Supplementary file to

“Spline-Lasso in High-Dimensional Linear Regression”

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In this supplementary file, we will give more detailed proofs to Theorems 1-3. The proof of Theorem 1 follows similar lines to those in Hebiri and Van de Geer (2011) while the proofs of Theorems 2-3 follow similar lines to those in Huang *et al.* (2011).

Proof of Theorem 1

We first give a lemma on a concentration result dealing with Gaussian noise:

Lemma 1. *Let $\eta \in (0, 1)$. Denote $\Lambda_{n,p} = \{\max_{j=1,\dots,p} 2|V_j| \leq \lambda_1/2\}$ where $V_j = n^{-1} \sum_{i=1}^n x_{i,j} \varepsilon_i$. Let λ_1 be defined as in Theorem 1. Then*

$$P\left(\max_{j=1,\dots,p} 2|V_j| \leq \lambda_1/2\right) \geq 1 - \eta.$$

The second lemma states that the squared risk and the l_1 estimation error can be controlled by the restricted l_2 estimation error $\|\beta_{\mathcal{B}} - \hat{\beta}_{\mathcal{B}}^{spline}\|_2$.

Lemma 2. *On the event $\Lambda_{n,p} = \{\max_{j=1,\dots,p} 2|V_j| \leq \lambda_1/2\}$, we have*

$$\frac{1}{n} \|\tilde{X}\beta - \tilde{X}\hat{\beta}^{spline}\|_2^2 + \frac{\lambda_1}{2} \|\beta - \hat{\beta}^{spline}\|_1 \leq r_n \|\beta_{\mathcal{B}} - \hat{\beta}_{\mathcal{B}}^{spline}\|_2,$$

where $\tilde{X} = (X, \sqrt{n\lambda_2}L)^T$, $r_n = 2\lambda_1\sqrt{|\mathcal{A}|} + 2\lambda_2\|\tilde{J}\beta\|_2$, and $\mathcal{B} \supseteq \mathcal{A}$.

Proof. Consider the augmented data set:

$$\tilde{X} = (X, \sqrt{n\lambda_2}L)^T, \text{ and } \tilde{Y} = (Y, \mathbf{0})^T, \text{ and } \tilde{\varepsilon} = (\varepsilon, -\sqrt{n\lambda_2}L\beta)^T.$$

Then the spline-lasso estimator is the minimizer of $n^{-1}\|\tilde{Y} - \tilde{X}\beta\|_2^2 + \lambda_1\|\beta\|_1$.

Therefore, we have

$$n^{-1}\|\tilde{Y} - \tilde{X}\hat{\beta}^{spline}\|_2^2 + \lambda_1\|\hat{\beta}^{spline}\|_1 \leq n^{-1}\|\tilde{Y} - \tilde{X}\beta\|_2^2 + \lambda_1\|\beta\|_1,$$

i.e.,

$$\frac{1}{n} \|\tilde{X}\beta - \tilde{X}\hat{\beta}^{spline}\|_2^2 \leq \lambda_1(\|\beta\|_1 - \|\hat{\beta}^{spline}\|_1) + \frac{2}{n} \tilde{\varepsilon}^T \tilde{X}(\beta - \hat{\beta}^{spline}).$$

Then on the event $\Lambda_{n,p} = \{\max_{j=1,\dots,p} 2|V_j| \leq \lambda_1/2\}$, we have

$$\begin{aligned} \frac{1}{n} \|\tilde{X}\beta - \tilde{X}\hat{\beta}^{spline}\|_2^2 &\leq \lambda_1(\|\beta\|_1 - \|\hat{\beta}^{spline}\|_1) + \frac{\lambda_1}{2} \|\beta - \hat{\beta}^{spline}\|_1 \\ &\quad - 2\lambda_2\beta^T \tilde{J}(\beta - \hat{\beta}^{spline}), \end{aligned}$$

where $\tilde{J} = L^T L$. Then consider a set $\mathcal{B}$ such that $\|\beta_{\mathcal{B}} - \hat{\beta}_{\mathcal{B}}^{spline}\|_2 = \|\beta - \hat{\beta}^{spline}\|_2$. Such a set exists, for instance, $\mathcal{B} = \{j \in \{1, \dots, p\} : \hat{\beta}^{spline} = 0 \text{ and } \beta = 0\}^C$, and it includes $\mathcal{A}$. Then we have

$$-\lambda_2\beta^T \tilde{J}(\beta - \hat{\beta}^{spline}) \leq \lambda_2\|\tilde{J}\beta\|_2\|\beta_{\mathcal{B}} - \hat{\beta}_{\mathcal{B}}^{spline}\|_2.$$

Hence, we have

$$\begin{aligned}
& \frac{1}{n} \|\tilde{X}\beta - \tilde{X}\hat{\beta}^{spline}\|_2^2 + \frac{\lambda_1}{2} \|\beta - \hat{\beta}^{spline}\|_1 \\
& \leq \lambda_1 [\|\beta\|_1 - \|\hat{\beta}^{spline}\|_1 + \|\beta - \hat{\beta}^{spline}\|_1] + 2\lambda_2 \|\tilde{J}\beta\|_2 \|\beta_{\mathcal{B}} - \hat{\beta}_{\mathcal{B}}^{spline}\|_2 \\
& \leq 2\lambda_1 \sum_{j \in \mathcal{A}} \|\beta_j - \hat{\beta}_j^{spline}\| + 2\lambda_2 \|\tilde{J}\beta\|_2 \|\beta_{\mathcal{B}} - \hat{\beta}_{\mathcal{B}}^{spline}\|_2 \\
& \leq r_n \|\beta_{\mathcal{B}} - \hat{\beta}_{\mathcal{B}}^{spline}\|_2,
\end{aligned}$$

where $r_n = 2\lambda_1 \sqrt{|\mathcal{A}|} + 2\lambda_2 \|\tilde{J}\beta\|_2$ and this proves the lemma. $\square$

Returning to the proof of Theorem 1, from the above lemmas, we have

$$\|\beta - \hat{\beta}^{spline}\|_1 \leq \varrho_n \|\beta_{\mathcal{B}} - \hat{\beta}_{\mathcal{B}}^{spline}\|_2,$$

where $\varrho_n = 2r_n/\lambda_1 = 4\sqrt{|\mathcal{A}|} + 4\lambda_2\lambda_1^{-1} \|\tilde{J}\beta\|_2$. Then the vector $\beta - \hat{\beta}^{spline}$ satisfies Assumption 1. Therefore, we get

$$\frac{1}{n} \|\tilde{X}\beta - \tilde{X}\hat{\beta}^{spline}\|_2^2 \leq \frac{r_n}{\sqrt{\phi_{\lambda_2}}} \sqrt{\frac{1}{n}} \|\tilde{X}\beta - \tilde{X}\hat{\beta}^{spline}\|_2,$$

which will easily lead to the first and the third part of the result

$$\frac{1}{n} \|X\beta - X\hat{\beta}^{spline}\|_2^2 \leq \phi_{\lambda_2}^{-1} (2\lambda_1 \sqrt{|\mathcal{A}|} + 2\lambda_2 \|\tilde{J}\beta\|_2)^2,$$

and when $\lambda_2 \neq 0$,

$$\|L(\beta - \hat{\beta}^{spline})\|_2 \leq \lambda_2^{-1/2} \phi_{\lambda_2}^{-1/2} \left(2\lambda_1 \sqrt{|\mathcal{A}|} + 2\lambda_2 \|\tilde{J}\beta\|_2 \right).$$

Then the first part, together with Assumption 1, yields the second inequality.

Proof of Theorem 2

Let $\tilde{X} = \sqrt{n}(\Sigma + \lambda_2 \tilde{J})^{1/2}$, $\tilde{Y} = \tilde{X}^{-1} X^T Y$ and

$$\tilde{M}(\mathbf{b}; \lambda, \gamma) = (2n)^{-1} \|\tilde{Y} - \tilde{X}\mathbf{b}\|_2^2 + \sum_{j=1}^p \rho(|b_j|; \lambda_1, \gamma).$$

Thus, $\hat{\beta}$ is also the minimizer of $\tilde{M}(\mathbf{b}; \lambda, \gamma)$.

In order to have $\hat{\beta} = \hat{\beta}^o$ with high probability, $\hat{\beta}^o$ should satisfy the KKT conditions of $\tilde{M}$. Therefore, $\hat{\beta}_{\mathcal{O}^o}$ and $\hat{\beta}_{\mathcal{O}^c}$ should satisfy the KKT conditions of $\tilde{M}$ for nonzero coefficients and for zero coefficients, respectively.

It is easy to see that $\hat{\beta}_{\mathcal{O}^c}$ satisfies the KKT condition of $\tilde{M}$ for zero coefficients on

$$\Omega_1 = \{ \|\tilde{X}_{\mathcal{O}^c}^T(\tilde{Y} - \tilde{X}\hat{\beta}^o/n)\|_\infty \leq \lambda_1 \}.$$

In addition, since $\hat{\beta}_{\mathcal{O}}^o$ is the solution for minimizing $(2n)^{-1}\|\tilde{Y} - \tilde{X}\mathbf{b}\|_2^2$ on the set $\mathcal{O}$ without the penalty term. Therefore, we need to have $\dot{\rho}(|\hat{\beta}_j^o|; \lambda_1, \gamma) = 0, j \in \mathcal{O}$. Since $|\hat{\beta}_j^o| \geq \gamma\lambda_1$ gives $\dot{\rho}(|\hat{\beta}_j^o|; \lambda_1, \gamma) = 0$, so $\hat{\beta}_{\mathcal{O}^o}$ would satisfy the KKT condition of $\tilde{M}$ for nonzero coefficients on the event

$$\Omega_2 = \{ \min_{j \in \mathcal{O}} \text{sgn}(\beta_j^*) \hat{\beta}_j^o \geq \gamma\lambda_1 \}.$$

To prove Theorem 2, it suffices to show $P(\Omega_1^c \cup \Omega_2^c) \leq 3\epsilon$.

By $P(\Omega_1^c \cup \Omega_2^c) \leq P(\Omega_1^c) + P(\Omega_2^c)$, we try to find larger events than Ω_1^c and Ω_2^c with small probability.

Note Ω_1 is related to the gradient in the linear model $\tilde{Y} = \tilde{X}\beta^* + \tilde{\varepsilon}^*$. Let $\tilde{\varepsilon}^* = \tilde{\varepsilon} + E\tilde{\varepsilon}^*$ with $\tilde{\varepsilon} = \tilde{Y} - E\tilde{Y}$. Since $\tilde{X}^T\tilde{Y} = X^TY$ and both true coefficient β^o and β^* are supported in $\mathcal{O}$, so for any subset B of indices $\{1, \dots, p\}$,

$$\begin{aligned} \tilde{X}_B^T E\tilde{\varepsilon}^*/n &= \tilde{X}_B^T (E\tilde{Y} - \tilde{X}\beta^*)/n \\ &= E(X_B^T Y)/n - \tilde{X}_B^T \tilde{X}\beta^*/n \\ &= X_B^T X\beta^o/n - \tilde{X}_B^T \tilde{X}_{\mathcal{O}}\beta_{\mathcal{O}}^*/n \\ &= \Sigma_{B,\mathcal{O}}\beta_{\mathcal{O}}^o - \Sigma_{B,\mathcal{O}}(\lambda_2)\Sigma_{\mathcal{O}}^{-1}(\lambda_2)\Sigma_{\mathcal{O}}\beta_{\mathcal{O}}^o \\ &= \lambda_2\{\Sigma_{B,\mathcal{O}}(\lambda_2)\Sigma_{\mathcal{O}}^{-1}(\lambda_2)\tilde{J}_{\mathcal{O}} - \tilde{J}_{B,\mathcal{O}}\}\beta_{\mathcal{O}}^o. \end{aligned}$$

Since $\tilde{X}_{\mathcal{O}}^T E \tilde{\varepsilon}^*/n = 0$, we have $\|\tilde{X}_{\mathcal{O}}^T E \tilde{\varepsilon}^*/n\|_{\infty} = \lambda_2 C_2$. Since $\tilde{X}^T \tilde{\varepsilon} = \tilde{X}^T \tilde{Y} - E \tilde{X}^T \tilde{Y} = X^T Y - E X^T Y = X^T \varepsilon$, we have

$$\Omega_1 \supseteq \{\|X_{\mathcal{O}^c}^T \varepsilon/n\|_{\infty} < \lambda_1 - \lambda_2 C_2\}.$$

If the event on the right-hand side holds, we have

$$\begin{aligned} \|\tilde{X}_{\mathcal{O}^c}^T (\tilde{Y} - \tilde{X} \hat{\beta}^o)/n\|_{\infty} &= \|\tilde{X}_{\mathcal{O}^c}^T (\tilde{Y} - E \tilde{Y} + E \tilde{Y} - \tilde{X} \hat{\beta}^o)/n\|_{\infty} \\ &= \|\tilde{X}_{\mathcal{O}^c}^T (\tilde{\varepsilon} + E \tilde{\varepsilon}^* + \tilde{X}(\hat{\beta}^o - \beta^*))/n\|_{\infty} \\ &= \|\tilde{X}_{\mathcal{O}^c}^T (\tilde{\varepsilon} + E \tilde{\varepsilon}^*)/n + \tilde{X}_{\mathcal{O}^c}^T \tilde{X}_{\mathcal{O}}(\hat{\beta}_{\mathcal{O}}^o - \beta_{\mathcal{O}}^*)/n\|_{\infty} \\ &= \|X_{\mathcal{O}^c}^T \varepsilon/n + \tilde{X}_{\mathcal{O}^c}^T E \tilde{\varepsilon}^*/n + \tilde{X}_{\mathcal{O}^c}^T \tilde{X}_{\mathcal{O}} \Sigma_{\mathcal{O}}^{-1}(\lambda_2) X_{\mathcal{O}}^T (Y - EY)/n^2\|_{\infty} \\ &= \|X_{\mathcal{O}^c}^T \varepsilon/n + \tilde{X}_{\mathcal{O}^c}^T E \tilde{\varepsilon}^*/n + \tilde{X}_{\mathcal{O}^c}^T \tilde{X}_{\mathcal{O}} \Sigma_{\mathcal{O}}^{-1}(\lambda_2) X_{\mathcal{O}}^T \varepsilon/n^2\|_{\infty} \\ &\leq \|X_{\mathcal{O}^c}^T \varepsilon/n\|_{\infty} + \|\tilde{X}_{\mathcal{O}^c}^T E \tilde{\varepsilon}^*/n\|_{\infty} \leq \lambda_1. \end{aligned}$$

Since $\beta^* = E \hat{\beta}^o$, $\hat{\beta}_{\mathcal{O}}^o = \Sigma_{\mathcal{O}}^{-1}(\lambda_2) X_{\mathcal{O}}^T Y/n$ can be written as $\beta_{\mathcal{O}}^* + ((v_j/n)^{1/2} \mathbf{u}_j^T \varepsilon, j \in \mathcal{O})^T$, where $\|\mathbf{u}_j\|_2 = 1$ and $\{v_j, j \in \mathcal{O}\}$ are the diagonal elements of $\Sigma_{\mathcal{O}}^{-1}(\lambda_2) \Sigma_{\mathcal{O}} \Sigma_{\mathcal{O}}^{-1}(\lambda_2)$. Thus,

$$\Omega_2^c \subseteq \bigcup_{j \in \mathcal{O}} \{\text{sgn}(\beta_j^*) \mathbf{u}_j^T \varepsilon \geq (n/v_j)^{1/2} |\beta_j^*| \geq \sigma \sqrt{2 \log(|\mathcal{O}|/\epsilon)}\}.$$

Since $\lambda_1 \geq \lambda_2 C_2 + \sigma \sqrt{2 \log((p - d^o)/\epsilon)} \max_{j \leq p} \|\mathbf{x}_j\|_2/n$, the sub-Gaussian Condition yields

$$\begin{aligned} P(\Omega_1^c) + P(\Omega_2^c) &\leq P(\|X_{\mathcal{O}^c}^T \varepsilon/n\|_{\infty} > \sigma \sqrt{2 \log((p - d^o)/\epsilon)} \max_{j \leq p} \|\mathbf{x}_j\|_2/n) \\ &\quad + \sum_{j \in \mathcal{O}} P(\text{sgn}(\beta_j^*) \mathbf{u}_j^T \varepsilon \geq \sigma \sqrt{2 \log(|\mathcal{O}|/\epsilon)}) \\ &\leq 2|\mathcal{O}^c|\epsilon/(p - |\mathcal{O}|) + |\mathcal{O}|\epsilon/|\mathcal{O}| = 3\epsilon. \end{aligned}$$

Proof of Theorem 3

$\hat{\beta}(\lambda)$ is the MC+ estimator based on data $(\tilde{X}, \tilde{Y})$ at penalty level λ_1 . From the proof of Theorem 5 in Zhang (2010), $\hat{\beta}(\lambda) = \hat{\beta}^o(\lambda)$ on $\Omega = \cap_{j=1}^3 \Omega_j$, where

$$\Omega_1 = \{\|\tilde{X}_{\mathcal{O}^c}^T(\tilde{Y} - \tilde{X}\hat{\beta}^o/n)\|_\infty \leq \lambda_1\},$$

$$\Omega_2 = \{\min_{j \in \mathcal{O}} \text{sgn}(\beta_j^*) \hat{\beta}_j^o \geq \gamma(2\sqrt{c^*}\lambda_1)\},$$

$$\Omega_3 = \{\zeta(\tilde{Y} - \tilde{\beta}^*; d^* - |\mathcal{O}|, \mathcal{O}, \lambda_2) \leq \lambda_1\},$$

where for $m \geq 1$ and vector $\mathbf{v}$ in the range of $\tilde{X}$,

$$\zeta(\mathbf{v}; m, \mathcal{O}, \lambda_2) = \max_B \left\{ \frac{\|(\tilde{P}_B - \tilde{P}_{\mathcal{O}})\mathbf{v}\|_2}{(mn)^{1/2}} : \mathcal{O} \subseteq B \subseteq \{1, \dots, p\}, |B| = m + |\mathcal{O}| \right\},$$

and $\tilde{P}_B = \tilde{X}_B(\tilde{X}_B^T \tilde{X}_B)^{-1} \tilde{X}_B^T$.

Similar to Theorem 2, we have $\Omega_1 \supseteq \{\|X_{\mathcal{O}^c}^T \varepsilon/n\|_\infty < \lambda_1 - \lambda_2 C_2\}$.

In addition, a slight modification on the proof of Theorem 2 leads to

$$\Omega_2^c \subseteq \bigcup_{j \in \mathcal{O}} \{\text{sgn}(\beta_j^*) \mathbf{u}_j^T \varepsilon \geq (n/v_j)^{1/2} (|\beta_j^*| - \gamma(2\sqrt{c^*}\lambda_1)) \geq \sigma \sqrt{2 \log(|\mathcal{O}|/\epsilon)}\}.$$

Then for $|B| \leq d^*$, we have $\|\tilde{P}_B E \tilde{\varepsilon}^*\|_2 / \sqrt{n} = \|\Sigma_B^{-1/2}(\lambda_2) \tilde{X}_B^T E \tilde{\varepsilon}^*\|_2 / n \leq \|\tilde{X}_B^T E \tilde{\varepsilon}^* / n\|_\infty \sqrt{|B|/c_*(\lambda_2)}$, and

$$\|\tilde{P}_B \tilde{\varepsilon}\|_2 / \sqrt{n} = \|\Sigma_B^{-1/2}(\lambda_2) \tilde{X}_B^T \tilde{\varepsilon}\|_2 / n \leq \|X_B^T \varepsilon / n\|_\infty \sqrt{|B|/c_*(\lambda_2)}.$$

Thus,

$$\begin{aligned} & \zeta(\tilde{Y} - \tilde{\beta}^*; d^* - |\mathcal{O}|, \mathcal{O}, \lambda_2) \\ &= \zeta(\tilde{\varepsilon} + E \tilde{\varepsilon}^*; d^* - |\mathcal{O}|, \mathcal{O}, \lambda_2) \\ &= \max \left\{ \frac{\|(\tilde{P}_B - \tilde{P}_{\mathcal{O}})(\tilde{\varepsilon} + E \tilde{\varepsilon}^*)\|_2}{((d^* - |\mathcal{O}|)n)^{1/2}} : \mathcal{O} \subseteq B \subseteq \{1, \dots, p\}, |B| = d^* \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\max_B \{ \|X_B^T \varepsilon/n\|_\infty \sqrt{|B|/c_*(\lambda_2)} + \|\tilde{X}_B^T E \tilde{\varepsilon}^*/n\|_\infty \sqrt{|B|/c_*(\lambda_2)} \}}{(d^* - |\mathcal{O}|)^{1/2}} \\
&\leq \frac{(\|X^T \varepsilon/n\|_\infty + \lambda_2 C_2) \sqrt{d^*}}{\sqrt{(d^* - |\mathcal{O}|) c_*(\lambda_2)}}.
\end{aligned}$$

Since $|\mathcal{O}| \leq d^*/(K_*+1)$, this gives $\Omega_3 \supseteq \{ \|X^T \varepsilon/n\|_\infty < \sqrt{c_*(\lambda_2) K_*/(K_*+1)} \lambda_1 - \lambda_2 C_2 \}$. Therefore,

$$\begin{aligned}
1 - P\{\Omega_1 \cap \Omega_3\} + P\{\Omega_2^c\} &\leq P(\|X^T \varepsilon/n\|_\infty > \sigma \sqrt{2 \log(p/\epsilon)} \max_{j \leq p} \|\mathbf{x}_j\|_2/n) \\
&\quad + \sum_{j \in \mathcal{O}} P(\text{sgn}(\beta_j^*) \mathbf{u}_j^T \varepsilon \geq \sigma \sqrt{2 \log(|\mathcal{O}|/\epsilon)}) \\
&\leq 2p(\epsilon/p) + |\mathcal{O}| \epsilon/|\mathcal{O}| = 3\epsilon.
\end{aligned}$$

This completes the proof, since $\hat{\beta}_j^o \neq 0$ for all $j \in \mathcal{O}$ in Ω_2 .