Simultaneous Tests for Homogeneity of Two Zero-inflated (Beta) Populations

Supplemental file

Luna Sun and Alix I. Gitelman

Department of Statistics, Oregon State University

Result 1 (Partition of multivariate normal distribution (Seely's Notes, unpublished)). Suppose $Y \sim N_p(\mu, V)$ where V is nonsingular. Let Y, μ and V be similarly partitioned in the form $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$, $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$, $V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$. Assume Y_1 is $q \times 1$ and that $s = p - q$ so that Y_2 is $s \times 1$. Set $V_{11.2} = V_{11} - V_{12}V_{22}^{-1}V_{21}$ and let $T = Y_1 - V_{12}V_{22}^{-1}Y_2$. Then

1. $T \sim N_q(\mu_1 - V_{12}V_{22}^{-1}\mu_2, V_{11.2})$ and $V_{11.2}$ is nonsingular.
2. T and Y_2 are independent random vectors.

Theorem 1 (Asymptotic Independence among Score Tests). $\hat{S}_1, \hat{S}_2$ and $\hat{S}_3$ (and also $\tilde{S}_1, \tilde{S}_2$ and $\tilde{S}_3$) are asymptotically independent.

Proof. Consider $\hat{S}_1, \hat{S}_2$ and $\hat{S}_3$ first: Denote

$$\omega = (p, \mu, \phi)', \zeta = (\delta, p, \mu, \phi)' = (\delta, \omega')', \eta = (\gamma, \delta, p, \mu, \phi)' = (\gamma, \zeta')' = (\gamma, \delta, \omega')',$$

$$\theta = (\beta, \gamma, \delta, p, \mu, \phi)' = (\beta, \eta')' = (\beta, \gamma, \zeta')' = (\beta, \gamma, \delta, \omega')'.$$

Recall that

$$\hat{S}_1 = \hat{s}_1'(\hat{A}_1 - \hat{C}_1\hat{D}_1^{-1}\hat{C}_1')^{-1}\hat{s}_1,$$

$$\hat{S}_2 = \hat{s}_2'(\hat{A}_2 - \hat{C}_2\hat{D}_2^{-1}\hat{C}_2')^{-1}\hat{s}_2$$

and

$$\hat{S}_3 = \hat{s}_3'(\hat{A}_3 - \hat{C}_3\hat{D}_3^{-1}\hat{C}_3')^{-1}\hat{s}_3.$$

Under H_0 ,

$$s_1 = \frac{\partial l_{1f}}{\partial \delta} = \frac{\partial l_f}{\partial \delta}, s_2 = \frac{\partial l_{2f}}{\partial \gamma} = \frac{\partial l_f}{\partial \gamma}, \text{ and } s_3 = \frac{\partial l_{3f}}{\partial \beta} = \frac{\partial l_f}{\partial \beta}.$$

And

$$\hat{s}_1 = s_1(\hat{\omega}), \hat{s}_2 = s_2(\hat{\zeta}), \hat{s}_3 = s_3(\hat{\eta}),$$

where $\hat{\omega}$, $\hat{\zeta}$ and $\hat{\eta}$ are maximum likelihood estimates of ω under H_0' , ζ under H_0'' and η under H_0''' , respectively (i.e., $\hat{s}_1 = \frac{\partial l_f}{\partial \delta} \Big|_{\omega=\hat{\omega}}$, $\hat{s}_2 = \frac{\partial l_f}{\partial \gamma} \Big|_{\zeta=\hat{\zeta}}$ and $\hat{s}_3 = \frac{\partial l_f}{\partial \beta} \Big|_{\eta=\hat{\eta}}$). Notice that when H_0 is true, H_0' , H_0'' and H_0''' are all true. Expanding $\hat{s}_1$, $\hat{s}_2$ and $\hat{s}_3$ around the true parameters θ_0 via Taylor expansion with $l_f = l$:

$$\hat{s}_1 = \frac{\partial l}{\partial \delta} - I_{\delta\omega_0} I_{\omega_0\omega_0}^{-1} \frac{\partial l}{\partial \omega_0} + \mathcal{O}_p(1),$$

$$\hat{s}_2 = \frac{\partial l}{\partial \gamma} - I_{\gamma\zeta_0} I_{\zeta_0\zeta_0}^{-1} \frac{\partial l}{\partial \zeta_0} + \mathcal{O}_p(1)$$

and

$$\hat{s}_3 = \frac{\partial l}{\partial \beta} - I_{\beta\eta_0} I_{\eta_0\eta_0}^{-1} \frac{\partial l}{\partial \eta_0} + \mathcal{O}_p(1),$$

where

$$I_{\delta\omega_0} = E\left(-\frac{\partial^2 l}{\partial \delta \partial \omega^T} \Big| H_0\right), I_{\omega_0\omega_0} = E\left(-\frac{\partial^2 l}{\partial \omega \partial \omega^T} \Big| H_0\right), I_{\gamma\zeta_0} = E\left(-\frac{\partial^2 l}{\partial \gamma \partial \zeta^T} \Big| H_0\right),$$

$$I_{\zeta_0\zeta_0} = E\left(-\frac{\partial^2 l}{\partial \zeta \partial \zeta^T} \Big| H_0\right), I_{\beta\eta_0} = E\left(-\frac{\partial^2 l}{\partial \beta \partial \eta^T} \Big| H_0\right), I_{\eta_0\eta_0} = E\left(-\frac{\partial^2 l}{\partial \eta \partial \eta^T} \Big| H_0\right),$$

and

$$\frac{\partial l}{\partial \omega_0} = \frac{\partial l}{\partial \omega} \Big|_{\omega=\omega_0}, \frac{\partial l}{\partial \zeta_0} = \frac{\partial l}{\partial \zeta} \Big|_{\zeta=\zeta_0}, \frac{\partial l}{\partial \eta_0} = \frac{\partial l}{\partial \eta} \Big|_{\eta=\eta_0}.$$

Let

$$s_{01} = \frac{\partial l}{\partial \delta} - I_{\delta\omega_0} I_{\omega_0\omega_0}^{-1} \frac{\partial l}{\partial \omega_0}, s_{02} = \frac{\partial l}{\partial \gamma} - I_{\gamma\zeta_0} I_{\zeta_0\zeta_0}^{-1} \frac{\partial l}{\partial \zeta_0} \text{ and } s_{03} = \frac{\partial l}{\partial \beta} - I_{\beta\eta_0} I_{\eta_0\eta_0}^{-1} \frac{\partial l}{\partial \eta_0}.$$

Since the score function can be written as

$$\frac{\partial l}{\partial \theta} = \begin{pmatrix} \frac{\partial l}{\partial \beta} \\ \frac{\partial l}{\partial \gamma} \\ \frac{\partial l}{\partial \delta} \\ \frac{\partial l}{\partial p} \\ \frac{\partial l}{\partial \mu} \\ \frac{\partial l}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial l}{\partial \beta} \\ \frac{\partial l}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial l}{\partial \beta} \\ \frac{\partial l}{\partial \gamma} \\ \frac{\partial l}{\partial \zeta} \end{pmatrix} = \begin{pmatrix} \frac{\partial l}{\partial \beta} \\ \frac{\partial l}{\partial \gamma} \\ \frac{\partial l}{\partial \delta} \\ \frac{\partial l}{\partial \omega} \end{pmatrix},$$

we have $\frac{\partial l}{\partial \theta} \xrightarrow{d} N(0, I(\theta_0))$. Or

$$\begin{pmatrix} \frac{\partial l}{\partial \beta} \\ \frac{\partial l}{\partial \eta} \end{pmatrix} \xrightarrow{d} N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, -E \left(\begin{array}{cc|c} \frac{\partial^2 l}{\partial \beta^2} & \frac{\partial^2 l}{\partial \beta \partial \eta^T} & H_0 \\ \frac{\partial^2 l}{\partial \eta \partial \beta} & \frac{\partial^2 l}{\partial \eta \partial \eta^T} & \end{array} \right) \right).$$

As the sample size going to $+\infty$, using the result from partition of multivariate normal distribution,

under H_0 , s_{03} is independent of $\frac{\partial l}{\partial \eta} = \begin{pmatrix} \frac{\partial l}{\partial \gamma} \\ \frac{\partial l}{\partial \zeta} \end{pmatrix} = \begin{pmatrix} \frac{\partial l}{\partial \delta} \\ \frac{\partial l}{\partial \omega} \end{pmatrix}$. Since s_{01} and s_{02} are only functions

of $(\frac{\partial l}{\partial \gamma}, \frac{\partial l}{\partial \zeta})$ and $(\frac{\partial l}{\partial \delta}, \frac{\partial l}{\partial \omega})$, s_{03} is independent of s_{01} and s_{02} . Similarly, it is easy to show s_{01} and s_{02} are independent as well.

For the independence among $\tilde{S}_1, \tilde{S}_2$ and $\tilde{S}_3$, similar arguments can be made. □

Theorem 2 (Size of Hybrid test). *The hypothesis in (8) is*

$$H_0 : p_1 = p_2 \text{ and } h_{Y_1} = h_{Y_2} \text{ vs } H_1 : \text{Either equality fails.}$$

Here, p_1 and p_2 are the population zero proportions for populations 1 and 2, and h_{Y_1} and h_{Y_2} are the pdf for the non-zero components of populations 1 and 2. To test these hypotheses at the level α , the Hybrid procedure involves the following steps:

1. Test $H_{01} : p_1 = p_2$ versus $H_{11} : p_1 \neq p_2$.
2. H_{01} is rejected if the p -value is $p < \alpha_1$. In this case, reject H_0 in (8); otherwise, test $H_{02} : h_{Y_1} = h_{Y_2}$ versus $H_{12} : h_{Y_1} \neq h_{Y_2}$.
3. H_{02} is rejected if the p -value is $p < \alpha_2$. In this case, reject H_0 in (8); otherwise, do not reject H_0 .

The size of Hybrid is less than or equal to $\alpha_1 + \alpha_2$.

Proof. Define

$$A = \{(p_1, p_2) \in (0, 1)^2 : p_1 = p_2\} \text{ and } B = \{h_{Y_1}, h_{Y_2} \in \mathcal{F} : h_{Y_1} = h_{Y_2}\}$$

where $\mathcal{F}$ is the space of a particular family of pdf. Under the set notation, $H_0, H_1, H_{01}, H_{11}, H_{02}$ and H_{12} can be expressed as $H_0 : A \cap B, H_1 : A^c \cup B^c, H_{01} : A, H_{11} : A^c, H_{02} : A \cap B$ and $H_{12} : A \cap B^c$.

Further define $\mathfrak{R}_1, \mathfrak{R}_2$ and $\mathfrak{R}$ as rejection region for H_{01}, H_{02} and H_0 , then

$$C = \{\mathbf{x} : \mathbf{x} \in \mathfrak{R}_1\} \text{ and } D = \{\mathbf{x} : \mathbf{x} \in \mathfrak{R}_2\},$$

then we have

$$C \cup (C^c \cap D) = \{\mathbf{x} : \mathbf{x} \in \mathfrak{R}\}.$$

It is reasonable to assume A and B are independent because the form of h_Y does not depend on p , neither the other way around. Also, it is reasonable to assume B and C are independent because C only concerns the p 's.

Let α_0 , α_1 and α_2 denote the type I error rates of H_0 vs H_1 , H_{01} vs H_{11} and H_{02} vs H_{12} , respectively.

Then

$$\alpha_0 = P(\text{Reject } H_0 | H_0 \text{ is true}) = P(C \cup (C^c \cap D) | A \cap B),$$

$$\alpha_1 = P(C | A) \text{ and } \alpha_2 = P(D | A \cap B).$$

Since $C \cap (C^c \cap D) = \emptyset$, we have $(C | A \cap B) \cap (C^c \cap D | A \cap B) = \emptyset$. Therefore,

$$P(C \cup (C^c \cap D) | A \cap B) = P(C | A \cap B) + P((C^c \cap D) | A \cap B).$$

Since $C^c \cap D \subset D$, there is $C^c \cap D | A \cap B \subset D | A \cap B$. Therefore,

$$P((C^c \cap D) | A \cap B) \leq P(D | A \cap B) = \alpha_2.$$

$$P(C | A \cap B) = \frac{P(C \cap A \cap B)}{P(A \cap B)} = \frac{P(B | A \cap C) \cdot P(A \cap C)}{P(B | A) \cdot P(A)} = P(C | A) \cdot \frac{P(B | A \cap C)}{P(B | A)}$$

Because we assume A and B are independent and B and C are independent, $P(B | A) = P(B)$ and $P(B | A \cap C) = P(B)$. Hence,

$$P(C | A \cap B) = P(C | A) = \alpha_1$$

As a result,

$$\alpha_0 = P(C \cup (C^c \cap D) | A \cap B) = P(C | A \cap B) + P((C^c \cap D) | A \cap B) \leq \alpha_1 + \alpha_2.$$

Under the Bonferroni correction, $\alpha_1 = \alpha_2 = \alpha/2$. □

Reference

Justus Seely. Linear Model Theory Notes, unpublished. *Department of Statistics, Oregon State University.*