Statistical Analysis of Non-Linear Least Squares Estimation for Harmonic Signals in Multiplicative and Additive Noise

Huiming Peng, Shaoquan Yu, Jiawen Bian, Yujie Zhang, Hongwei Li*

School of Mathematics and Physics, China University of Geosciences, Wuhan 430074, China

(*Corresponding Author: hwli@cug.edu.cn)

Supplementary Data

(A) Simulation results in case of zero-mean multiplicative noise

Table S1. The results of the frequencies

Sample size	50	100	150	200	250	300
ω_1^0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
AE	1.0088	1.0024	1.0011	1.0006	1.0000	1.0000
MSE	3.841e-4	4.630e-5	1.801e-5	5.866e-6	2.783e-6	1.578e-6
AVAR	3.120e-4	3.900e-5	1.156e-5	4.875e-6	2.496e-6	1.444e-6
Length	0.0579	0.0205	0.0111	0.0072	0.0052	0.0039
Coverage	0.8760	0.8600	0.8720	0.8820	0.8780	0.8860
ω_2^0	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
AE	1.9935	1.9984	1.9992	1.9998	2.0000	2.0000
MSE	7.152e-5	9.013e-6	3.853e-6	1.451e-6	4.995e-7	2.975e-7
AVAR	6.000e-5	7.500e-6	2.222e-6	9.375e-7	4.800e-7	2.778e-7
Length	0.0254	0.0090	0.0049	0.0032	0.0023	0.0017
Coverage	0.8660	0.8680	0.8900	0.8920	0.8840	0.8800

Table S2. The results of the phases

Sample size	50	100	150	200	250	300
ϕ_1^0	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
AE	2.0511	2.0332	2.0222	2.0167	2.0087	2.0014
MSE	0.2951	0.1465	0.0936	0.0693	0.0556	0.0445
AVAR	0.2600	0.1300	0.0867	0.0650	0.0520	0.0433
Length	1.6725	1.1826	0.9656	0.8362	0.7480	0.6828
Coverage	0.8760	0.8680	0.8860	0.8840	0.8760	0.8960
ϕ_2^0	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000
AE	2.9752	2.9815	2.9895	2.9910	2.9961	2.9992
MSE	0.0584	0.0297	0.0183	0.0136	0.0109	0.0088
AVAR	0.0500	0.0250	0.0167	0.0125	0.0100	0.0083
Length	0.7334	0.5186	0.4234	0.3667	0.3280	0.2994
Coverage	0.8820	0.8720	0.8940	0.8640	0.8900	0.8880

Table S3. The results of the noise variances

Sample size	50	100	150	200	250	300
σ_{s1}^2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
AE	1.1051	1.0774	1.0432	1.0230	1.0086	1.0041
MSE	0.3211	0.1613	0.1089	0.0795	0.0645	0.0526
AVAR	0.3000	0.1500	0.1000	0.0750	0.0600	0.0500
Length	1.7965	1.2703	1.0372	0.8983	0.8034	0.7334
Coverage	0.9020	0.8760	0.9120	0.8800	0.8940	0.8880
σ_{s2}^2	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000
AE	2.1112	2.0868	2.0657	2.0233	2.0095	2.0049
MSE	0.3876	0.1935	0.1298	0.0973	0.0761	0.0629
AVAR	0.3600	0.1800	0.1200	0.0900	0.0720	0.0600
Length	1.9680	1.3916	1.1362	0.9840	0.8801	0.8034
Coverage	0.9100	0.8580	0.8920	0.8960	0.9060	0.8980
σ^2	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000
AE	4.0985	4.0589	4.0247	4.0113	4.0062	4.0032
MSE	0.4537	0.2235	0.1479	0.1112	0.0883	0.0727
AVAR	0.4200	0.2100	0.1400	0.1050	0.0840	0.0700
Length	2.1257	1.5031	1.2273	1.0628	0.9506	0.8678
Coverage	0.9080	0.8940	0.9160	0.9100	0.8960	0.9080
σ_0^2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
AE	0.8822	0.8947	0.9158	0.9650	0.9881	0.9942
MSE	1.1624	0.5783	0.3866	0.2880	0.2289	0.1882
AVAR	1.0800	0.5400	0.3600	0.2700	0.2160	0.1800
Length	3.4087	2.4103	1.9680	1.7043	1.5244	1.3916
Coverage	0.9320	0.9260	0.9140	0.9120	0.9180	0.9080

(B) Some expressions in the proofs of Theorems 1-6
in case of nonzero-mean multiplicative noise

B1. The expression of $Q(\theta)$.

$$\begin{aligned}
Q(\theta) = & \sum_{t=1}^T \left\{ \sum_{k=1}^M \left[\mu_{sk}^0 \cos(\omega_k^0 t + \phi_k^0) - \mu_{sk} \cos(\omega_k t + \phi_k) \right] \right\}^2 \\
& + \sum_{t=1}^T \left\{ \sum_{k=1}^M \left[\mu_{sk}^0 \sin(\omega_k^0 t + \phi_k^0) - \mu_{sk} \sin(\omega_k t + \phi_k) \right] \right\}^2 \\
& + 2 \sum_{t=1}^T \sum_{k=1}^M \varepsilon_r(t) \left[\mu_{sk}^0 \cos(\omega_k^0 t + \phi_k^0) - \mu_{sk} \cos(\omega_k t + \phi_k) \right] \\
& + 2 \sum_{t=1}^T \sum_{k=1}^M \varepsilon_c(t) \left[\mu_{sk}^0 \sin(\omega_k^0 t + \phi_k^0) - \mu_{sk} \sin(\omega_k t + \phi_k) \right] \\
& + \sum_{t=1}^T |\varepsilon(t)|^2
\end{aligned}$$

where $\varepsilon(t) = \sum_{k=1}^M (s_k(t) - \mu_{sk}^0) e^{j(\omega_k^0 t + \phi_k^0)} + n(t)$.

B2. The elements of $Q'_k(\theta^0)$, for $k = 1, 2, \dots, M$.

$$\begin{aligned}
\frac{\partial Q(\theta^0)}{\partial \omega_k} &= 2 \sum_{t=1}^T t \mu_{sk}^0 \left[\varepsilon_r(t) \sin(\omega_k^0 t + \phi_k^0) - \varepsilon_c(t) \cos(\omega_k^0 t + \phi_k^0) \right] \\
\frac{\partial Q(\theta^0)}{\partial \phi_k} &= 2 \sum_{t=1}^T \mu_{sk}^0 \left[\varepsilon_r(t) \sin(\omega_k^0 t + \phi_k^0) - \varepsilon_c(t) \cos(\omega_k^0 t + \phi_k^0) \right] \\
\frac{\partial Q(\theta^0)}{\partial \mu_{sk}} &= 2 \sum_{t=1}^T \left[-\varepsilon_r(t) \cos(\omega_k^0 t + \phi_k^0) - \varepsilon_c(t) \sin(\omega_k^0 t + \phi_k^0) \right]
\end{aligned}$$

B3. The elements of $Q''_{kk}(\theta^0)$, for $k = 1, 2, \dots, M$.

$$\begin{aligned}
\frac{\partial^2 Q(\theta^0)}{\partial \omega_k^2} &= 2 \sum_{t=1}^T t^2 \mu_{sk}^0 \left[\varepsilon_r(t) \cos(\omega_k^0 t + \phi_k^0) + \varepsilon_c(t) \sin(\omega_k^0 t + \phi_k^0) \right] + 2(\mu_{sk}^0)^2 \sum_{t=1}^T t^2 \\
\frac{\partial^2 Q(\theta^0)}{\partial \phi_k^2} &= 2 \sum_{t=1}^T \mu_{sk}^0 \left[\varepsilon_r(t) \cos(\omega_k^0 t + \phi_k^0) + \varepsilon_c(t) \sin(\omega_k^0 t + \phi_k^0) \right] + 2T(\mu_{sk}^0)^2 \\
\frac{\partial^2 Q(\theta^0)}{\partial \mu_{sk}^2} &= 2T \\
\frac{\partial^2 Q(\theta^0)}{\partial \omega_k \partial \phi_k} &= 2 \sum_{t=1}^T t \mu_{sk}^0 \left[\varepsilon_r(t) \cos(\omega_k^0 t + \phi_k^0) + \varepsilon_c(t) \sin(\omega_k^0 t + \phi_k^0) \right] + 2(\mu_{sk}^0)^2 \sum_{t=1}^T t \\
\frac{\partial^2 Q(\theta^0)}{\partial \omega_k \partial \mu_{sk}} &= 2 \sum_{t=1}^T t \left[\varepsilon_r(t) \sin(\omega_k^0 t + \phi_k^0) - \varepsilon_c(t) \cos(\omega_k^0 t + \phi_k^0) \right] \\
\frac{\partial^2 Q(\theta^0)}{\partial \phi_k \partial \mu_{sk}} &= 2 \sum_{t=1}^T \left[\varepsilon_r(t) \sin(\omega_k^0 t + \phi_k^0) - \varepsilon_c(t) \cos(\omega_k^0 t + \phi_k^0) \right]
\end{aligned}$$

B4. The elements of $Q''_{kl}(\theta^0)$, for $k, l = 1, 2, \dots, M$ and $k \neq l$.

$$\begin{aligned}
\frac{\partial^2 Q(\theta^0)}{\partial \omega_k \partial \omega_l} &= 2 \sum_{t=1}^T t^2 \mu_{sk}^0 \mu_{sl}^0 \cos(\omega_k^0 t + \phi_k^0 - \omega_l^0 t - \phi_l^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial \omega_k \partial \phi_l} &= 2 \sum_{t=1}^T t \mu_{sk}^0 \mu_{sl}^0 \cos(\omega_k^0 t + \phi_k^0 - \omega_l^0 t - \phi_l^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial \omega_k \partial \mu_{sl}} &= 2 \sum_{t=1}^T t \mu_{sk}^0 \sin(\omega_k^0 t + \phi_k^0 - \omega_l^0 t - \phi_l^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial \phi_k \partial \omega_l} &= 2 \sum_{t=1}^T t \mu_{sk}^0 \mu_{sl}^0 \cos(\omega_k^0 t + \phi_k^0 - \omega_l^0 t - \phi_l^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial \phi_k \partial \phi_l} &= 2 \sum_{t=1}^T \mu_{sk}^0 \mu_{sl}^0 \cos(\omega_k^0 t + \phi_k^0 - \omega_l^0 t - \phi_l^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial \phi_k \partial \mu_{sl}} &= 2 \sum_{t=1}^T \mu_{sk}^0 \sin(\omega_k^0 t + \phi_k^0 - \omega_l^0 t - \phi_l^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial \mu_{sk} \partial \omega_l} &= 2 \sum_{t=1}^T t \mu_{sl}^0 \sin(\omega_l^0 t + \phi_l^0 - \omega_k^0 t - \phi_k^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial \mu_{sk} \partial \phi_l} &= 2 \sum_{t=1}^T \mu_{sl}^0 \sin(\omega_l^0 t + \phi_l^0 - \omega_k^0 t - \phi_k^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial \mu_{sk} \partial \mu_{sl}} &= 2 \sum_{t=1}^T \cos(\omega_k^0 t + \phi_k^0 - \omega_l^0 t - \phi_l^0)
\end{aligned}$$

B5. The expression of the remainder in (A.13) in the proof of Theorem 4.

$$\begin{aligned}
R_T(\hat{\theta}_k, \theta^0) &= \frac{1}{T} \sum_{t=1}^T s_k^2(t) [\cos(2\omega_k^0 t + 2\phi_k^0 - 2\hat{\omega}_k - 2\hat{\phi}_k) - 1] \\
&+ \frac{1}{T} \sum_{t=1}^T \sum_{i \neq k} s_i^2(t) \cos(2\omega_i^0 t + 2\phi_i^0 - 2\hat{\omega}_k - 2\hat{\phi}_k) \\
&+ \frac{2}{T} \sum_{t=1}^T \sum_{i=1}^M n_r(t) s_i(t) \cos(\omega_i^0 t + \phi_i^0 - 2\hat{\omega}_k - 2\hat{\phi}_k) \\
&+ \frac{2}{T} \sum_{t=1}^T \sum_{j=1}^M n_c(t) s_i(t) \sin(2\hat{\omega}_k + 2\hat{\phi}_k - \omega_i^0 t - \phi_i^0) \\
&+ \frac{1}{T} \sum_{t=1}^T \sum_{i \neq j} s_i(t) s_j(t) \cos(\omega_i^0 t + \phi_i^0 + \omega_j^0 t + \phi_j^0 - 2\hat{\omega}_k - 2\hat{\phi}_k) \\
&+ \frac{1}{T} \sum_{t=1}^T [n_r^2(t) - e_c^2(t)] \cos(2\hat{\omega}_k + 2\hat{\phi}_k) \\
&+ \frac{2}{T} \sum_{t=1}^T n_r(t) n_c(t) \sin(-2\hat{\omega}_k - 2\hat{\phi}_k).
\end{aligned}$$

(C) Some expressions in the proofs of Theorems 7-11
in case of zero-mean multiplicative noise

C1. The expression of $Q(\theta)$.

$$\begin{aligned}
Q(\theta) = & \sum_{t=1}^T \left\{ \sum_{k=1}^M \left[p_k^0 \cos(2\omega_k^0 t + 2\phi_k^0) - p_k \cos(2\omega_k t + 2\phi_k) \right] \right\}^2 \\
& + \sum_{t=1}^T \left\{ \sum_{k=1}^M \left[p_k^0 \sin(2\omega_k^0 t + 2\phi_k^0) - p_k \sin(2\omega_k t + 2\phi_k) \right] \right\}^2 \\
& + 2 \sum_{t=1}^T \sum_{k=1}^M \varepsilon_r(t) \left[p_k^0 \cos(2\omega_k^0 t + 2\phi_k^0) - p_k \cos(2\omega_k t + 2\phi_k) \right] \\
& + 2 \sum_{t=1}^T \sum_{k=1}^M \varepsilon_c(t) \left[p_k^0 \sin(2\omega_k^0 t + 2\phi_k^0) - p_k \sin(2\omega_k t + 2\phi_k) \right] \\
& + \sum_{t=1}^T |\varepsilon(t)|^2
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon(t) = & \sum_{k=1}^M \left(s_k^2(t) - p_k^0 \right) e^{j(2\omega_k^0 t + 2\phi_k^0)} + 2n(t) \sum_{k=1}^M s_k(t) e^{j(\omega_k^0 t + \phi_k^0)} \\
& + \sum_{k \neq l} s_k(t) s_l(t) e^{j(\omega_k^0 t + \phi_k^0 + \omega_l^0 t + \phi_l^0)} + n^2(t)
\end{aligned}$$

C2. The elements of $Q'_k(\theta^0)$, for $k = 1, 2, \dots, M$.

$$\begin{aligned}
\frac{\partial Q(\theta^0)}{\partial \omega_k} &= 4 \sum_{t=1}^T t p_k^0 \left[\varepsilon_r(t) \sin(2\omega_k^0 t + 2\phi_k^0) - \varepsilon_c(t) \cos(2\omega_k^0 t + 2\phi_k^0) \right] \\
\frac{\partial Q(\theta^0)}{\partial \phi_k} &= 4 \sum_{t=1}^T p_k^0 \left[\varepsilon_r(t) \sin(2\omega_k^0 t + 2\phi_k^0) - \varepsilon_c(t) \cos(2\omega_k^0 t + 2\phi_k^0) \right] \\
\frac{\partial Q(\theta^0)}{\partial p_k} &= 2 \sum_{t=1}^T \left[-\varepsilon_r(t) \cos(2\omega_k^0 t + 2\phi_k^0) - \varepsilon_c(t) \sin(2\omega_k^0 t + 2\phi_k^0) \right]
\end{aligned}$$

C3. The elements of $Q''_{kk}(\theta^0)$, for $k = 1, 2, \dots, M$.

$$\begin{aligned}
\frac{\partial^2 Q(\theta^0)}{\partial \omega_k^2} &= 8 \sum_{t=1}^T t^2 p_k^0 \left[\varepsilon_r(t) \cos(2\omega_k^0 t + 2\phi_k^0) + \varepsilon_c(t) \sin(2\omega_k^0 t + 2\phi_k^0) \right] + 8(p_k^0)^2 \sum_{t=1}^T t^2 \\
\frac{\partial^2 Q(\theta^0)}{\partial \phi_k^2} &= 8 \sum_{t=1}^T p_k^0 \left[\varepsilon_r(t) \cos(2\omega_k^0 t + 2\phi_k^0) + \varepsilon_c(t) \sin(2\omega_k^0 t + 2\phi_k^0) \right] + 8T(p_k^0)^2 \\
\frac{\partial^2 Q(\theta^0)}{\partial p_k^2} &= 2T
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 Q(\theta^0)}{\partial \omega_k \partial \phi_k} &= 8 \sum_{t=1}^T t p_k^0 \left[\varepsilon_r(t) \cos(2\omega_k^0 t + 2\phi_k^0) + \varepsilon_c(t) \sin(2\omega_k^0 t + 2\phi_k^0) \right] + 8(p_k^0)^2 \sum_{t=1}^T t \\
\frac{\partial^2 Q(\theta^0)}{\partial \omega_k \partial p_k} &= 4 \sum_{t=1}^T t \left[\varepsilon_r(t) \sin(2\omega_k^0 t + 2\phi_k^0) - \varepsilon_c(t) \cos(2\omega_k^0 t + 2\phi_k^0) \right] \\
\frac{\partial^2 Q(\theta^0)}{\partial \phi_k \partial p_k} &= 4 \sum_{t=1}^T \left[\varepsilon_r(t) \sin(2\omega_k^0 t + 2\phi_k^0) - \varepsilon_c(t) \cos(2\omega_k^0 t + 2\phi_k^0) \right]
\end{aligned}$$

C4. The elements of $Q''_{kl}(\theta^0)$, for $k, l = 1, 2, \dots, M$ and $k \neq l$.

$$\begin{aligned}
\frac{\partial^2 Q(\theta^0)}{\partial \omega_k \partial \omega_l} &= 8 \sum_{t=1}^T t^2 p_k^0 p_l^0 \cos(2\omega_k^0 t + 2\phi_k^0 - 2\omega_l^0 t - 2\phi_l^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial \omega_k \partial \phi_l} &= 8 \sum_{t=1}^T t p_k^0 p_l^0 \cos(2\omega_k^0 t + 2\phi_k^0 - 2\omega_l^0 t - 2\phi_l^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial \omega_k \partial p_l} &= 4 \sum_{t=1}^T t p_k^0 \sin(2\omega_k^0 t + 2\phi_k^0 - 2\omega_l^0 t - 2\phi_l^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial \phi_k \partial \omega_l} &= 8 \sum_{t=1}^T t p_k^0 p_l^0 \cos(2\omega_k^0 t + 2\phi_k^0 - 2\omega_l^0 t - 2\phi_l^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial \phi_k \partial \phi_l} &= 8 \sum_{t=1}^T p_k^0 p_l^0 \cos(2\omega_k^0 t + 2\phi_k^0 - 2\omega_l^0 t - 2\phi_l^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial \phi_k \partial p_l} &= 4 \sum_{t=1}^T p_k^0 \sin(2\omega_k^0 t + 2\phi_k^0 - 2\omega_l^0 t - 2\phi_l^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial p_k \partial \omega_l} &= 4 \sum_{t=1}^T t p_l^0 \sin(2\omega_l^0 t + 2\phi_l^0 - 2\omega_k^0 t - 2\phi_k^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial p_k \partial \phi_l} &= 4 \sum_{t=1}^T p_l^0 \sin(2\omega_l^0 t + 2\phi_l^0 - 2\omega_k^0 t - 2\phi_k^0) \\
\frac{\partial^2 Q(\theta^0)}{\partial p_k \partial p_l} &= 2 \sum_{t=1}^T \cos(2\omega_k^0 t + 2\phi_k^0 - 2\omega_l^0 t - 2\phi_l^0)
\end{aligned}$$