^1] J |d\eta\{u(\xi)\}_{,\xi} \\
& - \xi \int_{-1}^{+1} [B_u^2]^T [D][B_u^1] J |d\eta\{u(\xi)\}_{,\xi} + \xi \int_{-1}^{+1} [B_u^1]^T [D][B_u^2] J |d\eta\{u(\xi)\}_{,\xi} - \int_{-1}^{+1} [B_u^2]^T [D][B_u^2] J |d\eta\{u(\xi)\} \\
& + \omega^2 \xi^2 \int_{-1}^{+1} [N^u]^T \rho [N^u] J |d\eta\{u(\xi)\} + \xi \int_{-1}^{+1} ([B_u^1]^T \{m\} [N^p] - [B_u^2]^T \{m\} [N^p]) J |d\eta\{p(\xi)\} \\
& + \xi^2 \int_{-1}^{+1} [N^u]^T \rho \{b\} J |d\eta + \xi \left([N^u]^T \{\tau^\eta\} g^\eta \right) \Big|_{-1}^{+1} = 0
\end{aligned}
\tag{40}$$

The above derivation is valid for both the bounded and unbounded medium with the integration limits between $\xi = 0 \rightarrow 1$ and $\xi = 1 \rightarrow \infty$, respectively.

5.2 Weighted Residual Method Applied to the Continuity Equation

As with the equilibrium equation, the continuity equation is multiplied by a weighting function and then integrated over the domain V. The weighting function $\{w^p\}^T$ that is multiplied by the differential equation of continuity is represented by the pore pressure function $\{w^p\} = \{w^p(\xi, \eta)\} = [N^p(\eta)] \{w^p(\xi)\}$.

$$\begin{aligned}
& \int_V \left(-k\{w\}^T \{m\}^T ([b^1] [b^1])^T \{m\} p_{,\xi\xi} + \frac{1}{\xi} k\{m\}^T [b^1] [b^2]^T \{m\} p_{,\xi\eta} - \frac{1}{\xi^2} [b^1] [b^2]^T \{m\} p_{,\eta} \right. \\
& + \frac{1}{\xi} [b^2] [b^1]^T \{m\} p_{,\xi\eta} + \frac{1}{\xi} [b^2] [b^1]^T \{m\} p_{,\xi} + \frac{1}{\xi^2} [b^2] [b^2]^T \{m\} p_{,\eta\eta} + \frac{1}{\xi^2} [b^2] [b^2]^T \{m\} p_{,\eta} \left. \right) \\
& + (i\omega + \omega^2 k\rho_f) \{w\}^T \{m\}^T ([b^1] [u_{,\xi}]) + \frac{1}{\xi} [b^2] [u_{,\eta}] + k\rho_f \{w\}^T \{m\}^T ([b^1] [b_{,\xi}] + \frac{1}{\xi} [b^2] [b_{,\eta}]) \\
& + i\omega \{w\}^T \frac{n}{K_f} p) dV = 0
\end{aligned} \tag{41}$$

For the bounded/unbounded element, substituting $dV = \xi |J| d\xi d\eta$ leads to

$$\begin{aligned}
& \int_{-1}^{+1} \xi \left(\int_{-1}^{+1} \left(-k\{w\}^T \{m\}^T ([b^1] [b^1])^T \{m\} p_{,\xi\xi} + \frac{1}{\xi} k\{m\}^T [b^1] [b^2]^T \{m\} p_{,\xi\eta} - \frac{1}{\xi^2} [b^1] [b^2]^T \{m\} p_{,\eta} \right. \right. \\
& + \frac{1}{\xi} [b^2] [b^1]^T \{m\} p_{,\xi\eta} + \frac{1}{\xi} [b^2] [b^1]^T \{m\} p_{,\xi} + \frac{1}{\xi^2} [b^2] [b^2]^T \{m\} p_{,\eta\eta} + \frac{1}{\xi^2} [b^2] [b^2]^T \{m\} p_{,\eta} \left. \right) \\
& + (i\omega + \omega^2 k\rho_f) \{w\}^T \{m\}^T ([b^1] [u_{,\xi}]) + \frac{1}{\xi} [b^2] [u_{,\eta}] + k\rho_f \{w\}^T \{m\}^T ([b^1] [b_{,\xi}] + \frac{1}{\xi} [b^2] [b_{,\eta}]) \\
& + i\omega \{w\}^T \frac{n}{K_f} p) |J| d\eta) d\xi = 0
\end{aligned} \tag{42}$$

Setting the integrand of the integral over ξ to zero satisfies the above equation and effectively enforces the main differential equation of continuity in the radial direction.

$$\begin{aligned}
& \xi \int_{-1}^{+1} \left(-k\{w\}^T \{m\}^T [b^1] [b^1]^T \{m\} p_{,\xi\xi} + \frac{1}{\xi} \{w\}^T \{m\}^T [b^1] [b^2]^T \{m\} p_{,\xi\eta} \right. \\
& - \frac{1}{\xi^2} \{w\}^T \{m\}^T [b^1] [b^2]^T \{m\} p_{,\eta} + \frac{1}{\xi} \{w\}^T \{m\}^T [b^2] [b^1]^T \{m\} p_{,\xi\eta} \\
& + \frac{1}{\xi} \{w\}^T \{m\}^T [b^2] [b^1]^T \{m\} p_{,\xi} + \frac{1}{\xi^2} \{w\}^T \{m\}^T [b^2] [b^2]^T \{m\} p_{,\eta\eta} \\
& + \frac{1}{\xi^2} \{w\}^T \{m\}^T [b^2] [b^2]^T \{m\} p_{,\eta} \left. \right) + (i\omega + \omega^2 k\rho_f) (\{w\}^T \{m\}^T [b^1] [u_{,\xi}] \\
& + \frac{1}{\xi} \{w\}^T \{m\}^T [b^2] [u_{,\eta}] + k\rho_f (\{w\}^T \{m\}^T [b^1] [b_{,\xi}] + \frac{1}{\xi} \{w\}^T \{m\}^T [b^2] [b_{,\eta}]) \\
& + i\omega \{w\}^T \frac{n}{K_f} p) |J| d\eta = 0
\end{aligned} \tag{43}$$

This equation can be rewritten in the form (44). Substituting both the pore pressure and displacement functions into the continuity equation leads to equation (45).

$$\begin{aligned}
& -k(\xi^2 \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] [b^1]^T \{m\} p_{,\xi\xi} |J| d\eta + \xi \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] [b^2]^T \{m\} p_{,\xi\eta} |J| d\eta \\
& - \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] [b^2]^T \{m\} p_{,\eta} |J| d\eta + \xi \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] [b^1]^T \{m\} p_{,\xi\eta} |J| d\eta \\
& + \xi \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] [b^1]^T \{m\} p_{,\xi} |J| d\eta + \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] [b^2]^T \{m\} p_{,\eta\eta} |J| d\eta \\
& + \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] [b^2]^T \{m\} p_{,\eta} |J| d\eta) + (i\omega + \omega^2 k \rho_f) (\xi^2 \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] \{u_{,\xi}\} |J| d\eta \\
& + \xi \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] \{u_{,\eta}\} |J| d\eta) + k \rho_f (\xi^2 \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] \{b_{,\xi}\} |J| d\eta \\
& + \xi \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] \{b_{,\eta}\} |J| d\eta) + i\omega \xi^2 \int_{-1}^{+1} \{w\}^T \frac{n}{K_f} p |J| d\eta = 0
\end{aligned} \tag{44}$$

$$\begin{aligned}
& -k(\xi^2 \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] [b^1]^T \{m\} [N^p] |J| d\eta \{p(\xi)\}_{,\xi\xi} + \xi \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] [b^2]^T \{m\} [N^p]_{,\eta} |J| d\eta \{p(\xi)\}_{,\xi} \\
& - \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] [b^2]^T \{m\} [N^p]_{,\eta} |J| d\eta \{p(\xi)\} + \xi \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] [b^1]^T \{m\} [N^p]_{,\eta} |J| d\eta \{p(\xi)\}_{,\xi} \\
& + \xi \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] [b^1]^T \{m\} [N^p] |J| d\eta \{p(\xi)\}_{,\xi} + \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] [b^2]^T \{m\} [N^p]_{,\eta\eta} |J| d\eta \{p(\xi)\} \\
& + \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] [b^2]^T \{m\} [N^p]_{,\eta} |J| d\eta \{p(\xi)\}) + (i\omega + \omega^2 k \rho_f) (\xi^2 \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] [N^u] |J| d\eta \{u(\xi)\}_{,\xi} \\
& + \xi \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] [N^u]_{,\eta} |J| d\eta \{u(\xi)\}) + k \rho_f (\xi^2 \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] \{b_{,\xi}\} |J| d\eta \\
& + \xi \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] \{b_{,\eta}\} |J| d\eta) + i\omega \xi^2 \int_{-1}^{+1} \{w\}^T \frac{n}{K_f} [N^p] |J| d\eta \{p(\xi)\} = 0
\end{aligned} \tag{45}$$

Rearranging results in ;

$$\begin{aligned}
& k \left(\xi^2 \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] [b^1]^T \{m\} [N^p] J |d\eta| \{p(\xi)\}_{,\xi\xi} + \xi \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] [b^2]^T \{m\} [N^p]_{,\eta} J |d\eta| \right. \\
& + \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] [b^1]^T \{m\} [N^p]_{,\eta} J |d\eta| + \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] [b^1]^T \{m\} [N^p] J |d\eta| \{p(\xi)\}_{,\xi} \\
& + \left(- \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] [b^2]^T \{m\} [N^p]_{,\eta} J |d\eta| + \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] [b^2]^T \{m\} [N^p]_{,\eta\eta} J |d\eta| \right. \\
& \left. + \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] [b^2]^T \{m\} [N^p]_{,\eta} J |d\eta| \{p(\xi)\} \right) \\
& - (i\omega + \omega^2 k \rho_f) \left(\xi^2 \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] [N^u] J |d\eta| \{u(\xi)\}_{,\xi} + \xi \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] [N^u]_{,\eta} J |d\eta| \{u(\xi)\} \right) (46) \\
& - k \rho_f \left(\xi^2 \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] [b_{,\xi}] J |d\eta| + \xi \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2] [b_{,\eta}] J |d\eta| \right) \\
& - i\omega \xi^2 \int_{-1}^{+1} \{w\}^T \frac{n}{K_f} [N^p] J |d\eta| \{p(\xi)\} = 0
\end{aligned}$$

Applying Green's theorem to the surface integral leads to

$$\begin{aligned}
& k \left(\xi^2 \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1 \mathbb{I} b^1]^T \{m\} [N^p] J |d\eta| p(\xi) \right)_{,\xi\xi} + \xi \left(\int_{-1}^{+1} \{w\}^T \{m\}^T [b^1 \mathbb{I} b^2]^T \{m\} [N^p]_{,\eta} |J| d\eta \right. \\
& + \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2 \mathbb{I} b^1]^T \{m\} [N^p]_{,\eta} |J| d\eta + \{w\}^T \left(\{m\}^T [b^2 \mathbb{I} b^1]^T \{m\} [N^p] J \right)_{-1}^{+1} \\
& - \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2 \mathbb{I} b^1]^T \{m\} [N^p]_{,\eta} |J| d\eta - \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2]_{,\eta} [b^1]^T \{m\} [N^p] J |d\eta \\
& - \int_{-1}^{+1} \{w\}_{,\eta}^T \{m\}^T [b^2 \mathbb{I} b^1]^T \{m\} [N^p] J |d\eta \left. \right) \{p(\xi)\}_{,\xi} + \left(- \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1 \mathbb{I} b^2]^T \{m\} [N^p]_{,\eta} |J| d\eta \right. \\
& + \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2 \mathbb{I} b^2]^T \{m\} [N^p]_{,\eta\eta} |J| d\eta + \{w\}^T \left(\{m\}^T [b^2 \mathbb{I} b^2]^T \{m\} [N^p]_{,\eta} |J| \right)_{-1}^{+1} \\
& - \left[\int_{-1}^{+1} \{w\}^T \{m\}^T [b^2 \mathbb{I} b^2]^T \{m\} [N^p]_{,\eta\eta} |J| d\eta + \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2]_{,\eta} [b^2]^T \{m\} [N^p]_{,\eta} |J| d\eta \right. \\
& + \left. \int_{-1}^{+1} \{w\}_{,\eta}^T \{m\}^T [b^2 \mathbb{I} b^2]^T \{m\} [N^p]_{,\eta} |J| d\eta \right) \{p(\xi)\} \\
& - (i\omega + \omega^2 k \rho_f) \left(\xi^2 \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1 \mathbb{I} N^u] J |d\eta| u(\xi) \right)_{,\xi} + \xi \left(\{w\}^T \left(\{m\}^T [b^2 \mathbb{I} N^u] J \right)_{-1}^{+1} \right. \\
& - \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2]_{,\eta} [N^u] J |d\eta - \int_{-1}^{+1} (\{w\}_{,\eta}^T \{m\}^T [b^2 \mathbb{I} N^u] J |d\eta) \{u(\xi)\} \left. \right) - k \rho_f \left(\xi^2 \int_{-1}^{+1} \{w\}^T \{m\}^T [b^1] \{b_{,\xi}\} J |d\eta \right. \\
& + \xi \{w\}^T \left(\{m\}^T [b^2] \{b\} J \right)_{-1}^{+1} - \xi \int_{-1}^{+1} \{w\}^T \{m\}^T [b^2]_{,\eta} \{b\} J |d\eta - \xi \int_{-1}^{+1} \{w\}_{,\eta}^T \{m\}^T [b^2] \{b\} J |d\eta \left. \right) \\
& - i\omega \xi^2 \int_{-1}^{+1} \{w\}^T \frac{n}{K_f} [N^p] J |d\eta| p(\xi) = 0
\end{aligned}
\tag{47}$$

Representing the weighting function $\{w\}$ using the shape functions $[N^p(\eta)]$ in the radial direction $\{w(\xi)\}$ leads to

$$\{w(\xi, \eta)\} = [N^p(\eta)] \{w(\xi)\} = [N^p] \{w(\xi)\} \tag{48}$$

where $\{w\}^T = \{w(\xi)\}^T [N^p]^T$ and $\{w\}_{,\eta}^T = \{w(\xi)\}^T [N^p]_{,\eta}^T$

Substituting the above and grouping the relevant terms leads to equation (49).

$$\begin{aligned}
& \{w(\xi)\}^T \left(\xi^2 \int_{-1}^{+1} [N^p]^T \{m\}^T [b^1] k [b^1]^T \{m\} [N^p] J |d\eta \{p(\xi)\}_{,\xi} \right. \\
& + \xi \left(\int_{-1}^{+1} [N^p]^T \{m\}^T [b^1] k [b^2]^T \{m\} [N^p]_{,\eta} J |d\eta + \left([N^p]^T \{m\}^T [b^2] k [b^1]^T \{m\} [N^p] J \right)_{-1}^{+1} \right. \\
& - \int_{-1}^{+1} [N^p]^T \{m\}^T [b^2]_{,\eta} k [b^1]^T \{m\} [N^p] J |d\eta - \int_{-1}^{+1} [N^p]^T \{m\}^T [b^2] k [b^1]^T \{m\} [N^p] J |d\eta \{p(\xi)\}_{,\xi} \\
& + \left(- \int_{-1}^{+1} [N^p]^T \{m\}^T [b^1] k [b^2]^T \{m\} [N^p]_{,\eta} J |d\eta + \left([N^p]^T \{m\}^T [b^2] [b^2]^T \{m\} [N^p]_{,\eta} J \right)_{-1}^{+1} \right. \\
& - \left. \left. \left(\int_{-1}^{+1} [N^p]^T \{m\}^T [b^2]_{,\eta} k [b^2]^T \{m\} [N^p]_{,\eta} J |d\eta + \int_{-1}^{+1} [N^p]^T \{m\}^T [b^2] k [b^2]^T \{m\} [N^p]_{,\eta} J |d\eta \right) \{p(\xi)\} \right) \right. \\
& - (i\omega + \omega^2 k \rho_f) \{w(\xi)\}^T \left(\xi^2 \int_{-1}^{+1} [N^p]^T \{m\}^T [b^1] [N^u] J |d\eta \{u(\xi)\}_{,\xi} + \xi \left(\left([N^p]^T \{m\}^T [b^2] [N^u] J \right)_{-1}^{+1} \right. \right. \\
& - \left. \left. \int_{-1}^{+1} [N^p]^T \{m\}^T [b^2]_{,\eta} [N^u] J |d\eta - \int_{-1}^{+1} [N^p]^T \{m\}^T [b^2] [N^u] J |d\eta \right) \{u(\xi)\} \right) \\
& - k \rho_f \{w(\xi)\}^T \left(\xi^2 \int_{-1}^{+1} [N^p]^T \{m\}^T [b^1] \{b_{,\xi}\} J |d\eta + \xi \left([N^p]^T \{m\}^T [b^2] \{b\} J \right)_{-1}^{+1} \right. \\
& - \xi \int_{-1}^{+1} [N^p]^T \{m\}^T [b^2]_{,\eta} \{b\} J |d\eta - \xi \int_{-1}^{+1} [N^p]^T \{m\}^T [b^2] \{b\} J |d\eta \left. \right) \\
& - i\omega \{w(\xi)\}^T \xi^2 \int_{-1}^{+1} [N^p]^T \frac{n}{K_f} [N^p] J |d\eta \{p(\xi)\} = 0
\end{aligned} \tag{49}$$

Substituting $(J[b^2])_{,\eta} = -(J[b^1])$ results in;

$$\begin{aligned}
& \xi^2 \int_{-1}^{+1} [N^p]^T \{m\}^T [b^1] k [b^1]^T \{m\} [N^p] J |d\eta \{p(\xi)\}_{,\xi\xi} + \xi \left(\int_{-1}^{+1} [N^p]^T \{m\}^T [b^1] k [b^2]^T \{m\} [N^p] J |d\eta \right. \\
& + \left. \left([N^p]^T \{m\}^T [b^2] k [b^1]^T \{m\} [N^p] J \right) \right)_{-1}^{+1} + \int_{-1}^{+1} [N^p]^T \{m\}^T [b^1] k [b^1]^T \{m\} [N^p] J |d\eta \\
& - \int_{-1}^{+1} [N^p]^T \{m\}^T [b^2] k [b^1]^T \{m\} [N^p] J |d\eta \{p(\xi)\}_{,\xi} + \left(- \int_{-1}^{+1} [N^p]^T \{m\}^T [b^1] k [b^2]^T \{m\} [N^p] J |d\eta \right. \\
& + \left. \left([N^p]^T \{m\}^T [b^2] k [b^2]^T \{m\} [N^p] J \right) \right)_{-1}^{+1} - \left[- \int_{-1}^{+1} [N^p]^T \{m\}^T [b^1] k [b^2]^T \{m\} [N^p] J |d\eta \right. \\
& + \left. \int_{-1}^{+1} [N^p]^T \{m\}^T [b^2] k [b^2]^T \{m\} [N^p] J |d\eta \right) \{p(\xi)\} - (i\omega + \omega^2 k \rho_f) \left(\xi^2 \int_{-1}^{+1} [N^p]^T \{m\}^T [b^1] [N^u] J |d\eta \{u(\xi)\}_{,\xi} \right. \\
& + \xi \left(\left([N^p]^T \{m\}^T [b^2] [N^u] J \right)_{-1}^{+1} + \int_{-1}^{+1} [N^p]^T \{m\}^T [b^1] [N^u] J |d\eta - \int_{-1}^{+1} [N^p]^T \{m\}^T [b^2] [N^u] J |d\eta \right) \{u(\xi)\} \\
& - k \rho_f \left(\xi^2 \int_{-1}^{+1} [N^p]^T \{m\}^T [b^1] \{b_{,\xi}\} J |d\eta + \xi \left([N^p]^T \{m\}^T [b^2] \{b\} J \right)_{-1}^{+1} \right. \\
& + \left. \xi \int_{-1}^{+1} [N^p]^T \{m\}^T [b^1] \{b\} J |d\eta - \xi \int_{-1}^{+1} [N^p]^T \{m\}^T [b^2] \{b\} J |d\eta \right) - i\omega \xi^2 \int_{-1}^{+1} [N^p]^T \frac{n}{K_f} [N^p] J |d\eta \{p(\xi)\} = 0
\end{aligned}$$

(50)

The above equation may be rewritten in the form of equation 51.

$$\begin{aligned}
& \xi^2 \int_{-1}^{+1} [B_p^1]^T k [B_p^1] J |d\eta \{p(\xi)\}_{,\xi\xi} + \xi \left(\left([B_p^{2A}]^T k [B_p^1] J \right)_{-1}^{+1} + \int_{-1}^{+1} [B_p^1]^T k [B_p^1] J |d\eta - \int_{-1}^{+1} [B_p^2]^T k [B_p^1] J |d\eta \right. \\
& + \left. \int_{-1}^{+1} [B_p^1]^T k [B_p^2] J |d\eta d\xi \right) \{p(\xi)\}_{,\xi} + \left(\left([B_p^{2A}]^T k [B_p^2] J \right)_{-1}^{+1} - \int_{-1}^{+1} [B_p^2]^T k [B_p^2] J |d\eta \right) \{p(\xi)\} \\
& - (i\omega + \omega^2 k \rho_f) \xi^2 \int_{-1}^{+1} [N^p]^T \{m\}^T [B_u^1] J |d\eta \{u(\xi)\}_{,\xi} - (i\omega + \omega^2 k \rho_f) \xi \left(\int_{-1}^{+1} [N^p]^T \{m\}^T [B_u^1] J |d\eta \right. \\
& - \left. \int_{-1}^{+1} [B_p^2]^T [N^u] J |d\eta + \left([B_p^{2A}]^T [N^u] J \right)_{-1}^{+1} \right) \{u(\xi)\} - k \rho_f \left(\xi^2 \int_{-1}^{+1} [B_p^1]^T \{b_{,\xi}\} J |d\eta + \xi \int_{-1}^{+1} [B_p^1]^T \{b\} J |d\eta \right. \\
& - \left. \xi \int_{-1}^{+1} [B_p^2]^T \{b\} J |d\eta + \xi \left([B_p^{2A}]^T \{b\} J \right)_{-1}^{+1} \right) - i\omega \xi^2 \int_{-1}^{+1} [N^p]^T \frac{n}{K_f} [N^p] J |d\eta \{p(\xi)\} = 0
\end{aligned} \tag{51}$$

where

$$\begin{aligned}
[B_p^1] &= [b^1]^T \{m\} [N^p(\eta)] = [b^1]^T \{m\} [N^p], \\
[B_p^2] &= [b^2]^T \{m\} [N^p(\eta, \zeta)]_{,\eta} = [b^2]^T \{m\} [N^p]_{,\eta}, \\
[B_p^{2A}]^T &= [N^p(\eta, \zeta)]^T \{m\}^T [b^2] = [N^p]^T \{m\}^T [b^2] \\
[B_u^1] &= [b^1] [N^u(\eta)] = [b^1] [N^u] \\
[B_u^2] &= [b^2] [N^u(\eta)]_{,\eta} = [b^2] [N^u]_{,\eta}
\end{aligned}$$

6. Summary of the Finite Element Coupled Consolidation Equations

The final formulation of the finite element equations (40) and (51) may be re-written in the following compact form:

$$\begin{aligned}
[E^0] \xi^2 \{u(\xi)\}_{,\xi\xi} + ([E^0] - [E^1] + [E^1]^T) \xi \{u(\xi)\}_{,\xi} - [E^2] \{u(\xi)\} + \omega^2 [M^0] \xi^2 \{u(\xi)\} + [E^3] \xi^2 \{p(\xi)\}_{,\xi} \\
+ ([E^3] - [E^4]) \xi \{p(\xi)\} + \xi^2 \{F^b(\xi)\} + \xi \{F^t(\xi)\} = 0
\end{aligned} \quad (52)$$

$$\begin{aligned}
[E^5] \xi^2 \{p(\xi)\}_{,\xi\xi} + ([E^5] - [E^6] + [E^6]^T + \{F^{t1}\}) \xi \{p(\xi)\}_{,\xi} + (-[E^7] + \{F^{t2}\}) \{p(\xi)\} - i\omega [M^1] \xi^2 \{p(\xi)\} - \\
(i\omega + \omega^2 k \rho_f) [E^3]^T \xi^2 \{u(\xi)\}_{,\xi} - (i\omega + \omega^2 k \rho_f) ([E^3]^T - [E^{4P}] + \{F^{t3}\}) \xi \{u(\xi)\} - k \rho_f (\xi^2 \{F^{b1}(\xi)\} \\
+ \xi \{F^{b2}(\xi)\} - \xi \{F^{b3}(\xi)\} + \xi \{F^{b4}(\xi)\}) = 0
\end{aligned} \quad (53)$$

where

$$\begin{aligned}
[E^0] &= \int_{-1}^{+1} [B_u^1]^T [D] [B_u^1] J |d\eta| & [E^1] &= \int_{-1}^{+1} [B_u^2]^T [D] [B_u^2] J |d\eta| & [E^2] &= \int_{-1}^{+1} [B_u^2]^T [D] [B_u^2] J |d\eta| \\
[E^3] &= \int_{-1}^{+1} [B_u^1]^T \{m\} [N^p] J |d\eta| & [E^4] &= \int_{-1}^{+1} [B_u^2]^T \{m\} [N^p] J |d\eta| & [E^{4P}] &= \int_{-1}^{+1} [B_p^2]^T [N^u] J |d\eta| \\
[E^5] &= \int_{-1}^{+1} [B_p^1]^T k [B_p^1] J |d\eta| & [E^6] &= \int_{-1}^{+1} [B_p^2]^T k [B_p^1] J |d\eta| & [E^7] &= \int_{-1}^{+1} [B_p^2]^T k [B_p^2] J |d\eta|
\end{aligned}$$

$$[M^0] = \int_{-1}^{+1} [N^u]^T \rho [N^u] J |d\eta| \quad [M^1] = \int_{-1}^{+1} [N^p]^T \frac{n}{K_f} [N^p] J |d\eta|$$

$$\{F^b(\xi)\} = \int_{-1}^{+1} [N^u]^T \rho \{b\} J |d\eta| \quad \{F^t(\xi)\} = \left([N^u]^T \{t^\eta(\xi)\} g^\eta \right)_{-1}^{+1} \quad \{F^{t1}\} = \left([B_p^{2A}]^T k [B_p^1] J \right)_{-1}^{+1}$$

$$\{F^{t2}\} = \left([B_p^{2A}]^T k [B_p^2] J \right)_{-1}^{+1} \quad \{F^{t3}\} = \left([B_p^{2A}]^T [N^u] J \right)_{-1}^{+1} \quad \{F^{b1}(\xi)\} = \int_{-1}^{+1} [B_p^1]^T \{b_{,\xi}\} J |d\eta|$$

$$\{F^{b2}(\xi)\} = \int_{-1}^{+1} [B_p^1]^T \{b\} J |d\eta| \quad \{F^{b3}(\xi)\} = \int_{-1}^{+1} [B_p^2]^T \{b\} J |d\eta| \quad \{F^{b4}(\xi)\} = \left([B_p^{2A}]^T \{b\} J \right)_{-1}^{+1}$$

$$[B_u^1] = [b^1] [N^u(\eta)] = [b^1] [N^u]$$

$$[B_u^2] = [b^2] [N^u(\eta)]_{,\eta} = [b^2] [N^u]_{,\eta}$$

$$[B_p^1] = [b^1]^T \{m\} [N^p(\eta)] = [b^1]^T \{m\} [N^p]$$

$$[B_p^2] = [b^2]^T \{m\} [N^p(\eta)]_{,\eta} = [b^2]^T \{m\} [N^p]_{,\eta}$$

$$[B_p^{2A}]^T = [N^p(\eta)]^T \{m\}^T [b^2] = [N^p]^T \{m\}^T [b^2]$$

7. Conclusion

A numerical formulation to correctly model the dynamic unbounded far-field boundary for two-phase media in 2D has been developed. The method of analysis extends the existing single-phase scaled boundary finite element method into a two-phase coupled solid-fluid approach to produce a more realistic representation of saturated soil at the unbounded far-field boundary. Body forces and surface tractions are considered in the derivation. The concept of similarity, the compatibility equation and Biot's coupled consolidation equations have been used to derive the formulation for the governing equations. The main difference from the single-phase version is the presence of pore water pressures as additional parameters to be solved for, in addition to the displacements, strain and stress. These are incorporated into the static-stiffness matrices by producing fully coupled matrices. Solving the resulting equations yields a boundary condition satisfying the far-field radiation condition

exactly. The coupled systems of differential equations are formulated for each element (i.e. Eq. 52 and 53). These coupled systems are then solved to determine the displacement and pore pressure functions in the radial direction. The computed solutions are exact in a radial direction (perpendicular to the boundary and pointing towards infinity), while converge to the exact solution in the finite element sense in the circumferential direction parallel to the soil-structure boundary interface.

The major advantages gained over previous methods used to model dynamic soil-structure interaction with coupled consolidation are :

- The computed solutions are exact in a radial direction (from the soil-structure boundary pointing towards infinity), while converge to the exact solution in the finite element sense in the circumferential direction parallel to the soil-structure boundary interface.
- For the static case, the SBFEM results in symmetric dynamic-stiffness and unit-impulse response matrices, which would not be the case if the BEM or FBEM were used, making it much more efficient time- and storage-wise.
- The spatial discretisation of problems is reduced by one, e.g. for 2D problems, 1D surface finite elements are used to discretise the boundary, which result in reducing the spatial dimension by one, and in turn the number of degrees of freedom during the analysis.
- As with the boundary element method (BEM), by not discretising the domain, internal displacements and stresses are computed more accurately than with the finite element method (FEM). However, in contrast to the BEM, the behaviour of the bounded near-field soil can be represented using any advanced constitutive model (for example, based on plasticity). This is due to the SBFEM not requiring a fundamental solution, which also eliminates the need to evaluate singular integrals

occurring in most fundamental solutions. Anisotropy can also be modelled using the SBFEM without an increase in computational effort.

The effects of dynamic soil-structure interaction can be very pronounced. When comparing the response of a structure embedded in soft ground to that of the same structure founded on the surface of rigid rock, it is evident that (i) the presence of the soil makes the dynamic system more flexible and (ii) the radiation of energy of the propagating waves away from the structure, if occurring, increases the damping. If seismic motion is applied at the base of the structure then the free-field motion will differ from the *bed-rock* control motion; usually being amplified towards the free surface. The excavation of the soil and insertion of a stiff base results in some averaging of the translation plus a *rotation* of the foundation given horizontal earthquake motion and the inertial loads resulting from the motion of the structure will further modify the seismic motion along the base of the structure. The degree of this interaction depends not only on the foundation stiffness, but also the stiffness and mass properties of the structure and the nature of the applied excitation. Ignoring dynamic soil-structure interaction effects could lead to a significantly over-conservative design [18]. The coupled consolidation SBFEM formulation should therefore be used for saturated soil where there is a flow of water, in order to incorporate the effect of intermediate-term consolidation of soil under loads, which would not be accounted for if the standard SBFEM was used.

Acknowledgements

The results presented in this paper were obtained in the course of research sponsored by the British Engineering and Physical Sciences Research Council (EPSRC) under Grant number GR/R 29833/01 to the first author at Loughborough

University. The authors gratefully acknowledge this support. The authors also gratefully acknowledge Professor John Wolf's support at EPFL in Switzerland.

References

- [1] Lysmer J and Kuhlemeyer RL. Finite dynamic model for infinite media. ASCE J. Eng. Mech. 1969; 95: 859-877.
- [2] White W, Valliappan S and Lee, IK. Unified boundary for finite dynamic models. ASCE J. Eng. Mech. 1977; 103: 949-964.
- [3] Smith WD. A non-reflecting plain boundary for wave propagation problems. J. Comput. Phys. 1974; 15.
- [4] Bettess P and Zienkiewicz OC. Diffraction and refraction of surface waves using finite and infinite elements. Int. J. Num. Meth. Eng. 1977; 11: 1271-1290.
- [5] Chow YK and Smith IM. Static and periodic infinite solid elements. Int. J. Num. Meth. Eng. 1981; 17: 503-526.
- [6] Zhao C and Valliappan S. A dynamic infinite element for three dimensional infinite domain wave problems. Int J. Num. Meth. Eng. 1992; 36.
- [7] Simoni L and Schrefler B. Mapped infinite elements in soil consolidation. Int. J. Num. Meth. Eng. 1987; 24.
- [8] Khalili N, Valliappan S, Tabtabaie-Yazdi J and Yazdchi M. 1D infinite element for dynamic problems in saturated porous media. Comm. Num. Meth. Eng. 1997; 13: 727-738; Khalili N, Yazdchi M and Valliappan S. Wave propagation analysis of two-phase saturated porous media using coupled finite-infinite element method. Soil Dyn. Earthq. Eng. 1999; 18: 533-553.
- [9] Cheng AH-D and Liggett JA. Boundary integral equation method for linear porous-elasticity with applications to soil consolidation. Int. J. Num. Meth. Eng. 1984; 20: 255-278.

- [10] Cheng AH-D and Detournay E. A direct boundary element method for plane strain poroelasticity. *Int. J. Num. Anal. Meth. Geomech.* 1988; 12: 551-572.
- [11] Cleary MP. Fundamental solutions for a fluid saturated porous solid. *Int. J. Solids Struct.* 1977; 13: 785-806.
- [12] Dargush GF and Banerjee P. A new boundary element method for three-dimensional coupled problems of consolidation and thermoelasticity. *J. Appl. Mech.* 1991; 58: 28-36.
- [13] Bonnet G. Basic singular solutions for a poroelastic medium in the dynamic range. *J. Acoust. Soc. Am.* 1987; 82: 1758-1762.
- [14] Dominguez J. An integral formulation for dynamic poroelasticity. *J. Applied Mech.* 1991; 58: 588-591.
- [15] Yazdchi M, Khalili N and Valliappan S. Non-linear seismic behaviour of concrete gravity dams using coupled finite element-boundary element method. *Int. J. Num. Meth. Eng.* 1999; 44.
- [16] Wolf JP and Song C. Dynamic-stiffness matrix in time domain of unbounded medium by infinitesimal finite-element cell method. *Earthq. Eng. Struct. Dyn.* 1994; 23.
- [17] Song C and Wolf J. The scaled boundary finite-element method - alias consistent infinitesimal finite-element cell method for elastodynamics. *Computer Methods in Applied Mechanics and Engineering* 1997; 147.
- [18] Wolf JP. *Foundation Vibration Analysis using Simple Physical Models*: Prentice Hall, 1994.
- [19] Biot MA. General theory of three-dimensional consolidation. *Journal of Applied Physics* 1941; 12: 155-164.