Statistical Analysis of Parameter Estimation for 2-D Harmonics in Multiplicative and Additive Noise

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Supplementary Data

(A) The expression of $Q(\boldsymbol{\theta})$ in the proof of Theorem 2.

$$\begin{aligned}
 Q(\boldsymbol{\theta}) = & \sum_{m=1}^M \sum_{n=1}^N \left\{ \sum_{k=1}^p \left[\mu_{sk}^0 \cos(w_{mn}(\boldsymbol{\theta}_k^0)) - \mu_{sk} \cos(w_{mn}(\boldsymbol{\theta}_k)) \right] \right\}^2 \\
 & + \sum_{m=1}^M \sum_{n=1}^N \left\{ \sum_{k=1}^p \left[\mu_{sk}^0 \sin(w_{mn}(\boldsymbol{\theta}_k^0)) - \mu_{sk} \sin(w_{mn}(\boldsymbol{\theta}_k)) \right] \right\}^2 \\
 & + 2 \sum_{m=1}^M \sum_{n=1}^N \sum_{k=1}^p x_r(m, n) \left[\mu_{sk}^0 \cos(w_{mn}(\boldsymbol{\theta}_k^0)) - \mu_{sk} \cos(w_{mn}(\boldsymbol{\theta}_k)) \right] \\
 & + 2 \sum_{m=1}^M \sum_{n=1}^N \sum_{k=1}^p x_c(m, n) \left[\mu_{sk}^0 \sin(w_{mn}(\boldsymbol{\theta}_k^0)) - \mu_{sk} \sin(w_{mn}(\boldsymbol{\theta}_k)) \right] \\
 & + \sum_{m=1}^M \sum_{n=1}^N |x(m, n)|^2
 \end{aligned}$$

where

$$\begin{aligned}
 x(m, n) &= e(m, n) + \sum_{k=1}^p (s_k(m, n) - \mu_{sk}^0) e^{iw_{mn}(\boldsymbol{\theta}_k^0)} \\
 w_{mn}(\boldsymbol{\theta}_k) &= m\lambda_k + n\mu_k + \phi_k
 \end{aligned}$$

(B) The expressions of the elements of $Q'(\boldsymbol{\theta}^0)$ in the proof of Theorem 2.

$$\begin{aligned}
 Q'(\boldsymbol{\theta}^0) &= (Q'_k(\boldsymbol{\theta}^0))_{k=1,2,\dots,p} = (Q'_1(\boldsymbol{\theta}^0), Q'_2(\boldsymbol{\theta}^0), \dots, Q'_p(\boldsymbol{\theta}^0)) \\
 Q'_k(\boldsymbol{\theta}^0) &= \frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \boldsymbol{\theta}_k} = \left(\frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \mu_{sk}}, \frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \phi_k}, \frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \lambda_k}, \frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \mu_k} \right) \\
 \frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \mu_{sk}} &= 2 \sum_{m=1}^M \sum_{n=1}^N \left[-x_r(m, n) \cos(w_{mn}(\boldsymbol{\theta}_k^0)) - x_c(m, n) \sin(w_{mn}(\boldsymbol{\theta}_k^0)) \right] \\
 \frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \phi_k} &= 2 \sum_{m=1}^M \sum_{n=1}^N \mu_{sk}^0 \left[x_r(m, n) \sin(w_{mn}(\boldsymbol{\theta}_k^0)) - x_c(m, n) \cos(w_{mn}(\boldsymbol{\theta}_k^0)) \right] \\
 \frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \lambda_k} &= 2 \sum_{m=1}^M \sum_{n=1}^N m \mu_{sk}^0 \left[x_r(m, n) \sin(w_{mn}(\boldsymbol{\theta}_k^0)) - x_c(m, n) \cos(w_{mn}(\boldsymbol{\theta}_k^0)) \right] \\
 \frac{\partial Q(\boldsymbol{\theta}^0)}{\partial \mu_k} &= 2 \sum_{m=1}^M \sum_{n=1}^N n \mu_{sk}^0 \left[x_r(m, n) \sin(w_{mn}(\boldsymbol{\theta}_k^0)) - x_c(m, n) \cos(w_{mn}(\boldsymbol{\theta}_k^0)) \right]
 \end{aligned}$$

(C) The expressions of the elements of $Q''(\boldsymbol{\theta}^0)$ in the proof of Theorem 2.

$$Q''(\boldsymbol{\theta}^0) = \left(Q''_{kl}(\boldsymbol{\theta}^0) \right)_{k,l=1,2,\dots,p} = \begin{bmatrix} Q''_{11}(\boldsymbol{\theta}^0) & Q''_{12}(\boldsymbol{\theta}^0) & \dots & Q''_{1p}(\boldsymbol{\theta}^0) \\ Q''_{21}(\boldsymbol{\theta}^0) & Q''_{22}(\boldsymbol{\theta}^0) & \dots & Q''_{2p}(\boldsymbol{\theta}^0) \\ \vdots & \vdots & \ddots & \vdots \\ Q''_{p1}(\boldsymbol{\theta}^0) & Q''_{p2}(\boldsymbol{\theta}^0) & \dots & Q''_{pp}(\boldsymbol{\theta}^0) \end{bmatrix}$$

$$Q''_{kl}(\boldsymbol{\theta}^0) = \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \boldsymbol{\theta}_k \partial \boldsymbol{\theta}_l^T} = \begin{bmatrix} \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_{sk} \partial \mu_{sl}} & \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_{sk} \partial \phi_l} & \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_{sk} \partial \lambda_l} & \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_{sk} \partial \mu_l} \\ \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \phi_k \partial \mu_{sl}} & \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \phi_k \partial \phi_l} & \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \phi_k \partial \lambda_l} & \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \phi_k \partial \mu_l} \\ \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \lambda_k \partial \mu_{sl}} & \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \lambda_k \partial \phi_l} & \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \lambda_k \partial \lambda_l} & \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \lambda_k \partial \mu_l} \\ \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_k \partial \mu_{sl}} & \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_k \partial \phi_l} & \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_k \partial \lambda_l} & \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_k \partial \mu_l} \end{bmatrix}$$

(C1) The expression of the elements of $Q''_{kk}(\boldsymbol{\theta}^0)$, for $k = 1, 2, \dots, p$.

$$\begin{aligned} \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_{sk}^2} &= 2MN \\ \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \phi_k^2} &= 2 \sum_{m=1}^M \sum_{n=1}^N \mu_{sk}^0 \left[x_r(m, n) \cos(w_{mn}(\boldsymbol{\theta}_k^0)) + x_c(m, n) \sin(w_{mn}(\boldsymbol{\theta}_k^0)) \right] + 2MN(\mu_{sk}^0)^2 \\ \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \lambda_k^2} &= 2 \sum_{m=1}^M \sum_{n=1}^N m^2 \mu_{sk}^0 \left[x_r(m, n) \cos(w_{mn}(\boldsymbol{\theta}_k^0)) + x_c(m, n) \sin(w_{mn}(\boldsymbol{\theta}_k^0)) \right] + 2N(\mu_{sk}^0)^2 \sum_{m=1}^M m^2 \\ \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_k^2} &= 2 \sum_{m=1}^M \sum_{n=1}^N n^2 \mu_{sk}^0 \left[x_r(m, n) \cos(w_{mn}(\boldsymbol{\theta}_k^0)) + x_c(m, n) \sin(w_{mn}(\boldsymbol{\theta}_k^0)) \right] + 2M(\mu_{sk}^0)^2 \sum_{n=1}^N n^2 \\ \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_{sk} \partial \phi_k} &= 2 \sum_{m=1}^M \sum_{n=1}^N \left[x_r(m, n) \sin(w_{mn}(\boldsymbol{\theta}_k^0)) - x_c(m, n) \cos(w_{mn}(\boldsymbol{\theta}_k^0)) \right] \\ \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_{sk} \partial \lambda_k} &= 2 \sum_{m=1}^M \sum_{n=1}^N m \left[x_r(m, n) \sin(w_{mn}(\boldsymbol{\theta}_k^0)) - x_c(m, n) \cos(w_{mn}(\boldsymbol{\theta}_k^0)) \right] \\ \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_{sk} \partial \mu_k} &= 2 \sum_{m=1}^M \sum_{n=1}^N n \left[x_r(m, n) \sin(w_{mn}(\boldsymbol{\theta}_k^0)) - x_c(m, n) \cos(w_{mn}(\boldsymbol{\theta}_k^0)) \right] \\ \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \phi_k \partial \lambda_k} &= 2 \sum_{m=1}^M \sum_{n=1}^N m \mu_{sk}^0 \left[x_r(m, n) \cos(w_{mn}(\boldsymbol{\theta}_k^0)) + x_c(m, n) \sin(w_{mn}(\boldsymbol{\theta}_k^0)) \right] + 2N(\mu_{sk}^0)^2 \sum_{m=1}^M m \\ \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \phi_k \partial \mu_k} &= 2 \sum_{m=1}^M \sum_{n=1}^N n \mu_{sk}^0 \left[x_r(m, n) \cos(w_{mn}(\boldsymbol{\theta}_k^0)) + x_c(m, n) \sin(w_{mn}(\boldsymbol{\theta}_k^0)) \right] + 2M(\mu_{sk}^0)^2 \sum_{n=1}^N n \\ \frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \lambda_k \partial \mu_k} &= 2 \sum_{m=1}^M \sum_{n=1}^N mn \mu_{sk}^0 \left[x_r(m, n) \cos(w_{mn}(\boldsymbol{\theta}_k^0)) + x_c(m, n) \sin(w_{mn}(\boldsymbol{\theta}_k^0)) \right] + 2(\mu_{sk}^0)^2 \sum_{m=1}^M \sum_{n=1}^N mn \end{aligned}$$

(C2) The expressions of the elements of $Q''_{kl}(\boldsymbol{\theta}^0)$, for $k, l = 1, 2, \dots, p$ and $k \neq l$.

$$\begin{aligned}
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_{sk} \partial \mu_{sl}} &= 2 \sum_{m=1}^M \sum_{n=1}^N \cos(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)) \\
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_{sk} \partial \phi_l} &= -2\mu_{sl}^0 \sum_{m=1}^M \sum_{n=1}^N \sin(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)) \\
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_{sk} \partial \lambda_l} &= -2\mu_{sl}^0 \sum_{m=1}^M \sum_{n=1}^N m \sin(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)) \\
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_{sk} \partial \mu_l} &= -2\mu_{sl}^0 \sum_{m=1}^M \sum_{n=1}^N n \sin(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)) \\
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \phi_k \partial \mu_{sl}} &= 2\mu_{sk}^0 \mu_{sl}^0 \sum_{m=1}^M \sum_{n=1}^N \sin(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)) \\
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \phi_k \partial \phi_l} &= 2\mu_{sk}^0 \mu_{sl}^0 \sum_{m=1}^M \sum_{n=1}^N \cos(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)) \\
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \phi_k \partial \lambda_l} &= 2\mu_{sk}^0 \mu_{sl}^0 \sum_{m=1}^M \sum_{n=1}^N m \cos(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)) \\
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \phi_k \partial \mu_l} &= 2\mu_{sk}^0 \mu_{sl}^0 \sum_{m=1}^M \sum_{n=1}^N n \cos(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)) \\
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \lambda_k \partial \mu_{sl}} &= 2\mu_{sk}^0 \sum_{m=1}^M \sum_{n=1}^N m \sin(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)) \\
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \lambda_k \partial \phi_l} &= 2\mu_{sk}^0 \mu_{sl}^0 \sum_{m=1}^M \sum_{n=1}^N m \cos(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)) \\
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \lambda_k \partial \lambda_l} &= 2\mu_{sk}^0 \mu_{sl}^0 \sum_{m=1}^M \sum_{n=1}^N m^2 \cos(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)) \\
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \lambda_k \partial \mu_l} &= 2\mu_{sk}^0 \mu_{sl}^0 \sum_{m=1}^M \sum_{n=1}^N mn \cos(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)) \\
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_k \partial \mu_{sl}} &= 2\mu_{sk}^0 \sum_{m=1}^M \sum_{n=1}^N n \sin(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)) \\
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_k \partial \phi_l} &= 2\mu_{sk}^0 \mu_{sl}^0 \sum_{m=1}^M \sum_{n=1}^N n \cos(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)) \\
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_k \partial \lambda_l} &= 2\mu_{sk}^0 \mu_{sl}^0 \sum_{m=1}^M \sum_{n=1}^N mn \cos(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)) \\
\frac{\partial^2 Q(\boldsymbol{\theta}^0)}{\partial \mu_k \partial \mu_l} &= 2\mu_{sk}^0 \mu_{sl}^0 \sum_{m=1}^M \sum_{n=1}^N n^2 \cos(w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\boldsymbol{\theta}_l^0)).
\end{aligned}$$

(D) The expression of the remainder $R_{MN}(\hat{\boldsymbol{\theta}}_k, \boldsymbol{\theta}^0)$ in the proof of Theorem 4.

$$\begin{aligned}
R_{MN}(\hat{\boldsymbol{\theta}}_k, \boldsymbol{\theta}^0) &= \frac{1}{MN} \sum_{m=1}^M \sum_{n=1}^N s_k^2(m, n) \left[\cos \left(2w_{mn}(\boldsymbol{\theta}_k^0) - 2w_{mn}(\hat{\boldsymbol{\theta}}_k) \right) - 1 \right] \\
&+ \frac{1}{MN} \sum_{m=1}^M \sum_{n=1}^N \sum_{j \neq k} s_j^2(m, n) \cos \left(2w_{mn}(\boldsymbol{\theta}_j^0) - 2w_{mn}(\hat{\boldsymbol{\theta}}_k) \right) \\
&+ \frac{2}{MN} \sum_{m=1}^M \sum_{n=1}^N \sum_{j=1}^p e_r(m, n) s_j(m, n) \cos \left(w_{mn}(\boldsymbol{\theta}_j^0) - 2w_{mn}(\hat{\boldsymbol{\theta}}_k) \right) \\
&+ \frac{2}{MN} \sum_{m=1}^M \sum_{n=1}^N \sum_{j=1}^p e_c(m, n) s_j(m, n) \sin \left(2w_{mn}(\hat{\boldsymbol{\theta}}_k) - w_{mn}(\boldsymbol{\theta}_j^0) \right) \\
&+ \frac{1}{MN} \sum_{m=1}^M \sum_{n=1}^N \sum_{i \neq j} s_i(m, n) s_j(m, n) \cos \left(w_{mn}(\boldsymbol{\theta}_i^0) + w_{mn}(\boldsymbol{\theta}_j^0) - 2w_{mn}(\hat{\boldsymbol{\theta}}_k) \right) \\
&+ \frac{1}{MN} \sum_{m=1}^M \sum_{n=1}^N \left[e_r^2(m, n) - e_c^2(m, n) \right] \cos \left(2w_{mn}(\hat{\boldsymbol{\theta}}_k) \right) \\
&+ \frac{2}{MN} \sum_{m=1}^M \sum_{n=1}^N e_r(m, n) e_c(m, n) \sin \left(-2w_{mn}(\hat{\boldsymbol{\theta}}_k) \right)
\end{aligned}$$

(E) The proof of " $f_{MN}(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}^0) \rightarrow 0$ a.s." in the proof of Theorem 5.

$$\begin{aligned}
&\frac{1}{MN} \sum_{m=1}^M \sum_{n=1}^N \left\{ \sum_{k=1}^p \left[\mu_{sk}^0 \cos \left(w_{mn}(\boldsymbol{\theta}_k^0) \right) - \hat{\mu}_{sk} \cos \left(w_{mn}(\hat{\boldsymbol{\theta}}_k) \right) \right] \right\}^2 \\
&\leq \frac{p}{MN} \sum_{k=1}^p \sum_{m=1}^M \sum_{n=1}^N \left[\mu_{sk}^0 \cos \left(w_{mn}(\boldsymbol{\theta}_k^0) \right) - \hat{\mu}_{sk} \cos \left(w_{mn}(\hat{\boldsymbol{\theta}}_k) \right) \right]^2 \\
&\leq \frac{2p}{MN} \sum_{k=1}^p \sum_{m=1}^M \sum_{n=1}^N \left\{ (\mu_{sk}^0 - \hat{\mu}_{sk})^2 \cos^2 \left(w_{mn}(\boldsymbol{\theta}_k^0) \right) + \hat{\mu}_{sk}^2 \left[\cos \left(w_{mn}(\boldsymbol{\theta}_k^0) \right) - \cos \left(w_{mn}(\hat{\boldsymbol{\theta}}_k) \right) \right]^2 \right\} \\
&\leq 2p \sum_{k=1}^p (\mu_{sk}^0 - \hat{\mu}_{sk})^2 + \frac{2p}{MN} \sum_{k=1}^p \sum_{m=1}^M \sum_{n=1}^N \hat{\mu}_{sk}^2 \left[w_{mn}(\boldsymbol{\theta}_k^0) - w_{mn}(\hat{\boldsymbol{\theta}}_k) \right]^2 \\
&\rightarrow 0 \text{ a.s.}
\end{aligned}$$

where the first inequality follows from Cauchy-Schwarz inequality, the second inequality follows from the fact $2x^2 + 2y^2 - (x + y)^2 \geq 0$, the third inequality follows from Taylor series expansion and the last step follows from the results in Theorems 1 and 2. Along the same line, it also can be shown that

$$\frac{1}{MN} \sum_{m=1}^M \sum_{n=1}^N \left\{ \sum_{k=1}^p \left[\mu_{sk}^0 \sin \left(w_{mn}(\boldsymbol{\theta}_k^0) \right) - \hat{\mu}_{sk} \sin \left(w_{mn}(\hat{\boldsymbol{\theta}}_k) \right) \right] \right\}^2 \rightarrow 0 \text{ a.s.}$$

Therefore, it can be obtained from (A.4) in the proof of Theorem 1 that $f_{MN}(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta}^0) \rightarrow 0$ a.s.