Online Supplement for “Systemic Risk Components in a Network
Model of Contagion” by Jeremy Staum et al.

Jeremy Staum Mingbin Feng

Ming Liu

Dept. of IEMS

Magnetar Capital, LLC

McCormick School of Engineering

1603 Orrington Avenue, 14th Floor

Northwestern University

Evanston, IL 60201, USA

2145 Sheridan Road

Evanston, IL 60208-3119, USA

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A Proof of Proposition 1

The algorithm terminates by iteration n because each node can appear in $\mathcal{D}'_k$ (i.e., be identified as a defaulting node) during at most one iteration, so $\mathcal{D}'_k$ can be non-empty for at most n iterations. The remaining sections of this appendix show by induction that:

- (i) For every iteration $k \geq 1$, the inverse matrix $(\mathbf{I} - (\mathbf{\Pi}_{\mathcal{D}_k \mathcal{D}_k})^\top)^{-1}$ exists, and it transforms non-negative vectors to non-negative vectors.
- (ii) For every iteration $k \geq 0$, for defaulting nodes, payment equals cash inflow: $\mathbf{p}_{\mathcal{D}_k}^k = (\mathbf{\Pi}_{\mathcal{D}_k})^\top \mathbf{p}^k + \mathbf{e}_{\mathcal{D}_k}$.
- (iii) For every iteration $k \geq 0$, all nodes make nonnegative payments: $\mathbf{p}^k \geq \mathbf{0}$.
- (iv) For every iteration $k \geq 0$, all nodes satisfy the outflow capacity constraint $\mathbf{p}^k \leq \bar{\mathbf{p}}$.
- (v) For every iteration $k \geq 0$, all nodes satisfy the priority constraint $\text{diag}(\mathbf{v}^k)(\bar{\mathbf{p}} - \mathbf{p}^k)^+ = \mathbf{0}$, where $\mathbf{v}^k = \mathbf{e} + \mathbf{\Pi}^\top \mathbf{p}^k - \mathbf{p}^k$.

At iteration 0, $\mathcal{N}_0 = \{1, \dots, n\}$, $\mathbf{p}^0 = \bar{\mathbf{p}}$, and $\mathcal{D}_0 = \emptyset$, so Equation (8) does not need to be evaluated and the remaining assertions on the list clearly hold.

The list includes all the constraints that characterize a clearing payment vector, except for the cash inflow capacity constraint for non-defaulting nodes, $\mathbf{p}_{\mathcal{N}_k}^k \leq (\mathbf{\Pi}_{\mathcal{N}_k})^\top \mathbf{p}^k + \mathbf{e}_{\mathcal{N}_k}$. When the algorithm terminates, $\mathcal{D}'_k = \emptyset$. Based on Step 3 of this terminal iteration k , it must be that $\mathbf{p}_{\mathcal{N}_k}^k \leq (\mathbf{\Pi}_{\mathcal{N}_k})^\top \mathbf{p}^k + \mathbf{e}_{\mathcal{N}_k}$. Therefore $\mathbf{p}^k$ is indeed a clearing payment vector at termination.

The logic of the inductive proof is as follows:

- Because Assertion (i) holds for iterations $1, \dots, k$, Assertions (ii) and (iii) hold for iteration k .
- Because Assertion (ii) holds for iteration k , Assertion (v) holds for iteration k .
- Because Assertions (ii) and (iii) hold for iteration k , Assertion (i) holds for iteration $k + 1$.
- Because Assertion (iv) holds for iteration k and Assertion (i) holds for iteration $k + 1$, Assertion (iv) holds for iteration $k + 1$.

A.1 Assertion (i)

We assume that Assertions (ii) and (iii) are satisfied at iteration k and show that Assertion (i) holds for iteration $k + 1$. Because Assertion (ii) holds for iteration k ,

$$\mathbf{p}_{\mathcal{D}_k}^k = \mathbf{e}_{\mathcal{D}_k} + (\mathbf{\Pi}_{\mathcal{D}_k})^\top \mathbf{p}^k. \quad (1)$$

From Steps 2 and 3 in iteration k , it follows that

$$\mathbf{p}_{\mathcal{D}'_k}^k = \bar{\mathbf{p}}_{\mathcal{D}'_k} > \mathbf{e}_{\mathcal{D}'_k} + (\mathbf{\Pi}_{\mathcal{D}'_k})^\top \mathbf{p}^k. \quad (2)$$

Because Step 4 in iteration k makes $\mathcal{D}_{k+1} = \mathcal{D}_k \cup \mathcal{D}'_k$, combining Equations (1) and (2) yields

$$\mathbf{p}_{\mathcal{D}_{k+1}}^k \geq \mathbf{e}_{\mathcal{D}_{k+1}} + (\mathbf{\Pi}_{\mathcal{D}_{k+1}})^\top \mathbf{p}^k. \quad (3)$$

Therefore

$$\begin{aligned}
(\mathbf{I} - (\mathbf{\Pi}_{\mathcal{D}_{k+1}\mathcal{D}_{k+1}})^\top) \mathbf{p}_{\mathcal{D}_{k+1}}^k &\geq \mathbf{e}_{\mathcal{D}_{k+1}} + (\mathbf{\Pi}_{\cdot\mathcal{D}_{k+1}})^\top \mathbf{p}^k - (\mathbf{\Pi}_{\mathcal{D}_{k+1}\mathcal{D}_{k+1}})^\top \mathbf{p}_{\mathcal{D}_{k+1}}^k \\
&= \mathbf{e}_{\mathcal{D}_{k+1}} + (\mathbf{\Pi}_{\mathcal{N}_{k+1}\mathcal{D}_{k+1}})^\top \mathbf{p}_{\mathcal{N}_{k+1}}^k \\
&\geq \mathbf{e}_{\mathcal{D}_{k+1}} > \mathbf{0},
\end{aligned}$$

where the final line follows from Assumption 1 and the inductive hypothesis that Assertion (iii) holds for iteration k . This shows that the condition of Theorem 8.3.2 of Karlin (1959) holds: there exists a vector (here $\mathbf{p}_{\mathcal{D}_{k+1}}^k$) that is mapped by $\mathbf{I} - (\mathbf{\Pi}_{\mathcal{D}_{k+1}\mathcal{D}_{k+1}})^\top$ to a strictly positive vector. According to this theorem of Karlin (1959), the inverse of $\mathbf{I} - (\mathbf{\Pi}_{\mathcal{D}_{k+1}\mathcal{D}_{k+1}})^\top$ exists, and this inverse matrix transforms non-negative vectors to non-negative vectors.

A.2 Assertion (ii)

We assume that Assertion (i) holds for iteration k and show that Assertion (ii) holds for iteration k . Premultiplying by $\mathbf{I} - (\mathbf{\Pi}_{\mathcal{D}_k\mathcal{D}_k})^\top$ on both sides of Equation (8) at iteration k yields

$$(\mathbf{I} - (\mathbf{\Pi}_{\mathcal{D}_k\mathcal{D}_k})^\top) \mathbf{p}_{\mathcal{D}_k}^k = \mathbf{e}_{\mathcal{D}_k} + (\mathbf{\Pi}_{\mathcal{N}_k\mathcal{D}_k})^\top \bar{\mathbf{p}}_{\mathcal{N}_k}.$$

From this and $\mathbf{p}_{\mathcal{N}_k}^k = \bar{\mathbf{p}}_{\mathcal{N}_k}$, it follows that

$$\mathbf{p}_{\mathcal{D}_k}^k = \mathbf{e}_{\mathcal{D}_k} + (\mathbf{\Pi}_{\mathcal{N}_k\mathcal{D}_k})^\top \bar{\mathbf{p}}_{\mathcal{N}_k} + (\mathbf{\Pi}_{\mathcal{D}_k\mathcal{D}_k})^\top \mathbf{p}_{\mathcal{D}_k}^k = \mathbf{e}_{\mathcal{D}_k} + (\mathbf{\Pi}_{\cdot\mathcal{D}_k})^\top \mathbf{p}^k.$$

A.3 Assertion (iii)

We assume that Assertion (i) holds for iteration k and show that Assertion (iii) holds for iteration k . By Assumption 1, $\mathbf{e}_{\mathcal{D}_k} + (\mathbf{\Pi}_{\mathcal{N}_k\mathcal{D}_k})^\top \bar{\mathbf{p}}_{\mathcal{N}_k}$ is non-negative. By Assertion (i), $(\mathbf{I} - (\mathbf{\Pi}_{\mathcal{D}_k\mathcal{D}_k})^\top)^{-1}$ transforms non-negative vectors to non-negative vectors. Therefore Step 2 at iteration k makes $\mathbf{p}^k \geq \mathbf{0}$.

A.4 Assertion (iv)

We assume that Assertion (iv) holds for iteration k and Assertion (i) holds for iteration $k+1$, and show that Assertion (iv) holds for iteration $k+1$. Define the map $f^k : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ by $f^k(\mathbf{p}) = \mathbf{p}'$ where

$$\mathbf{p}'_{\mathcal{N}_{k+1}} = \bar{\mathbf{p}}_{\mathcal{N}_{k+1}} \quad (4)$$

$$\mathbf{p}'_{\mathcal{D}_{k+1}} = \mathbf{e}_{\mathcal{D}_{k+1}} + (\mathbf{\Pi}_{\mathcal{N}_{k+1}\mathcal{D}_{k+1}})^\top \bar{\mathbf{p}}_{\mathcal{N}_{k+1}} + (\mathbf{\Pi}_{\mathcal{D}_{k+1}\mathcal{D}_{k+1}})^\top \mathbf{p}_{\mathcal{D}_{k+1}}. \quad (5)$$

This map f^k is monotone increasing. In iteration $k+1$, by Assertion (i), Step 2 is valid because the matrix inverse exists. It follows that $\mathbf{p}^{k+1} = f^k(\mathbf{p}^{k+1})$, i.e., $\mathbf{p}^{k+1}$ is a fixed point of f^k . By Tarski's fixed point theorem (Tarski, 1955), if $\mathbf{p} \geq f^k(\mathbf{p})$, then $\mathbf{p} \geq \mathbf{p}^{k+1}$. If we show that $\mathbf{p}^k \geq f^k(\mathbf{p}^k)$, it will follow that $\mathbf{p}^k \geq \mathbf{p}^{k+1}$. Together with Assertion (iv) for iteration k , this will establish that $\mathbf{p}^{k+1} \leq \bar{\mathbf{p}}$. It remains to show that $\mathbf{p}^k \geq f^k(\mathbf{p}^k)$.

The rows $i = 1, \dots, n$ are partitioned into three disjoint sets: $\mathcal{N}_{k+1}$, $\mathcal{D}'_k$, and $\mathcal{D}_k$. We have $\mathcal{D}_{k+1} = \mathcal{D}_k \cup \mathcal{D}'_k$ and $\mathcal{D}_k$ is the set of nodes marked as defaulting in both iterations k and $k+1$. Likewise, we have $\mathcal{N}_k = \mathcal{N}_{k+1} \cup \mathcal{D}'_k$ and $\mathcal{N}_{k+1}$ is the set of nodes marked as not defaulting in both iterations k and $k+1$.

Clearly, $\mathbf{p}^k_{\mathcal{N}_{k+1}} = \bar{\mathbf{p}}_{\mathcal{N}_{k+1}} = (f^k(\mathbf{p}^k))_{\mathcal{N}_{k+1}}$.

Because $\mathbf{p}^k_{\mathcal{D}'_k} = \bar{\mathbf{p}}_{\mathcal{D}'_k}$, from Equation (5) we get

$$\begin{aligned} (f^k(\mathbf{p}^k))_{\mathcal{D}_{k+1}} &= \mathbf{e}_{\mathcal{D}_{k+1}} + (\mathbf{\Pi}_{\mathcal{N}_{k+1}\mathcal{D}_{k+1}})^\top \bar{\mathbf{p}}_{\mathcal{N}_{k+1}} + (\mathbf{\Pi}_{\mathcal{D}_{k+1}\mathcal{D}_{k+1}})^\top \mathbf{p}^k_{\mathcal{D}_{k+1}} \\ &= \mathbf{e}_{\mathcal{D}_{k+1}} + (\mathbf{\Pi}_{\mathcal{N}_k\mathcal{D}_{k+1}})^\top \bar{\mathbf{p}}_{\mathcal{N}_k} + (\mathbf{\Pi}_{\mathcal{D}_k\mathcal{D}_{k+1}})^\top \mathbf{p}^k_{\mathcal{D}_k}. \end{aligned} \quad (6)$$

Because $\mathcal{D}_k \subseteq \mathcal{D}_{k+1}$, Equation (6) implies $(f^k(\mathbf{p}^k))_{\mathcal{D}_k} = (f^{k-1}(\mathbf{p}^k))_{\mathcal{D}_k}$. Together with $f^{k-1}(\mathbf{p}^k) = \mathbf{p}^k$, this yields $(f^k(\mathbf{p}^k))_{\mathcal{D}_k} = \mathbf{p}^k_{\mathcal{D}_k}$.

By Step 3 of iteration k , $\mathbf{p}^k_{\mathcal{D}'_k} = \bar{\mathbf{p}}_{\mathcal{D}'_k} > \mathbf{e}_{\mathcal{D}'_k} + (\mathbf{\Pi}_{\mathcal{D}'_k})^\top \mathbf{p}^k$. Combining this with $\mathcal{D}_k \subseteq \mathcal{D}_{k+1}$ and Equation (6), we get $\mathbf{p}^k_{\mathcal{D}'_k} > (f^k(\mathbf{p}^k))_{\mathcal{D}'_k}$.

We have shown $\mathbf{p}^k \geq f^k(\mathbf{p}^k)$ for all the rows: those in $\mathcal{N}_{k+1}$, $\mathcal{D}'_k$, and $\mathcal{D}_k$.

A.5 Assertion (v)

We assume that Assertion (ii) holds for iteration k and show that Assertion (v) holds for iteration k . Step 2 of iteration k implies $\bar{\mathbf{p}}_{\mathcal{N}_k} - \mathbf{p}_{\mathcal{N}_k}^k = \mathbf{0}$. Assertion (ii) implies $\mathbf{v}_{\mathcal{D}_k}^k = \mathbf{0}$. Because $\mathcal{N}_k \cup \mathcal{D}_k = \{1, \dots, n\}$, the priority constraint $\text{diag}(\mathbf{v}^k)(\bar{\mathbf{p}} - \mathbf{p}^k)^+ = \mathbf{0}$ holds at each iteration.

B Proof of Proposition 2

Because $\boldsymbol{\delta}$ and $\mathbf{\Pi}^0$ are insensitive to perturbations of $\mathbf{e}$, Equation (10) follows directly from Equation (7). Consider the LP

$$\min \mathbf{d}^\top \mathbf{x} \text{ such that } \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0} \quad (7)$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \\ \mathbf{v} \\ \boldsymbol{\ell} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \text{diag}(\boldsymbol{\delta} + \mathbf{L}\mathbf{1}) - \mathbf{L}^\top & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \text{diag}(\boldsymbol{\delta}) & \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} \mathbf{e} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}.$$

There is a one-to-one correspondence between feasible solutions $\mathbf{x}$ of LP (7) and payment vectors $\mathbf{p}$ that satisfy the capacity constraints (5). The correspondence $\mathbf{p} = \text{diag}(\bar{\mathbf{p}})\mathbf{f}$ is one-to-one for any value of $\bar{\mathbf{p}}$ because we stipulated $f_i = 1$ if $\bar{p}_i = 0$. Given $\mathbf{f}$, the only values of other coordinates of $\mathbf{x}$ that satisfy $\mathbf{A}\mathbf{x} = \mathbf{b}$ are $\mathbf{g} = \mathbf{1} - \mathbf{f}$, $\boldsymbol{\ell} = \text{diag}(\boldsymbol{\delta})(\mathbf{1} - \mathbf{f})$, and $\mathbf{v} = \mathbf{e} + \mathbf{L}^\top \mathbf{f} - \text{diag}(\boldsymbol{\delta} + \mathbf{L}\mathbf{1})\mathbf{f}$. The non-negativity of $\mathbf{f}$ and $\mathbf{g}$ is equivalent to the capacity constraint $\mathbf{0} \leq \mathbf{p} \leq \bar{\mathbf{p}}$, and the non-negativity of $\mathbf{v}$ is equivalent to the capacity constraint $(\mathbf{I} - \mathbf{\Pi}^\top)\mathbf{p} \leq \mathbf{e}$.

We now establish that the clearing payment vector $\mathbf{p}^*$ corresponds to an optimal solution $\mathbf{x}^*$ of LP (7). According to Proposition 1 of Liu and Staum (2010), for any non-zero, non-negative row vector $\mathbf{c}$ of length n , $\mathbf{c}\mathbf{p}^*$ is the maximal value of $\mathbf{c}\mathbf{p}$ subject to the capacity constraints (5). Therefore the clearing payment vectors is greater than or equal to any payment vector $\mathbf{p}$ which satisfies the capacity constraints (5), i.e., $\mathbf{p}^* \geq \mathbf{p}$. On the feasible set, the objective $\mathbf{d}^\top \mathbf{x} = \mathbf{1}^\top \boldsymbol{\ell} = \mathbf{1}^\top \text{diag}(\boldsymbol{\delta})(\mathbf{1} - \mathbf{f}) = \mathbf{1}^\top (\boldsymbol{\delta} - \text{diag}(\mathbf{\Pi}^0)\mathbf{p})$. Therefore, for any feasible $\mathbf{x}$, $\mathbf{d}^\top \mathbf{x} = \mathbf{1}^\top (\boldsymbol{\delta} - \text{diag}(\mathbf{\Pi}^0)\mathbf{p}) \geq \mathbf{1}^\top (\boldsymbol{\delta} - \text{diag}(\mathbf{\Pi}^0)\mathbf{p}^*) = \mathbf{d}^\top \mathbf{x}^*$ by Assumption 1.

Next we perform a sensitivity analysis of the LP (7) at its optimal solution $\mathbf{x}^*$. By definition $\mathbf{g}_{\mathcal{G}} = \mathbf{0}$ and $\mathbf{v}_{\mathcal{R}} = \mathbf{0}$. Note that the LP (7) has $4n$ variables and $3n$ constraints. One possible basis $\mathcal{B}$ for the solution contains $3n$ variables in $\mathbf{f}$, $\mathbf{g}_{\mathcal{R}}$, $\mathbf{v}_{\mathcal{G}}$, and $\mathbf{\ell}$. Because there are no borderline nodes, the chosen basis remains optimal when the LP data $\mathbf{A}$ and $\mathbf{b}$ are perturbed. The non-basic variables are zero. The basic variables are given by $\mathbf{x}_{\mathcal{B}} = \mathbf{A}_{\mathcal{B}}^{-1}\mathbf{b}$ and the sensitivities of the optimal objective value $\mathbf{d}^\top \mathbf{x}^*$ to $\mathbf{b}$ are given by the dual-optimal solution $\mathbf{y}^* = \mathbf{d}_{\mathcal{B}}^\top \mathbf{A}_{\mathcal{B}}^{-1}$. The sensitivity of the optimal objective value to A_{ij} is $-y_i^* x_j^*$.

We need to know the first $2n$ components of $\mathbf{y}^*$. Combining Equation (10) with the LP sensitivity analysis here, the sensitivity of the optimal objective value to $\mathbf{e}$ is $\mathbf{y}^* \nabla_{\mathbf{e}} \mathbf{b} = -\boldsymbol{\zeta}$. Because $\nabla_{\mathbf{e}} \mathbf{b} = \text{diag} \left(\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right)$, the first n components of $\mathbf{y}^*$ are $-\boldsymbol{\zeta}$. To find the second n components of $\mathbf{y}^*$, observe that the first n and last n components of $\mathbf{b}$ determine $\mathbf{f}^*$, $\mathbf{g}^*$, and $\mathbf{v}^*$; given these, the second n components of $\mathbf{b}$ determine $\boldsymbol{\ell}^*$ via $\ell_i^* = \delta_i g_i^* - b_{n+i}$ for all $i = 1, \dots, n$. Therefore the sensitivity of the optimal objective value to b_{n+i} is $y_{n+i}^* = \nabla_{b_{n+i}} \boldsymbol{\ell}^* = -1$, for $i = 1, \dots, n$.

Equation (11) comes from looking at the sensitivity of the optimal objective value to A_{ii} and $A_{n+i, n+i}$ for $i = 1, \dots, n$. Clearly $\nabla_{\delta_i} A_{ii} = \nabla_{\delta_i} A_{n+i, n+i} = 1$, therefore the sensitivity of the optimal objective value to δ_i is $-y_i^* x_i^* - y_{n+i}^* x_{n+i}^* = \zeta_i f_i^* + g_i^* = \zeta_i f_i^* + (1 - f_i^*)$ for $i = 1, \dots, n$.

Equation (12) comes from looking at the sensitivity of the optimal objective value to A_{ii} and A_{ji} for $i \neq j$. These coefficients have sensitivity 1 and -1 , respectively, to L_{ij} . Therefore the sensitivity of the optimal objective value to L_{ij} is $-y_i^* x_i^* + y_j^* x_i^* = \zeta_i f_i^* - \zeta_j f_i^*$.

C Derivations of Aumann-Shapley Values

In the network model of contagion, the cost function whose gradient appears in Equation (1) is given by $c(\boldsymbol{\lambda}) = \ell(\mathbf{e}(\boldsymbol{\lambda}), \boldsymbol{\delta}(\boldsymbol{\lambda}), \mathbf{L}(\boldsymbol{\lambda}))$, where external asset value $\mathbf{e}(\boldsymbol{\lambda}) = \boldsymbol{\eta}(\boldsymbol{\lambda}) + \boldsymbol{\Theta}(\boldsymbol{\lambda})\mathbf{a}$. Equation (14) provides an explicit formula for the cost in terms of the balance-sheet quantities $\boldsymbol{\delta}(\boldsymbol{\lambda})$, $\mathbf{e}(\boldsymbol{\lambda})$, and $\mathbf{L}(\boldsymbol{\lambda})$, and the sensitivities $\boldsymbol{\zeta}(\boldsymbol{\lambda})$ analyzed in Section 3.2. The balance-sheet schemes that we propose make the balance-sheet quantities $\boldsymbol{\delta}(\boldsymbol{\lambda})$, $\mathbf{e}(\boldsymbol{\lambda})$, and $\mathbf{L}(\boldsymbol{\lambda})$ differentiable in $\boldsymbol{\lambda}$, and the sensitivities $\boldsymbol{\zeta}(\boldsymbol{\lambda})$ piecewise constant in $\boldsymbol{\lambda}$. The sensitivities $\boldsymbol{\zeta}(\boldsymbol{\lambda})$ are non-differentiable at $\boldsymbol{\lambda}$ only if there is a borderline node in the system described by $\boldsymbol{\lambda}$. Therefore, in computing the gradient $\nabla c(\boldsymbol{\lambda})$, we need not consider the local sensitivity of $\boldsymbol{\zeta}(\boldsymbol{\lambda})$ to $\boldsymbol{\lambda}$, unless there are borderline nodes. In the schemes

that we consider, borderline nodes are present only for a finite number of values of t in the integral in Equation (1), so the integral can be computed while ignoring the issue of borderline nodes.

Proposition 1 gives the gradient of the cost function for schemes in which internal loan sizes are fixed or depend only on the participation levels of the borrower or lender. In the latter case, the number n of components in the participation vector $\boldsymbol{\lambda}$ equals the number n of nodes. Let $\nabla \boldsymbol{\delta}$, $\nabla \mathbf{e}$, $\nabla \mathbf{L}$, and $\nabla \mathbf{L}^\top$ denote the $n \times n$ matrices whose (i, j) th elements are $\partial \delta_j / \partial \lambda_i$, $\partial e_j / \partial \lambda_i$, $\partial L_{ij} / \partial \lambda_i$, and $\partial L_{ji} / \partial \lambda_i$, respectively.

Proposition 1. *Suppose that there are no borderline nodes in the counterfactual system specified by participation vector $\boldsymbol{\lambda}$, and that the external debt $\boldsymbol{\delta}$, external asset value $\mathbf{e}$, and internal liability $\mathbf{L}$ are differentiable at $\boldsymbol{\lambda}$. If for all $i, j \in \{1, \dots, n\}$ and $\boldsymbol{\lambda}' \in [0, 1]^n$, the counterfactual principal $L_{ij}(\boldsymbol{\lambda}')$ of the liability of node i to node j depends only on the participation levels λ'_i and λ'_j , then the gradient of the cost function is given by*

$$\begin{aligned} \nabla c(\boldsymbol{\lambda}) &= (\nabla \boldsymbol{\delta}(\boldsymbol{\lambda})) ((\mathbf{1} - \mathbf{f}^*(\boldsymbol{\lambda})) + \text{diag}(\mathbf{f}^*(\boldsymbol{\lambda}))\boldsymbol{\zeta}(\boldsymbol{\lambda})) \\ &\quad - (\nabla \mathbf{e}(\boldsymbol{\lambda}))\boldsymbol{\zeta}(\boldsymbol{\lambda}) \\ &\quad + (\nabla \mathbf{L}^\top(\boldsymbol{\lambda}))\text{diag}(\mathbf{f}^*(\boldsymbol{\lambda}))\boldsymbol{\zeta}(\boldsymbol{\lambda}) - \text{diag}(\boldsymbol{\zeta}(\boldsymbol{\lambda}))(\nabla \mathbf{L}^\top(\boldsymbol{\lambda}))\mathbf{f}^*(\boldsymbol{\lambda}) \\ &\quad + \text{diag}(\boldsymbol{\zeta}(\boldsymbol{\lambda}))\text{diag}(\mathbf{f}^*(\boldsymbol{\lambda}))(\nabla \mathbf{L}(\boldsymbol{\lambda}))\mathbf{1} - \text{diag}(\mathbf{f}^*(\boldsymbol{\lambda}))(\nabla \mathbf{L}(\boldsymbol{\lambda}))\boldsymbol{\zeta}(\boldsymbol{\lambda}). \end{aligned}$$

Proof. This is an application of the chain rule and Proposition 2. The first line, involving sensitivity to external debt, comes from Equation (11); the second line, involving sensitivity to external asset value, comes from Equation (10); the third and fourth lines, involving sensitivity to internal lending and borrowing, respectively, come from Equation (12). $\square$

C.1 External Assets

The points of potential discontinuity $\tilde{t}_2, \dots, \tilde{t}_m$ can be found by starting from $\tilde{t}_1 = 1$ and descending to the next value of $\tilde{t}$ for which a node becomes borderline.

A red node j in the counterfactual system specified by $\tilde{t}_i \mathbf{1}$ pays less than its total liabilities, i.e., $p_j(\tilde{t}_i \mathbf{1}) < \bar{p}_i$. It becomes borderline when its payment increases to its total liabilities. The left derivatives of clearing payments are $\mathbf{p}'(\tilde{t}_i \mathbf{1}) = (\nabla_{\boldsymbol{\lambda}} \mathbf{p}(\tilde{t}_i \mathbf{1}))\mathbf{1} = (\nabla_{\mathbf{e}} \mathbf{p}(\tilde{t}_i \mathbf{1}))(\nabla_{\boldsymbol{\lambda}} \mathbf{e})\mathbf{1} = (\nabla_{\mathbf{e}} \mathbf{p}(\tilde{t}_i \mathbf{1}))\boldsymbol{\Theta}(\mathbf{a} - \mathbf{1})$. A green node j satisfies $p_j(\tilde{t}_i \mathbf{1}) = \bar{p}_j$ and has positive time-1 equity $v_j(\tilde{t}_i \mathbf{1})$.

It becomes borderline when its time-1 equity drops to 0. The left derivatives of time-1 equity are $\mathbf{v}'(\tilde{t}_i \mathbf{1}) = (\nabla_{\mathbf{e}} \mathbf{v}(\tilde{t}_i \mathbf{1})) \boldsymbol{\Theta}(\mathbf{a} - \mathbf{1})$ where $\nabla_{\mathbf{e}} \mathbf{v}(\tilde{t}_i \mathbf{1}) = \mathbf{I} + (\boldsymbol{\Pi}^\top - \mathbf{I}) \nabla_{\mathbf{e}} \mathbf{p}(\tilde{t}_i \mathbf{1})$ (Liu and Staum, 2010).

Because we are decreasing t , let $\partial p_i / \partial e_j$ be the left derivative if $\boldsymbol{\Theta}_j \cdot (\mathbf{a} - \mathbf{1}) > 0$, in which case e_j decreases as t decreases, and otherwise be the right derivative (see Liu and Staum, 2010). The sensitivities $\zeta(\mu_i \mathbf{1}) = \zeta(\tilde{t}_i \mathbf{1})$ can be computed at participation $\tilde{t}_i \mathbf{1}$ by evaluating Equation (10) using these one-sided derivatives for $\partial p_i / \partial e_j$. The distance ω_{ij} associated with node j in the counterfactual system specified by $\tilde{t}_i \mathbf{1}$ is as follows.

- If node j is red and $p'_j(\tilde{t}_i \mathbf{1}) < 0$, then $\omega_{ij} = -(\bar{p}_j - p_j(\tilde{t}_i \mathbf{1})) / p'_j(\tilde{t}_i \mathbf{1})$.
- If node j is green and $v'_j(\tilde{t}_i \mathbf{1}) > 0$, then $\omega_{ij} = v_j / v'_j(\tilde{t}_i \mathbf{1})$.
- Otherwise, $\omega_{ij} = \infty$ because there is no local movement of node j towards the borderline.

For $i < m$, t can decrease from $\tilde{t}_i$ by $\omega_i = \min_{j=1, \dots, n} \omega_{ij}$ until some node becomes borderline. The process ends at $i = m$ because $\min_{j=1, \dots, n} \omega_{ij} = \infty$ indicating that no more borderlines are encountered for $t \in [0, \tilde{t}_1]$. Then $\omega_m = \tilde{t}_m$.

A proof that there are finitely many breakpoints $\tilde{t}_2, \dots, \tilde{t}_m$ comes from parametric linear programming (Alevras and Padberg, 2001). Consider a parametric LP in which the scalar parameter t affects $\mathbf{e}$ in the LP of Appendix B, in the way described in Section 4.2. The breakpoints $\tilde{t}_1, \dots, \tilde{t}_m$ corresponds to critical points of $t \in [0, 1]$ where the optimal basis in the parametric LP changes. The parameter t affects only the right hand side of the LP, which is strictly increasing in t by Assumption 1, so no basis can be repeated when t varies monotonically. Since there are only a finite number of bases in LP (7), the number of breakpoints is finite.

C.2 Transmission and Leverage

In Section 4.3, the outflow fractions do not change: $\boldsymbol{\Pi}(\boldsymbol{\lambda}) = \boldsymbol{\Pi}$. Therefore the cost function is $c(\boldsymbol{\lambda}) = \boldsymbol{\delta}(\boldsymbol{\lambda})^\top \mathbf{1} - \boldsymbol{\Pi}^0 \mathbf{p}(\boldsymbol{\lambda}) = \sum_{i \in \mathcal{R}(\boldsymbol{\lambda})} (\delta_i(\boldsymbol{\lambda}) - \Pi_i^0 p_i(\boldsymbol{\lambda}))$. If node i is red in the counterfactual system specified by $\boldsymbol{\lambda}$, then from Equation (8) and Corollary (1), we get

$$p_i(\boldsymbol{\lambda}) = \sum_{j \in \mathcal{R}(\boldsymbol{\lambda})} \frac{\partial p_i}{\partial e_j}(\boldsymbol{\lambda}) \left(e_j(\boldsymbol{\lambda}) + \sum_{k \notin \mathcal{R}(\boldsymbol{\lambda})} \bar{p}_k(\boldsymbol{\lambda}) \Pi_{kj} \right) = \sum_{j \in \mathcal{R}(\boldsymbol{\lambda})} \frac{\partial p_i}{\partial e_j}(\boldsymbol{\lambda}) \left(e_j(\boldsymbol{\lambda}) + \sum_{k \notin \mathcal{R}(\boldsymbol{\lambda})} L_{kj}(\boldsymbol{\lambda}) \right).$$

The balance-sheet scheme specifies that external debt $\delta_i(\boldsymbol{\lambda}) = \lambda_i \delta_i$, internal debt $L_{ij}(\boldsymbol{\lambda}) = \lambda_i L_{ij}$, and external income $e_i(\boldsymbol{\lambda}) = e_i + \sum_{j=1}^n (1 - \lambda_j) L_{ji}$. The payment made by node i is

$$p_i(\boldsymbol{\lambda}) = \sum_{j \in \mathcal{R}(\boldsymbol{\lambda})} \frac{\partial p_i}{\partial e_j}(\boldsymbol{\lambda}) \left(e_j + \sum_{k \in \mathcal{R}(\boldsymbol{\lambda})} L_{kj}(1 - \lambda_k) + \sum_{k \notin \mathcal{R}(\boldsymbol{\lambda})} L_{kj} \right).$$

Rearranging and using $\zeta_j = \sum_{i \in \mathcal{R}} \Pi_i^0 \partial p_i / \partial e_j$, as follows from Equation (10), the cost is

$$c(\boldsymbol{\lambda}) = \sum_{i \in \mathcal{R}(\boldsymbol{\lambda})} \delta_i \lambda_i - \sum_{j \in \mathcal{R}(\boldsymbol{\lambda})} \left(e_j + \sum_{k \in \mathcal{R}(\boldsymbol{\lambda})} L_{kj}(1 - \lambda_k) + \sum_{k \notin \mathcal{R}(\boldsymbol{\lambda})} L_{kj} \right) \zeta_j(\boldsymbol{\lambda}).$$

The sensitivity of cost to the participation of node i is 0 if node i is green and if node i is red, it is

$$\frac{\partial c}{\partial \lambda_i}(\boldsymbol{\lambda}) = \delta_i + \sum_{j \in \mathcal{R}(\boldsymbol{\lambda})} L_{ij} \zeta_j(\boldsymbol{\lambda}) = \delta_i + \sum_{j=1}^n L_{ij} \zeta_j(\boldsymbol{\lambda})$$

because $\zeta_j(\boldsymbol{\lambda}) = 0$ if node j is green in the counterfactual system specified by $\boldsymbol{\lambda}$.

The points of potential discontinuity $\tilde{t}_2, \dots, \tilde{t}_m$ are found using a procedure similar to that in Appendix C.1. If node i is red, then the left derivative is $p'_i(\tilde{t}_h \mathbf{1}) = - \sum_{j \in \mathcal{R}(\tilde{t}_h \mathbf{1})} \frac{\partial^+ p_i}{\partial e_j}(\tilde{t}_h \mathbf{1}) \sum_{k \in \mathcal{R}(\tilde{t}_h \mathbf{1})} L_{kj}$. In this expression, right derivatives $\nabla_e^+ \mathbf{p}(\tilde{t}_i \mathbf{1}) = \nabla_e \mathbf{p}(\mu_i \mathbf{1})$ are used, for the following reason. Decreasing the scale t in constructing the counterfactual system specified by $t \mathbf{1}$ decreases the nodes' time-1 debts and increases their time-1 income. It replaces some of their internal loans with cash, and in this model, the return on internal loans is negative or zero, whereas the return on cash is zero. This shows also that decreasing the scale can convert red nodes to green nodes, but cannot convert green nodes to red nodes. The distance ω_{ij} associated with node j in the counterfactual system specified by $\tilde{t}_i \mathbf{1}$ is as follows.

- If node j is red and $p'_j(\tilde{t}_i \mathbf{1}) < \bar{p}_j$, then $\omega_{ij} = (\tilde{t}_i \bar{p}_j - p_j(\tilde{t}_i \mathbf{1})) / (\bar{p}_j - p'_j(\tilde{t}_i \mathbf{1}))$.
- Otherwise, $\omega_{ij} = \infty$ because there is no local movement of node j towards the borderline.

Appendix B shows that the partitions of red and green nodes determines an optimal basis of the LP (7). In the iterative method above the number of red nodes strictly decreases in each iteration before termination. Therefore only finitely many iterations can be performed, so there are only finitely many breakpoints $\tilde{t}_2, \dots, \tilde{t}_m$.

C.3 Solvency

External asset value $e_i(\boldsymbol{\lambda}) = \lambda_i(\varsigma_i + \sum_{j=1}^n \Theta_{ij}(a_j - 1)) - \sum_{j=1}^n \lambda_j L_{ji}$. The cost $c(\cdot)$ is homogeneous.

By Proposition 1, if there are no borderline nodes, the Aumann-Shapley value in Section 4.5 is

$$\begin{aligned} \nabla c(\mathbf{1}) &= \text{diag}(\boldsymbol{\delta})\text{diag}(\mathbf{1}_{\mathcal{R}})(\mathbf{1} - \mathbf{f}^*) + \text{diag}(\boldsymbol{\delta})\text{diag}(\mathbf{f}^*)\boldsymbol{\zeta} \\ &\quad - \text{diag}(\boldsymbol{\delta} + \mathbf{L}\mathbf{1} + \boldsymbol{\gamma} + \boldsymbol{\Theta}(\mathbf{a} - \mathbf{1}))\boldsymbol{\zeta} + \mathbf{L}\boldsymbol{\zeta} + \text{diag}(\boldsymbol{\zeta})\text{diag}(\mathbf{f}^*)\mathbf{L}\mathbf{1} - \text{diag}(\mathbf{f}^*)\mathbf{L}\boldsymbol{\zeta}. \end{aligned} \quad (8)$$

Again, the allocation to green nodes is zero. If node i is red, then from Equation (13) for red nodes and the definitions $\Pi_{ij} = L_{ij}/(\delta_i + \sum_{k=1}^n L_{ik})$ and $\Pi_i^0 = \delta_i/(\delta_i + \sum_{k=1}^n L_{ik})$, it follows that

$$\delta_i + \sum_{j=1}^n L_{ij}\zeta_j = \left(\delta_i + \sum_{j=1}^n L_{ij} \right) \zeta_i \quad (9)$$

Using this equation in Equation (8) and simplifying yields the allocation given in Section 4.5.

C.4 Absorption

Similar to Appendix C.3, the cost $c(\cdot)$ is homogeneous. By Proposition 1, if there are no borderline nodes then the Aumann-Shapley value in Section 4.6 is

$$\begin{aligned} \nabla c(\mathbf{1}) &= (\text{diag}(\boldsymbol{\delta} + \mathbf{L}\mathbf{1}) - \mathbf{L}^\top)(\mathbf{1} - \mathbf{f}^* + \text{diag}(\mathbf{f}^*)\boldsymbol{\zeta}) - \text{diag}(\mathbf{e})\boldsymbol{\zeta} + \mathbf{L}^\top \text{diag}(\mathbf{f}^*)\boldsymbol{\zeta} - \text{diag}(\boldsymbol{\zeta})\mathbf{L}^\top \mathbf{f}^* \\ &= (\text{diag}(\boldsymbol{\delta} + \mathbf{L}\mathbf{1}) - \mathbf{L}^\top)(\mathbf{1} - \mathbf{f}^*) + \text{diag}(\boldsymbol{\zeta})(\text{diag}(\mathbf{f}^*)(\boldsymbol{\delta} + \mathbf{L}\mathbf{1}) - \mathbf{e} - \mathbf{L}^\top \mathbf{f}^*). \end{aligned}$$

The capacity constraint (5) implies that the total outflow from a node can not exceed the total inflow: $f_i^*(\delta_i + \sum_{j=1}^n L_{ij}) \leq e_i + \sum_{j=1}^n L_{ji}f_j^*$. The priority constraint (6) implies that equality holds for rows corresponding to defaulting nodes, which are obligated to pay out as much as they can. As stated in Equation (13), green nodes have zero marginal price of wealth: $\boldsymbol{\zeta}_{\mathcal{G}} = \mathbf{0}$. Therefore

$$\text{diag}(\boldsymbol{\zeta})\text{diag}(\mathbf{f}^*)(\boldsymbol{\delta} + \mathbf{L}\mathbf{1}) = \text{diag}(\boldsymbol{\zeta})(\mathbf{e} + \mathbf{L}^\top \mathbf{f}^*). \quad (10)$$

The formula given in Section 4.6 follows from this.

C.5 Funding

Cash and external asset value are

$$\begin{aligned}\boldsymbol{\eta}(\boldsymbol{\lambda}) &= \text{diag}(\boldsymbol{\lambda})(\boldsymbol{\gamma} + \boldsymbol{\delta} - \boldsymbol{\Theta}\mathbf{1} - \boldsymbol{L}^\top \mathbf{1}) + \boldsymbol{L}\boldsymbol{\lambda} = \text{diag}(\boldsymbol{\lambda})\boldsymbol{\eta} - \text{diag}(\boldsymbol{\lambda})\boldsymbol{L}\mathbf{1} + \boldsymbol{L}\boldsymbol{\lambda} \quad \text{and} \\ \boldsymbol{e}(\boldsymbol{\lambda}) &= \text{diag}(\boldsymbol{\lambda})(\boldsymbol{\gamma} + \boldsymbol{\delta} + \boldsymbol{\Theta}(\boldsymbol{a} - \mathbf{1}) - \boldsymbol{L}^\top \mathbf{1}) + \boldsymbol{L}\boldsymbol{\lambda} = \text{diag}(\boldsymbol{\lambda})\boldsymbol{e} - \text{diag}(\boldsymbol{\lambda})\boldsymbol{L}\mathbf{1} + \boldsymbol{L}\boldsymbol{\lambda}.\end{aligned}$$

The cost function $c(\cdot)$ is homogeneous. By Proposition 1, if there are no borderline nodes then the Aumann-Shapley value in Section 4.7 is

$$\begin{aligned}\nabla c(\mathbf{1}) &= \text{diag}(\boldsymbol{\delta})((\mathbf{1} - \boldsymbol{f}^*) + \text{diag}(\boldsymbol{f}^*)\boldsymbol{\zeta}) - \text{diag}(\boldsymbol{e} - \boldsymbol{L}\mathbf{1})\boldsymbol{\zeta} - \boldsymbol{L}^\top \boldsymbol{\zeta} + \boldsymbol{L}^\top \text{diag}(\boldsymbol{f}^*)\boldsymbol{\zeta} - \text{diag}(\boldsymbol{\zeta})\boldsymbol{L}^\top \boldsymbol{f}^* \\ &= \text{diag}(\boldsymbol{\delta})(\mathbf{1} - \boldsymbol{f}^*) + \text{diag}(\boldsymbol{\delta})\text{diag}(\boldsymbol{f}^*)\boldsymbol{\zeta} + \text{diag}(\boldsymbol{L}\mathbf{1})\boldsymbol{\zeta} - \boldsymbol{L}^\top \text{diag}(\mathbf{1} - \boldsymbol{f}^*)\boldsymbol{\zeta} - \text{diag}(\boldsymbol{\zeta})(\boldsymbol{e} + \boldsymbol{L}^\top \boldsymbol{f}^*).\end{aligned}$$

The formula given in Section 4.6 follows from this and Equation (10).

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