Online Supplement for “Structural Properties of the Optimal Policy for
Dual-Sourcing Systems with General Lead Times” by Zhongsheng Hua, Wei
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Proof of Lemma 2. It is clear that $\tilde{\mathbf{x}} - \delta_{\max} \mathbf{e} \leq \mathbf{x} \leq \tilde{\mathbf{x}} - \delta_{\min} \mathbf{e}$. Since $\eta(\mathbf{x})$ is increasing in $\mathbf{x}$,

$$\eta(\tilde{\mathbf{x}} - \delta_{\max} \mathbf{e}) \leq \eta(\mathbf{x}) \leq \eta(\tilde{\mathbf{x}} - \delta_{\min} \mathbf{e}). \quad (\text{A.1})$$

If $\delta_{\min} < 0$, then $\eta(\tilde{\mathbf{x}} - \delta_{\min} \mathbf{e}) = \eta[\tilde{\mathbf{x}} + (-\delta_{\min})\mathbf{e}] \leq \eta(\tilde{\mathbf{x}}) - \delta_{\min}$. Otherwise (i.e., $\delta_{\min} \geq 0$), $\eta(\tilde{\mathbf{x}} - \delta_{\min} \mathbf{e}) \leq \eta(\tilde{\mathbf{x}})$ because (1) $\eta(\mathbf{x})$ is increasing in $\mathbf{x}$, and (2) $\tilde{\mathbf{x}} - \delta_{\min} \mathbf{e} \leq \tilde{\mathbf{x}}$. So,

$$\eta(\tilde{\mathbf{x}} - \delta_{\min} \mathbf{e}) \leq \eta(\tilde{\mathbf{x}}) - \min(0, \delta_{\min}). \quad (\text{A.2})$$

If $\delta_{\max} > 0$, then $\eta(\tilde{\mathbf{x}}) = \eta[(\tilde{\mathbf{x}} - \delta_{\max} \mathbf{e}) + \delta_{\max} \mathbf{e}] \leq \eta(\tilde{\mathbf{x}} - \delta_{\max} \mathbf{e}) + \delta_{\max}$. Otherwise (i.e., $\delta_{\max} \leq 0$), $\eta(\tilde{\mathbf{x}} - \delta_{\max} \mathbf{e}) \geq \eta(\tilde{\mathbf{x}})$ because (1) $\eta(\mathbf{x})$ is increasing in $\mathbf{x}$, and (2) $\tilde{\mathbf{x}} - \delta_{\max} \mathbf{e} \geq \tilde{\mathbf{x}}$. So,

$$\eta(\tilde{\mathbf{x}} - \delta_{\max} \mathbf{e}) \geq \eta(\tilde{\mathbf{x}}) - \max(0, \delta_{\max}). \quad (\text{A.3})$$

Concluding equations (A.1-3), $\eta(\tilde{\mathbf{x}}) - \max(0, \delta_{\max}) \leq \eta(\mathbf{x}) \leq \eta(\tilde{\mathbf{x}}) - \min(0, \delta_{\min})$.

Proof of Proposition 1. It is easy to verify that $f_0^v(\mathbf{v})$ is $L^{\natural}$ -convex because $f_0^v(\mathbf{v}) \equiv 0$. Suppose that

$f_{n-1}^v(\mathbf{v})$ is $L^{\natural}$ -convex. We next prove that $f_n^v(\mathbf{v})$ is also $L^{\natural}$ -convex.

Define

$$F_n^v(\mathbf{v}, \zeta, v_+^0) := c^e(v_+^0 - v^0) - c^r \zeta + p(v_+^0 - v^1) + \alpha E_D f_{n-1}^v[(v_+^0 - D, v^2, \dots, v^{l-1}, 0) - \zeta \mathbf{e}] \text{ and}$$

$$G_n^v(\mathbf{v}, \zeta) := \min_{v_+^0 \geq v^0} F_n^v(\mathbf{v}, \zeta, v_+^0), \text{ where } v_+^0 = v^0 + q^e.$$

It is clear that $f_n^v(\mathbf{v}) = \min_{\zeta \leq 0} G_n^v(\mathbf{v}, \zeta)$. Under any given demand realization, $L^{\natural}$ -convexity of $f_{n-1}^v(\mathbf{v})$

implies that $f_{n-1}^v[(v_+^0 - D, v^2, \dots, v^{l-1}, 0) - \zeta \mathbf{e}]$ is submodular in $(v_+^0, v^2, \dots, v^{l-1}, \zeta)$. Since submodularity is

preserved under expectation (Corollary 2.6.2 of Topkis 1998), $E_D f_{n-1}^v[(v_+^0 - D, v^2, \dots, v^{l-1}, 0) - \zeta \mathbf{e}]$ is also submodular in $(v_+^0, v^2, \dots, v^{l-1}, \zeta)$. It is clear $c^e q^e$ and $p(\cdot)$ are convex. Lemma 2.6.2 of Topkis (1998) implies that (1) $c^e(v_+^0 - v^0)$ is submodular in (v_+^0, v^0) , and (2) $p(v_+^0 - v^1)$ is submodular in (v_+^0, v^1) . Consequently, $F_n^v(\mathbf{v}, \zeta, v_+^0)$ is submodular because $-c^r \zeta$ is separable. It follows that $F_n^v(\mathbf{v}, \zeta, v_+^0)$ is $L^\natural$ -convex from Lemma 1 of Zipkin (2008b). Note that it is easy to verify that $\{(\mathbf{v}, \zeta, v_+^0) : v^1 \geq v^2 \geq \dots \geq v^{l-1}, v_+^0 - v^0 \geq 0, \zeta \leq 0\}$ is a lattice. This indicates that both $G_n^v(\mathbf{v}, \zeta)$ and $f_n^v(\mathbf{v})$ are $L^\natural$ -convex from Lemma 2 of Zipkin (2008b).

Define $\bar{\zeta}_n(\mathbf{v}) := \max\{\arg \min_{\zeta \leq 0} G_n^v(\mathbf{v}, \zeta)\}$. Lemma 1 implies that $\bar{\zeta}_n(\mathbf{v})$ is increasing in $\mathbf{v}$, but $\bar{\zeta}_n(\mathbf{v} + u\mathbf{e}) \leq \bar{\zeta}_n(\mathbf{v}) + u$ for $u > 0$. Since $\bar{q}_n^r(\mathbf{v}) = -\bar{\zeta}_n(\mathbf{v})$, $\bar{q}_n^r(\mathbf{v})$ is decreasing in $\mathbf{v}$, but $\bar{q}_n^r(\mathbf{v} + u\mathbf{e}) \geq \bar{q}_n^r(\mathbf{v}) - u$ for $u \geq 0$. By assuming smooth $\bar{q}_n^r(\mathbf{v})$, we have $\partial \bar{q}_n^r(\mathbf{v}) / \partial v^i \leq 0$ for $i = 0, 1, \dots, l-1$ and $\sum_{i=0}^{l-1} \partial \bar{q}_n^r(\mathbf{v}) / \partial v^i \geq -1$. Since $v^i = \sum_{k=i}^{l-1} z^k$, $i = 0, \dots, l-1$, we have that $\partial \hat{q}_n^r(\mathbf{z}) / \partial z^i = \sum_{k=0}^i \partial \bar{q}_n^r(\mathbf{v}) / \partial v^k$, $i = 0, \dots, l-1$. It follows that

$$\partial \hat{q}_n^r(\mathbf{z}) / \partial z^0 = \partial \bar{q}_n^r(\mathbf{v}) / \partial v^0 \leq 0,$$

$$\partial \hat{q}_n^r(\mathbf{z}) / \partial z^i - \partial \hat{q}_n^r(\mathbf{z}) / \partial z^{i-1} = \partial \bar{q}_n^r(\mathbf{v}) / \partial v^i \leq 0, \quad i = 1, \dots, l-1.$$

$$\partial \hat{q}_n^r(\mathbf{z}) / \partial z^{l-1} = \sum_{k=0}^{l-1} \partial \bar{q}_n^r(\mathbf{v}) / \partial v^k \geq -1.$$

That is, $-1 \leq \partial \hat{q}_n^r(\mathbf{z}) / \partial z^{l-1} \leq \dots \leq \partial \hat{q}_n^r(\mathbf{z}) / \partial z^1 \leq \partial \hat{q}_n^r(\mathbf{z}) / \partial z^0 \leq 0$.

For all $\tilde{\mathbf{v}}$ and $\mathbf{v}$, Lemma 2 indicates that

$$\bar{\zeta}_n(\tilde{\mathbf{v}}) - \max\{0, \max_{i=0, \dots, l-1} (\tilde{v}^i - v^i)\} \leq \bar{\zeta}_n(\mathbf{v}) \leq \bar{\zeta}_n(\tilde{\mathbf{v}}) - \min\{0, \min_{i=0, \dots, l-1} (\tilde{v}^i - v^i)\}.$$

From $\bar{q}_n^r(\mathbf{v}) = -\bar{\zeta}_n(\mathbf{v})$, we have

$$\bar{q}_n^r(\tilde{\mathbf{v}}) + \min\{0, \min_{i=0, \dots, l-1} (\tilde{v}^i - v^i)\} \leq \bar{q}_n^r(\mathbf{v}) \leq \bar{q}_n^r(\tilde{\mathbf{v}}) + \max\{0, \max_{i=0, \dots, l-1} (\tilde{v}^i - v^i)\}.$$

Let $s^{r,l} = \bar{q}_n^r(\tilde{\mathbf{v}}) \geq 0$ and $s^{r,i} = \bar{q}_n^r(\tilde{\mathbf{v}}) + v^i$, $i=0, \dots, l-1$. Recall that $\bar{q}_n^r(\mathbf{v}) \geq 0$. So,

$$\min\{s^{r,l}, \min_{i=0, \dots, l-1} (s^{r,i} - v^i)^+\} \leq \bar{q}_n^r(\mathbf{v}) \leq \max\{s^{r,l}, \max_{i=0, \dots, l-1} (s^{r,i} - v^i)\}. \quad ,$$

Proof of Proposition 2. It is clear that $f_0^w(\mathbf{w})$ is $L^{\natural}$ -convex because $f_0^w(\mathbf{w}) \equiv 0$. Suppose that $f_{n-1}^w(\mathbf{w})$ is $L^{\natural}$ -convex. In what follows, we show the $L^{\natural}$ -convexity of $f_n^w(\mathbf{w})$. Define

$$F_n^w(\mathbf{w}, \xi, w_+^{l-1}) := -c^e \xi + c^r (w_+^{l-1} - w^{l-1}) + p(w^0 - \xi) + \alpha E_D f_{n-1}^w[(w^1, w^2, \dots, w^{l-1}, w_+^{l-1}) - (\xi + D)\mathbf{e}],$$

$$G_n^w(\mathbf{w}, \xi) := \min_{w_+^{l-1} \geq w^{l-1}} F_n^w(\mathbf{w}, \xi, w_+^{l-1}), \text{ where } w_+^{l-1} = w^{l-1} + q^r.$$

So, $f_n^w(\mathbf{w}) = \min_{\xi \leq 0} G_n^w(\mathbf{w}, \xi)$. Since (1) $f_{n-1}^w(\mathbf{w})$ is $L^{\natural}$ -convex, and (2) submodularity is preserved under expectation (Corollary 2.6.2 of Topkis 1998), we have that $E_D f_{n-1}^w[(w^1, w^2, \dots, w^{l-1}, w_+^{l-1}) - (\xi + D)\mathbf{e}]$ is submodular in $(w^1, w^2, \dots, w^{l-1}, w_+^{l-1}, \xi)$. It can be also verified that $-c^e \xi$ is separable, $c^r (w_+^{l-1} - w^{l-1})$ is submodular in (w^{l-1}, w_+^{l-1}) , and $p(w^0 - \xi)$ is submodular in (w^0, ξ) . Consequently, $F_n^w(\mathbf{w}, \xi, w_+^{l-1})$ is submodular, and is also $L^{\natural}$ -convex from Lemma 1 of Zipkin (2008b). Note that it is easy to verify that $\{(\mathbf{w}, \xi, w_+^{l-1}) : w^0 \leq w^1 \leq \dots \leq w^{l-1} \leq w_+^{l-1}, \xi \leq 0\}$ is a lattice. It follows that both $G_n^w(\mathbf{w}, \xi)$ and $f_n^w(\mathbf{w})$ are $L^{\natural}$ -convex from Lemma 2 of Zipkin (2008b).

Define $\bar{\xi}_n^e(\mathbf{w}) := \max\{\arg \min_{\xi \leq 0} G_n^w(\mathbf{w}, \xi)\}$. From Lemma 1 and $\bar{q}_n^e(\mathbf{w}) = -\bar{\xi}_n^e(\mathbf{w})$, we have $\bar{q}_n^e(\mathbf{w})$ is decreasing in $\mathbf{w}$, but $\bar{q}_n^e(\mathbf{w} + u\mathbf{e}) \geq \bar{q}_n^e(\mathbf{w}) - u$ for $u \geq 0$. By assuming smooth $\bar{q}_n^e(\mathbf{w})$, we have $\bar{q}_n^e(\mathbf{w})/\partial w^i \leq 0$ for $i=0, 1, \dots, l-1$ and $\sum_{i=0}^{l-1} \partial \bar{q}_n^e(\mathbf{w})/\partial w^i \geq -1$. Since (1) $w^i = \sum_{k=0}^i z^k$, $i=0, \dots, l-1$, and (2) $\hat{q}_n^e(\mathbf{z}) = \bar{q}_n^e(\mathbf{w})$, we have $\partial \hat{q}_n^e(\mathbf{z})/\partial z^i = \sum_{k=i}^{l-1} \partial \bar{q}_n^e(\mathbf{w})/\partial w^k$, $i=0, \dots, l-1$. Consequently,

$$\partial \hat{q}_n^e(\mathbf{z})/\partial z^0 = \sum_{k=0}^{l-1} \partial \bar{q}_n^e(\mathbf{w})/\partial w^k \geq -1,$$

$$\partial \hat{q}_n^e(\mathbf{z})/\partial z^{i-1} - \partial \hat{q}_n^e(\mathbf{z})/\partial z^i = \partial \bar{q}_n^e(\mathbf{w})/\partial w^{i-1} \leq 0, \quad i=1, \dots, l-1,$$

$$\partial \hat{q}_n^e(\mathbf{z})/\partial z^{l-1} = \partial \bar{q}_n^e(\mathbf{w})/\partial w^{l-1} \leq 0.$$

That is, $-1 \leq \partial \hat{q}_n^e(\mathbf{z}) / \partial z^0 \leq \partial \hat{q}_n^e(\mathbf{z}) / \partial z^1 \leq \dots \leq \partial \hat{q}_n^e(\mathbf{z}) / \partial z^{l-1} \leq 0$.

For all $\tilde{\mathbf{w}}$ and $\mathbf{w}$, it is easy to verify from Lemma 2 that

$$\bar{q}_n^e(\tilde{\mathbf{w}}) + \min\{0, \min_{i=0, \dots, l-1}(\tilde{w}^i - w^i)\} \leq \bar{q}_n^e(\mathbf{w}) \leq \bar{q}_n^e(\tilde{\mathbf{w}}) + \max\{0, \max_{i=0, \dots, l-1}(\tilde{w}^i - w^i)\}.$$

Let $s^{e,l} = \bar{q}_n^e(\tilde{\mathbf{w}}) \geq 0$ and $s^{e,i} = \bar{q}_n^e(\tilde{\mathbf{w}}) + v^i$, $i = 0, \dots, l-1$. Since $\bar{q}_n^r(\mathbf{v}) \geq 0$, we have

$$\min\{s^{e,l}, \min_{i=0, \dots, l-1}(s^{e,i} - w^i)^+\} \leq \bar{q}_n^e(\mathbf{w}) \leq \max\{s^{e,l}, \max_{i=0, \dots, l-1}(s^{e,i} - w^i)\}, \quad ,$$

Proof of Proposition 3. Obviously, we have $f_1(\mathbf{z}) \geq 0 = f_0(\mathbf{z})$ for any given $\mathbf{z}$. Suppose that

$f_n(\mathbf{z}) \geq f_{n-1}(\mathbf{z})$ for any given $\mathbf{z}$. Then

$$\begin{aligned} f_{n+1}(\mathbf{z}) &= \min_{q^e, q^r \geq 0} \{c^e q^e + c^r q^r + p(z^0 + q^e) + \alpha E_D f_n(z^0 + q^e - D + z^1, z^2, \dots, z^{l-1}, q^r)\} \\ &\geq \min_{q^e, q^r \geq 0} \{c^e q^e + c^r q^r + p(z^0 + q^e) + \alpha E_D f_{n-1}(z^0 + q^e + z^1 - D, z^2, \dots, z^{l-1}, q^r)\} \\ &= f_n(\mathbf{z}). \end{aligned} \quad (\text{A.4})$$

That is, $\{f_n(\mathbf{z})\}_n$ is an ascending functional sequence in n for any $\mathbf{z}$.

Denote by $(y_n(\mathbf{z}) + z^1 - D, \tilde{\mathbf{z}}_+) = (z^0 + \hat{q}_n^e(\mathbf{z}) + z^1 - D, z^2, \dots, z^{l-1}, \hat{q}_n^r(\mathbf{z}))$ the state in the next period.

Recall that $\hat{q}_n^r(\mathbf{z})$ and $\hat{q}_n^e(\mathbf{z})$ are optimal solutions to $f_n(\mathbf{z})$. It is clear that $\hat{q}_n^r(\mathbf{z})$ and $\hat{q}_n^e(\mathbf{z})$ are also feasible to $f_{n+1}(\mathbf{z})$. So,

$$f_{n+1}(\mathbf{z}) \leq c^e \hat{q}_n^e(\mathbf{z}) + c^r \hat{q}_n^r(\mathbf{z}) + p(z^0 + \hat{q}_n^e(\mathbf{z})) + \alpha E_D f_n[y_n(\mathbf{z}) + z^1 - D, \tilde{\mathbf{z}}_+]. \quad (\text{A.5})$$

Equations (A.4) and (A.5) indicates that

$$0 \leq f_{n+1}(\mathbf{z}) - f_n(\mathbf{z}) \leq \alpha E_D \{f_n[y_n(\mathbf{z}) + z^1 - D, \tilde{\mathbf{z}}_+] - f_{n-1}[y_n(\mathbf{z}) + z^1 - D, \tilde{\mathbf{z}}_+]\}. \quad (\text{A.6})$$

When the demand is finite, it is easy to verify that $y_n(\mathbf{z})$ and all elements of $\tilde{\mathbf{z}}_+$ are finite over the entire planning horizon. Denote by Y and Q the upper bounds on $y_n(\mathbf{z})$ and all elements of $\tilde{\mathbf{z}}_+$ for all n , respectively. Let $\Xi = \{(z^1, \dots, z^{l-1}) : 0 \leq z^i \leq Q, i = 1, \dots, l-1\}$ be the space of the outstanding orders.

Recall that any order from each of the two suppliers is nonnegative, and any demand realization over the planning horizon is nonnegative. For all $x \in [-M, M]$ (M is any positive number), we have

$$y_n(\mathbf{z}) + z^1 - D \in [-M - d_{\max}, M' + Q], \quad (\text{A.7})$$

where $M' = \max(M, Y)$. From equations (A.6) and (A.7), we have

$$\begin{aligned} & \max_{[-M, M] \times \Xi} |f_{n+1}(\mathbf{z}) - f_n(\mathbf{z})| \\ & \leq \alpha \max_{[-M, M] \times \Xi} E_D \{f_n[y_n(\mathbf{z}) + z^1 - D, \tilde{\mathbf{z}}_+] - f_{n-1}[y_n(\mathbf{z}) + z^1 - D, \tilde{\mathbf{z}}_+]\} \\ & \leq \alpha \max_{[-M - d_{\max}, M' + Q] \times \Xi} |f_n(\mathbf{z}) - f_{n-1}(\mathbf{z})| \\ & \leq \dots \leq \alpha^n \max_{[-M - nd_{\max}, M' + nQ] \times \Xi} f_1(\mathbf{z}) \leq \alpha^n \max_{[-M - nd_{\max}, M' + nQ] \times \Xi} p(z^0). \end{aligned} \quad (\text{A.8})$$

Note that the last inequality holds because ordering nothing from two suppliers is a feasible solution in the last period.

From Federgruen and Zipkin (1986), there are positive constants A and B such that $p(z^0) \leq A + B|z^0|$ for all z^0 . So, equation (A.8) implies

$$\max_{[-M, M] \times \Xi} |f_{n+1}(x, \mathbf{z}) - f_n(x, \mathbf{z})| \leq \alpha^n \max\{A + B(M + nd_{\max}), A + B(M' + nQ)\}.$$

when $\alpha < 1$,

$$\lim_{n \rightarrow \infty} \frac{\alpha^{n+1} \max\{A + B[M + (n+1)d_{\max}], A + B[M' + (n+1)Q]\}}{\alpha^n \max\{A + B(M + nd_{\max}), A + B(M' + nQ)\}} = \alpha < 1.$$

Hence, $\sum_{n=1}^{\infty} [f_{n+1}(\mathbf{z}) - f_n(\mathbf{z})]$ converges absolutely and uniformly for any $\mathbf{z} \in [-M, M] \times \Xi$. It follows that $f_n(\mathbf{z})$ converges monotonically and uniformly for all $\mathbf{z} \in [-M, M] \times \Xi$.

Denote by $f(\mathbf{z})$ the convergence of $\{f_n(\mathbf{z})\}_n$. It is easy to verify that

$$f(\mathbf{z}) = \min_{q^e, q^r \geq 0} \{c^e q^e + c^r q^r + p(z^0 + q^e) + \alpha E_D f(z^0 + q^e - D + z^1, z^2, \dots, z^{l-1}, q^r)\}.$$

That is, the convergence of the optimal finite-horizon cost function satisfies the function equation (3).

Consequently, there exist some convergent subsequences in the optimal finite-horizon policies, and the limit of any convergent subsequence is optimal for the infinite horizon discounted cost problem. In addition, it is easy to verify that the stationary function $f(\mathbf{z})$ is also $L^{\natural}$ -convex, so the optimal orders in the infinite horizon discounted cost problem have the same structural properties as shown in Propositions 1 and 2.

Proof of Proposition 4. We first consider the finite horizon discounted cost problem. Given the state $\ddot{\mathbf{z}} = (OV, z^1, \dots, z^{l-1})$, the state in the next period is $((OV + z^1 - D)^+, z^2, \dots, z^{l-1}, q^r)$. If the demand equals 0, then the overshoot in the next period is $OV + z^1$. Therefore, the expedited order in the next period under a given demand D is $D - \min(OV + z^1, D)$, where $\min(OV + z^1, D)$ is the part of demand covered by $OV + z^1$. It follows that the finite horizon discounted model is

$$\begin{aligned} \ddot{f}_n(\ddot{\mathbf{z}}) = \min_{q^r \geq 0} \{ & \alpha E_D c^e [D - \min(OV + z^1, D)] + c^r q^r + p(S^e + OV) \\ & + \alpha E_D \ddot{f}_{n-1}[(OV + z^1 - D)^+, z^2, \dots, z^{l-1}, q^r] \}, \end{aligned} \quad (\text{A.9})$$

where the first term is the expected ordering cost of expedited orders in the next period. Let $\ddot{\mathbf{v}} = (\ddot{v}^0, \dots, \ddot{v}^{l-1})$. Under the new state $\ddot{\mathbf{v}}$, Model (A.9) can be rewritten as:

$$\begin{aligned} \ddot{f}_n^v(\ddot{\mathbf{v}}) = \min_{q^r \geq 0} \{ & \alpha E_D c^e [D - \min(\ddot{v}^0 - \ddot{v}^2, D)] + c^r q^r + p(S^e + \ddot{v}^0 - \ddot{v}^1) \\ & + \alpha E_D \ddot{f}_{n-1}^v[(\ddot{v}^0 - \ddot{v}^2 - D)^+ + \ddot{v}^2, \ddot{v}^2, \dots, \ddot{v}^{l-1}, 0) + q^r \mathbf{e}] \}. \end{aligned} \quad (\text{A.10})$$

The $L^{\natural}$ -convexity of $\ddot{f}_n^v(\ddot{\mathbf{v}})$ for all n can be proved by induction. It is clear that $\ddot{f}_0^v(\ddot{\mathbf{v}}) \equiv 0$ is $L^{\natural}$ -convex. Suppose that $\ddot{f}_{n-1}^v(\ddot{\mathbf{v}})$ is $L^{\natural}$ -convex. We next prove that $\ddot{f}_n^v(\ddot{\mathbf{v}})$ is $L^{\natural}$ -convex.

Reformulate Model (A.10) with two optimization steps. The first step is that, after knowing the demand, to decide how much demand will be covered by $OV + z^1 = \ddot{v}^0 - \ddot{v}^2$. Let a be this decision, and

ε be the remaining part of $\ddot{v}^0 - \ddot{v}^2$. The problem is

$$\begin{aligned} \phi_n^v(\ddot{\mathbf{v}}, \zeta | D) = \min_{a, \varepsilon} \{ & \alpha c^e(D - a) + h(S^e + \ddot{v}^0 - \ddot{v}^1 - D)^+ + b(D - S^e - \ddot{v}^0 + \ddot{v}^1)^+ \\ & + \alpha \ddot{f}_{n-1}[(\varepsilon + \ddot{v}^2, \ddot{v}^2, \dots, \ddot{v}^{l-1}, 0) - \zeta \mathbf{e}] : a + \varepsilon = \ddot{v}^0 - \ddot{v}^2, 0 \leq a \leq D, \varepsilon \geq 0 \}. \end{aligned} \quad (\text{A.11})$$

The second step is $\ddot{f}_n^v(\ddot{\mathbf{v}}) = \min_{\zeta \leq 0} \{-c^r \zeta + E_D \phi_n^v(\ddot{\mathbf{v}}, \zeta | D)\}$. For the first step, it is not hard to show that

$a = \min(\ddot{v}^0 - \ddot{v}^2, D)$. Further, set $\ddot{v}_+^2 = \varepsilon + \ddot{v}^2$, and use $a + \varepsilon = \ddot{v}^0 - \ddot{v}^2$ to eliminate a . Then, equation

(A.11) can be rewritten as:

$$\begin{aligned} \phi_n^v(\ddot{\mathbf{v}}, \zeta | D) = \min_{\ddot{v}_+^2} \{ & \phi_n^v(\ddot{\mathbf{v}}, \ddot{v}_+^2, \zeta) = \alpha c^e(D - \ddot{v}^0 + \ddot{v}_+^2) + h(S^e + \ddot{v}^0 - \ddot{v}^1 - D)^+ + b(D - S^e - \ddot{v}^0 + \ddot{v}^1)^+ \\ & + \alpha \ddot{f}_{n-1}[(\ddot{v}_+^2, \ddot{v}^2, \dots, \ddot{v}^{l-1}, 0) - \zeta \mathbf{e}] : -D \leq \ddot{v}_+^2 - \ddot{v}^0 \leq 0, \ddot{v}_+^2 - \ddot{v}^2 \geq 0 \}. \end{aligned}$$

It is easy to verify that $\{(\ddot{\mathbf{v}}, \ddot{v}_+^2, \zeta) : \ddot{v}^0 \geq \ddot{v}^1 \geq \dots \geq \ddot{v}^{l-1}, -D \leq \ddot{v}_+^2 - \ddot{v}^0 \leq 0, \ddot{v}_+^2 - \ddot{v}^2 \geq 0, \zeta \leq 0\}$ is a lattice.

Since $\ddot{f}_{n-1}^v(\ddot{\mathbf{v}})$ is $L^\natural$ -convex, $\ddot{f}_{n-1}[(\ddot{v}_+^2, \ddot{v}^2, \dots, \ddot{v}^{l-1}, 0) - \zeta \mathbf{e}]$ is submodular in $(\ddot{v}_+^2, \ddot{v}^2, \dots, \ddot{v}^{l-1}, \zeta)$. From

Lemma 2.6.2 of Topkis (1998), we have that $\alpha c^e(D - \ddot{v}^0 + \ddot{v}_+^2)$, $h(S^e + \ddot{v}^0 - \ddot{v}^1 - D)^+$, and

$b(D - S^e - \ddot{v}^0 + \ddot{v}^1)^+$ are submodular in $(\ddot{v}^0, \ddot{v}_+^2)$, $(\ddot{v}^0, \ddot{v}^1)$, and $(\ddot{v}^0, \ddot{v}^1)$, respectively. Consequently,

$\phi_n^v(\ddot{\mathbf{v}}, \ddot{v}_+^2, \zeta)$ is submodular in $(\ddot{\mathbf{v}}, \ddot{v}_+^2, \zeta)$, and is also $L^\natural$ -convex from Lemma 1 of Zipkin (2008b). It

follows that both $\phi_n^v(\ddot{\mathbf{v}}, \zeta | D)$ and $\ddot{f}_n^v(\ddot{\mathbf{v}})$ are $L^\natural$ -convex from Lemma 2 of Zipkin (2008b).

Based on the $L^\natural$ -convexity of $\ddot{f}_n^v(\ddot{\mathbf{v}})$, similar to the proof of Proposition 1, we can prove that

Proposition 4 holds for the finite horizon discounted cost problem. Similar to the proof of Proposition 3,

we can also show that the finite-horizon cost function converges absolutely and uniformly to the

stationary cost function in the infinite horizon discounted cost problem. Consequently, Proposition 4 also

holds for the infinite horizon discounted cost problem. As the discount factor approaches 1, Proposition 4

also holds for the infinite horizon average cost problem.