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SHARP SPECTRAL BOUNDS FOR COMPLEX PERTURBATIONS OF THE INDEFINITE LAPLACIAN

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ABSTRACT. We derive quantitative bounds for eigenvalues of complex perturbations of the indefinite Laplacian on the real line. Our results substantially improve existing results even for real potentials. For L^1 -potentials, we obtain optimal spectral enclosures which accommodate also embedded eigenvalues, while our result for L^p -potentials yield sharp spectral bounds on the imaginary parts of eigenvalues of the perturbed operator for all $p \in [1, \infty)$. The sharpness of the results are demonstrated by means of explicit examples.

1. INTRODUCTION AND MAIN RESULTS

Spectral estimates for non-self-adjoint Schrödinger operators and related questions have been a very active area of mathematical research recently. By now several robust methods have been developed and the corresponding literature is extensive. To name a few, we mention [9–16, 19, 22–28, 31, 33, 36, 36–38, 42] and also [17, 20, 21] for related recent results obtained in an abstract operator theoretic setting.

To goal of this paper is to establish sharp spectral estimates for the operator

$$H_V := \operatorname{sgn}(x)(-\partial_x^2 + V) \quad \text{in} \quad L^2(\mathbb{R}). \quad (1.1)$$

The unperturbed operator H_0 is not symmetric in the Hilbert space $L^2(\mathbb{R})$ due to the sign change of the weight function; it can be interpreted as a self-adjoint operator with respect to the Krein space inner product $(\operatorname{sgn} \cdot, \cdot)$ in $L^2(\mathbb{R})$. The sum in (1.1) is defined in the form sense (see Section 2 for details)

Unlike its definite counterpart (i.e. without the sign function), the existing literature on eigenvalue bounds for operators of type (1.1) seems to be much more sparse even in the one-dimensional setting with real potentials, see *e.g.* [4, 6, 8], and a number of interesting questions remain open. One of them is a conjecture of Behrndt in [3] according to which the eigenvalues of singular indefinite Sturm-Liouville operators (*i.e.* (1.1) with real-valued V) accumulate to the real axis whenever the eigenvalues of the corresponding definite Sturm-Liouville operator accumulate to the bottom of the essential spectrum from below. So far the conjecture of Behrndt has been confirmed only for a particular family of (shifted Coulomb type) potentials in [39].

The present paper is partially motivated by the recent work [7], where the following (non optimal) indefinite analogue of the celebrated result of [1] was shown: every non-real eigenvalue λ of the singular indefinite Sturm-Liouville operator with a real-valued potential $V \in L^1(\mathbb{R})$ obeys the estimate

$$|\lambda|^{1/2} \leq \|V\|_1. \quad (1.2)$$

Our standing assumption is that there exist numbers $a \in (0, 1)$ and $b \in \mathbb{R}$ such that, for all $\psi \in H^1(\mathbb{R})$,

$$\int_{\mathbb{R}} |V| |\psi|^2 \leq a \int_{\mathbb{R}} |\psi'|^2 + b \int_{\mathbb{R}} |\psi|^2. \quad (1.3)$$

In particular, potentials $V \in L^p(\mathbb{R})$ are covered by this hypothesis for all $p \in [1, \infty]$, see Section 2 for more details. Furthermore, in the same way, it is also possible to proceed in a greater generality and give a meaning to the distributional Dirac delta potential δ , which is explicitly solvable. Putting $\|\delta\|_{L^1(\mathbb{R})} := 1$ by convention, Theorem 1 remains valid in this more general setting. The feature of our results are that we work with complex potentials of minimal regularity, our bounds are quantitative and substantially improve existing results. Moreover, for L^1 -potentials our spectral enclosures are sharp and we cover embedded eigenvalues, too.

Throughout the paper we denote by $\|\cdot\|_p$ the standard L^p -norm for $p \in [1, \infty]$. Our first result quantitatively improves the known bound (1.2) of [7] and holds also for possibly embedded eigenvalues.

Theorem 1. *For $V \in L^1(\mathbb{R})$, every eigenvalue λ of H_V satisfies*

$$\sqrt{2}|\lambda| \leq \sqrt{|\lambda| + |\Re(\lambda)|} \|V\|_1. \quad (1.4)$$

Remark 1. (i) The bound in (1.4) is sharp in the sense that, for any $Q > 0$ and any point $\lambda \in \mathbb{C} \setminus \mathbb{R}$ which fulfills the equation

$$2|\lambda|^2 = (|\lambda| + |\Re(\lambda)|) Q^2,$$

there exists $V \in L^1(\mathbb{R})$ such that $Q = \|V\|_1$ and λ is an eigenvalue of the corresponding operator H_V , see Section 5.2. This means that every boundary point of the spectral enclosure corresponding to (1.4), with the exception of points located in the real line, is an eigenvalue of H_V for some L^1 -potential V . Hence the obtained spectral enclosure cannot be squeezed any further.

- (ii) It readily follows from Theorem 1 that the intervals $(-\infty, \|V\|_1^2)$ and $(\|V\|_1^2, \infty)$ are free of embedded eigenvalues of H_V for $V \in L^1(\mathbb{R})$.
- (iii) In view of the elementary inequality $|\Re(\lambda)| \leq |\lambda|$, it is obvious that the bound (1.4) provides a strictly tighter spectral enclosure than (1.2).

Corollary 1. *For $V \in L^1(\mathbb{R})$, every eigenvalue λ of H_V satisfies the sharp bound*

$$|\Im(\lambda)| \leq \frac{3\sqrt{3}}{8} \|V\|_1^2. \quad (1.5)$$

The spectral enclosure corresponding to (1.4) is a compact set, which is symmetric with respect to both the real and the imaginary axes. The geometry of its boundary is quite easy to understand. We refer to Figure 1 for the plots of the boundary curves corresponding to (1.2) and (1.4) with $\|V\|_1 = 1$.

Our next result provides a spectral estimate in terms of the L^p -norm of the potential for $p \in (1, \infty)$. The proof interpolates the bound of Theorem 1 with the standard operator norm bound for the resolvent of a self-adjoint operator. The strategy was first used in [27].

Theorem 2. *Let $p \in (1, \infty)$. For $V \in L^p(\mathbb{R})$, every eigenvalue λ of H_V satisfies*

$$2^{\frac{3}{2p}-1} |\lambda|^{\frac{1}{p}} |\Im(\lambda)|^{1-\frac{1}{p}} \leq (|\lambda| + |\Re(\lambda)|)^{\frac{1}{2p}} \|V\|_p. \quad (1.6)$$

FIGURE 1. The blue circle and the red curve on the left correspond to the boundary curves of the spectral enclosures respectively given by (1.2) and (1.4) for $\|V\|_{L^1(\mathbb{R})} = 1$. The picture on the right illustrates plots of the expanding boundary curves corresponding to the spectral enclosure (1.4) for $\|V\|_1 = (1.35)^j/7$, $j = 1, 2, \dots, 9$.

Remark 2. (i) The bound (1.6) trivially holds for embedded eigenvalues. Letting $p \searrow 1$ in (1.6), one arrives at the bound (1.4). An analogous version of the bound (1.6) for the case $p = \infty$ is very easy to derive and reads as

$$|\Im(\lambda)| \leq 2\|V\|_\infty.$$

(ii) Using the elementary inequalities $|\Re(\lambda)| \leq |\lambda|$ and $|\Im(\lambda)| \leq |\lambda|$, one can easily read off from (1.6) that the following rough bound holds

$$|\Im(\lambda)| \leq 2^{\frac{2p-2}{2p-1}} \|V\|_p^{\frac{2p}{2p-1}}. \quad (1.7)$$

This improves the recent result in [40] where, for real-valued $V \in L^p(\mathbb{R})$ with $p \geq 2$, the estimate (1.7) was shown to hold with a constant factor $C(p)$ such that $C(p) > 2^{\frac{2p-2}{2p-1}}$. While the result of [40] is a considerable improvement of the analogous result of [8], the function $C(p)$ therein is a strictly decreasing function of the parameter p with the limit equal to $2 + \sqrt{2}$ at infinity.

As a matter of fact, by an optimization trick under the constraint (1.6), one can substantially improve the bound (1.7), thus generalizing the sharp bound (1.5) for L^p -potentials for all $p \in (1, \infty)$.

Corollary 2. *Let $p \in (1, \infty]$. For $V \in L^p(\mathbb{R})$, every eigenvalue λ of H_V satisfies the bound*

$$|\Im(\lambda)| \leq 2 \left(\frac{3\sqrt{3}}{16} \right)^{\frac{1}{2p-1}} \|V\|_p^{\frac{2p}{2p-1}}. \quad (1.8)$$

Below we provide various plots of the expanding boundary curves corresponding to the spectral enclosure from Theorem 2.

The next result concerns with spectral bounds for L^p -potentials again. This time, however, it is given in terms of the L^p -sizes of the restrictions to the intervals $(0, \infty)$ and $(-\infty, 0)$

FIGURE 2. The plots of the expanding boundary curves corresponding to the spectral enclosure from Theorem 2 for $p = 1.25$ and $\|V\|_p = (1.25)^j/10$, $j = 1, 2, \dots, 10$.

of the potential. In the sequel, for $p \in (1, \infty)$, we denote by $\|V\|_{p,\pm}$ the standard L^p -norms over the intervals $(0, \infty)$ and $(-\infty, 0)$, respectively.

Theorem 3. *Let $p \in (1, \infty)$. For $V \in L^p(\mathbb{R})$, every non-real eigenvalue of H_V satisfies*

$$\sqrt{2} \left(\frac{p}{p-1} \right)^{1-\frac{1}{p}} \sqrt{|\lambda|} \leq \max \left\{ \frac{\|V\|_{p,-}}{|\Re(\sqrt{\lambda})|^{1-\frac{1}{p}}} + \frac{\sqrt{2}\|V\|_{p,+}}{|\Im(\sqrt{\lambda})|^{1-\frac{1}{p}}}, \frac{\sqrt{2}\|V\|_{p,-}}{|\Re(\sqrt{\lambda})|^{1-\frac{1}{p}}} + \frac{\|V\|_{p,+}}{|\Im(\sqrt{\lambda})|^{1-\frac{1}{p}}} \right\}. \quad (1.9)$$

Remark 3. (i) In general, neither of the spectral bounds of Theorems 2 and 3 is better than the other. Numerical experiments, however, indicate that the one of Theorem 2 is better than that of Theorem 3 for sufficiently small values of $p > 1$, while the opposite is true for sufficiently large values of p .

(ii) Unlike that of Theorem 1, the spectral enclosures of Theorems 2 and 3 are non-compact sets, yet being symmetric with respect to both the real and the imaginary axes.

Note that the bounds (1.2), (1.4)–(1.6) are scale-invariant. More precisely, if U_ρ is the unitary operator $U_\rho f(x) = \rho^{-1/2} f(x/\rho)$, $\rho > 0$, then

$$U_\rho^* H_V U_\rho = \rho^{-2} \operatorname{sgn}(x) (-\partial_x^2 + V_\rho), \quad V_\rho(x) = \rho^2 V(\rho x), \quad (1.10)$$

has the same eigenvalues as H_V . Hence,

$$\lambda \in \sigma_p(H_V) \iff \rho^2 \lambda \in \sigma_p(H_{V_\rho}).$$

Since $\|V_\rho\|_p = \rho^{2-1/p} \|V\|_p$ and $|\cdot|$, $\Im(\cdot)$, $\Re(\cdot)$ are homogeneous of degree one, applying any of the inequalities (1.2), (1.4)–(1.6) to H_{V_ρ} instead of H_V does not change the outcome.

Next, by a Wigner-von Neumann type example we show that the scale-invariant inequality

$$|\lambda|^{p-\frac{1}{2}} \leq C \|V\|_p^p, \quad (1.11)$$

which holds for $p = 1$ by (1.2), cannot hold for $p > 1$. The idea to consider complex-valued Wigner-von Neumann potentials in the context of spectral estimates for non-self-adjoint operators is due to Frank and Simon [30].

Theorem 4. *Let $\lambda > 0$ be arbitrary. There exists a positive constant $C = C(\lambda)$ and a sequence of potentials $V_n : \mathbb{R} \rightarrow \mathbb{R}$ such that, for all n , λ is an eigenvalue of H_{V_n} and*

$$|V_n(x)| \leq \frac{C}{n + |x|}, \quad x \in \mathbb{R}. \quad (1.12)$$

In particular, for any $p > 1$, we have

$$\lim_{n \rightarrow \infty} \|V_n\|_p = 0.$$

We now modify the above result to produce a *non-real* eigenvalue instead of an embedded one, showing that the scale-invariant inequality (1.11) also fails in this case. Additionally, we demonstrate that the exponents in (1.6) cannot be improved.

Theorem 5. *Given $\varepsilon > 0$ sufficiently small and $\mu > 0$, there exists a potential $V = V(\varepsilon, \mu) \in L^\infty(\mathbb{R})$ with compact support and such that H_V has eigenvalue $\lambda = \lambda(\varepsilon, \mu)$ with $\Re(\lambda) = \mu(1 + \mathcal{O}(\varepsilon))$, $\Im(\lambda) = \mu\varepsilon(2 + \mathcal{O}(\varepsilon^2))$, and for every $p \geq 1$,*

$$\|V(\varepsilon, \mu)\|_p \approx |\mu|^{1 - \frac{1}{2p}} \varepsilon^{1 - \frac{1}{p}} |\ln \varepsilon|^{\frac{1}{p}}. \quad (1.13)$$

In particular, for $p > 1$, we have

$$\lim_{\varepsilon \rightarrow 0} \|V(\varepsilon, \mu)\|_p = 0.$$

Remark 4. Theorem 5 can be reformulated in the following way: Consider the sector

$$\Sigma_{\varepsilon_0} := \{\lambda \in \mathbb{C} : |\Im(\lambda)| \leq \varepsilon_0 |\Re(\lambda)|\},$$

where ε_0 is sufficiently small but fixed. Then, given λ in the intersection of Σ_{ε_0} with the first quadrant, there exists a potential $V = V(\lambda) \in L_c^\infty(\mathbb{R})$ such that H_V has eigenvalue λ and, for every $p \geq 1$,

$$\|V(\lambda)\|_p \approx |\lambda|^{\frac{1}{2p}} |\Im(\lambda)|^{1 - \frac{1}{p}} \left| \ln \left(\frac{|\Im(\lambda)|}{|\lambda|} \right) \right|^{\frac{1}{p}}.$$

This is easily seen by observing that we can find $\lambda = \lambda(\varepsilon, \mu)$ as in the theorem with

$$\mu = \Re(\lambda)(1 + \mathcal{O}(\varepsilon_0)), \quad \varepsilon = \frac{\Im(\lambda)}{2\Re(\lambda)}(1 + \mathcal{O}(\varepsilon_0)), \quad |\lambda| = \Re(\lambda)(1 + \mathcal{O}(\varepsilon_0)).$$

It is also clear that the assumption of λ lying in the first quadrant can be omitted.

In the next result we study the weak coupling limit, i.e. we replace V by εV in H_V and establish existence, uniqueness and asymptotics of an eigenvalue $\lambda(\varepsilon)$ as $\varepsilon \rightarrow 0^+$. This will yield another confirmation that the spectral bound (1.4) is sharp. In the following we set

$$v_+ = \int_0^\infty V(x) dx, \quad v_- = \int_{-\infty}^0 V(x) dx, \quad v_{\text{sgn}} = \int_{-\infty}^\infty \text{sgn}(x) V(x) dx.$$

Theorem 6. *Assume that $\Re(v_{\text{sgn}}) + \Im(v_{\text{sgn}}) < 0$ and $\Re(v_{\text{sgn}}) < \Im(v_{\text{sgn}})$. Then, for all sufficiently small $\varepsilon > 0$, there exists a unique eigenvalue $\lambda(\varepsilon) \in \mathbb{C}^+$ of $H_{\varepsilon V}$ satisfying*

$$\lambda(\varepsilon) = \frac{\varepsilon^2}{(1 - i)^2} v_{\text{sgn}}^2 + o(\varepsilon^2), \quad \varepsilon \rightarrow 0^+.$$

Our next result is a particular case of a Lieb-Thirring type bound. This is an analogue to a special case of [27, Thm 1.3] for the definite Schrödinger operator. In fact, the conclusions of [27, Thm 1.2] and [27, Thm 1.3] in the case $d = 1$ there continue to hold for the indefinite operator H_V considered here.

Theorem 7. *For $V \in L^1(\mathbb{R})$, we have*

$$\sum_j |\Im(\lambda_j)| \leq C \|V\|_1^2,$$

where λ_j are the eigenvalues of H_V repeated according to their algebraic multiplicities.

We use the method of [29] to bound the number of eigenvalues of H_V in terms of an exponentially weighted L^1 norm of V as follows.

Theorem 8. *The number of eigenvalues $N(V)$ of H_V , counting algebraic multiplicities, satisfies, for any $\varepsilon > 0$,*

$$N(V) \leq \frac{1}{\varepsilon^2} \left(\int_{\mathbb{R}} e^{\varepsilon|x|} |V(x)| dx \right)^2.$$

The outline of the paper is as follows. In Section 2, we rigorously introduce the perturbed operator (1.1) using form methods. In Section 3, we derive a sharp estimate on the integral kernel of the free resolvent as well as a bound on a Krein type resolvent. In Section 4, we establish a limiting absorption principle; this can be viewed as a one-sided conventional Birman-Schwinger principle which holds also for embedded eigenvalues. The proofs of Theorems 1-8 and Corollaries 1, 2 are given in Section 5.

2. DEFINITION OF THE OPERATOR H_V

Let T_0 be the self-adjoint operator in $L^2(\mathbb{R})$ associated with the quadratic form

$$t_0[\psi] := \int_{\mathbb{R}} |\psi'|^2, \quad D(t_0) := H^1(\mathbb{R}). \quad (2.1)$$

One has $D(T_0) = H^2(\mathbb{R})$ and $T_0 = -\partial_x^2$. The spectrum of T_0 is purely absolutely continuous and coincides with the semi-axis $[0, +\infty)$.

Let v be a quadratic form in $L^2(\mathbb{R})$, which is relatively bounded with respect to t_0 with the relative bound less than one. That is, $D(v) \supset H^1(\mathbb{R})$ and there exist numbers $a \in (0, 1)$ and $b \in \mathbb{R}$ such that, for all $\psi \in H^1(\mathbb{R})$,

$$|v[\psi]| \leq a \int_{\mathbb{R}} |\psi'|^2 + b \int_{\mathbb{R}} |\psi|^2. \quad (2.2)$$

Then the sum $t_V := t_0 + v$ is a closed sectorial form with $D(t_V) = H^1(\mathbb{R})$, which gives rise to an m-sectorial operator T_V in $L^2(\mathbb{R})$ via the representation theorem (cf. [35, Thm. VI.2.1]).

For example, if $V \in L^1_{\text{loc}}(\mathbb{R})$ is such that

$$v[\psi] := \int_{\mathbb{R}} V |\psi|^2, \quad D(v) := \left\{ \psi \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |V| |\psi|^2 < \infty \right\},$$

verifies (2.2) (which coincides with (1.3) in this case), then we write $T_V = T_0 + V$.

Let us now discuss sufficient conditions which guarantee (2.2).

By the Sobolev embedding theorem ([2, Thm. 5.4]), every function $\psi \in H^1(\mathbb{R})$ is bounded and continuous. More specifically (cf. [41, Theorem IX.28]), for any positive α there is $\beta \in \mathbb{R}$ such that, for all $\psi \in H^1(\mathbb{R})$,

$$\|\psi\|_{\infty} \leq \alpha \|\psi'\|_2 + \beta \|\psi\|_2. \quad (2.3)$$

Consequently, any potential $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ satisfies (1.3) with the relative bound equal to zero (*i.e.* a can be chosen arbitrarily small).

3. FREE RESOLVENT AND A KREIN TYPE RESOLVENT FORMULA

For real-valued L^p -potentials, the perturbed operator can be viewed as a self-adjoint operator in the Krein space with the indefinite inner product $(\operatorname{sgn} \cdot, \cdot)$. Consequently, the spectrum of the perturbed operator is symmetric with respect to the real axis, see *e.g.* [18, 34]. Certainly, this property no longer holds for general complex-valued potentials. As it was mentioned in the Introduction, it turns out, however, that our spectral enclosures are symmetric with respect to the both the real and the imaginary axes. Unless specified otherwise, for the rest of the paper we will work with a fixed spectral parameter λ from the *upper half-plane* $\mathbb{C}^+$ since the analysis of the lower part of the spectrum of H_V is identical to that of the part in the upper half-plane.

3.1. Free resolvent. In the Hilbert space $L^2(\mathbb{R})$, let us consider the unperturbed operator

$$H_0 := \operatorname{sgn}(x)(-\partial_x^2), \quad \mathcal{D}(H_0) := H^2(\mathbb{R}).$$

The spectrum of H_0 is continuous and coincides with $\mathbb{R}$. For $\lambda \in \mathbb{C}^+$, we denote by G_λ the *Green's function* of $H_0 - \lambda$, *i.e.* the integral kernel of the free resolvent $(H_0 - \lambda)^{-1}$, which can be determined explicitly using the well-known form of the Green's function for the definite counterpart of H_0 . In fact, we have¹

$$G_\lambda(x, y) = \frac{1}{2\alpha\sqrt{\lambda}} \begin{cases} \alpha e^{i\sqrt{\lambda}(x+y)} + \bar{\alpha} e^{i\sqrt{\lambda}|x-y|} & x \geq 0, y \geq 0, \\ -e^{\sqrt{\lambda}(ix+y)} & x \geq 0, y < 0, \\ e^{\sqrt{\lambda}(x+iy)} & x < 0, y \geq 0, \\ -\bar{\alpha} e^{\sqrt{\lambda}(x+y)} - \alpha e^{-\sqrt{\lambda}|x-y|} & x < 0, y < 0, \end{cases} \quad (3.1)$$

where $\alpha := \frac{1-i}{2}$ (see also [7]). Observe that, for all non-zero x, y , we have

$$\operatorname{sgn}(x)G_\lambda(x, y) = \operatorname{sgn}(y)G_\lambda(y, x). \quad (3.2)$$

The pointwise estimate of the Green's function obtained in the next lemma plays a crucial role in establishing Theorem 1.

Lemma 1. *Let $\lambda \in \mathbb{C}^+$. The Green's function in (3.1) obeys the sharp pointwise estimate*

$$|G_\lambda(x, y)|^2 \leq \frac{1}{2|\lambda|} + \frac{|\Re(\lambda)|}{2|\lambda|^2}. \quad (3.3)$$

Proof. Let $a := \Re(\sqrt{\lambda}) > 0$ and $b := \Im(\sqrt{\lambda}) > 0$. We distinguish the four cases.

Case $x \geq 0, y \geq 0$. In view of (3.2), there is no loss of generality in assuming that $x \geq y$. Then elementary calculations yield

$$|G_\lambda(x, y)|^2 = \frac{1}{4|\lambda|} \Phi(x, y), \quad (3.4)$$

where

$$\Phi(x, y) := e^{-2b(x+y)} + e^{-2b(x-y)} + 2e^{-2bx} \sin(2ay). \quad (3.5)$$

We fix $y \geq 0$ and consider Φ as a function of the variable x on the interval $[y, \infty)$. Since Φ is non-negative for obvious reasons, and

$$\frac{\partial}{\partial x} \Phi(x, y) = -2b\Phi(x, y) \leq 0, \quad (3.6)$$

¹Here and in the sequel we choose the branch of the square root with $\Im(\sqrt{\lambda}) > 0, \Re(\sqrt{\lambda}) > 0$ ($\lambda \in \mathbb{C}^+$).

we conclude that $\Phi(\cdot, y)$ is non-increasing on $[y, \infty)$. Therefore,

$$\Phi(x, y) \leq \Phi(y, y) = 1 + e^{-4by} + 2e^{-2by} \sin(2ay) =: \varphi(y). \quad (3.7)$$

In view of the identities $\Re(\lambda) = a^2 - b^2$ and $\Im(\lambda) = 2ab$, it thus suffices to show that

$$\max_{y \geq 0} \varphi(y) \leq 2 \left(1 + \frac{|a^2 - b^2|}{a^2 + b^2} \right). \quad (3.8)$$

It is not difficult to check that

$$\varphi'(y) = 0 \iff e^{-2by} = \frac{a}{b} \cos(2ay) - \sin(2ay). \quad (3.9)$$

In view of this, elementary calculations show that, if $y_0 \geq 0$ is a critical point of φ , then it must hold that

$$\varphi(y_0) = \left(1 + \frac{a^2}{b^2} \right) \cos^2(2ay_0). \quad (3.10)$$

Let $t_0 := \cos(2ay_0)$. If $t_0 \leq 0$, then (3.9) implies that $\sin(2ay_0) < 0$. Then $\varphi(y_0) < 1 + e^{-4by_0} \leq 2$ and (3.8) holds. If $b \geq at_0$, then $\varphi(y_0) \leq 1 + t_0^2 \leq 2$ and (3.8) follows. If $\sin(2ay_0) < 0$, then again $\varphi(y_0) < 2$, implying (3.8). So let us assume that $t_0 > 0$, $at_0 > b$ and $\sin(2ay_0) \geq 0$. Then $\sin(2ay_0) = \sqrt{1 - t_0^2}$ and (3.9) implies that

$$0 < \frac{at_0}{b} - 1 \leq \sqrt{1 - t_0^2}. \quad (3.11)$$

Squaring both sides of the latter, we conclude

$$0 < t_0 \leq \frac{2ab}{a^2 + b^2}. \quad (3.12)$$

On the other hand, the condition $at_0 > b$ together with (3.12) imply that $b \leq a$. Therefore, in this case, we have

$$\varphi(y_0) = t_0^2 \left(1 + \frac{a^2}{b^2} \right) \leq \frac{4a^2}{a^2 + b^2} = 2 \left(1 + \frac{|a^2 - b^2|}{a^2 + b^2} \right). \quad (3.13)$$

Finally, by noticing that $\varphi(y) \rightarrow 2$ as $y \downarrow 0$ and that $\varphi(y) \rightarrow 1$ as $y \uparrow +\infty$, and summing up the above observations, we conclude (3.8).

Case $x \geq 0, y < 0$. In this case, we have

$$|G_\lambda(x, y)| = \frac{e^{ay-bx}}{2|\alpha|\sqrt{|\lambda|}} \leq \frac{1}{\sqrt{2|\lambda|}}. \quad (3.14)$$

Case $x < 0, y \geq 0$. In this case, the result follows from the previous step and the observation (3.2).

Case $x < 0, y < 0$. In view of (3.2), there is no loss of generality in assuming that $x \leq y$. Then elementary calculations show that

$$|G_\lambda(x, y)|^2 = \frac{1}{4|\lambda|} \Psi(x, y), \quad (3.15)$$

where

$$\Psi(x, y) := e^{-2a(y-x)} + e^{2a(y+x)} - 2e^{2ax} \sin(2by). \quad (3.16)$$

In the same way as in the first case, by fixing $y \geq 0$ and considering Ψ as a function of the variable x on the interval $(-\infty, y]$, we come to the conclusion that

$$\max_{x \leq y \leq 0} \Psi(x, y) \leq 2 \max \left(1 + \frac{|\Re(\lambda)|}{|\lambda|} \right), \quad (3.17)$$

completing the proof. $\square$

3.2. Krein resolvent formula. Let $B_{\pm} = \mp \partial_x^2$ on $L^2(\mathbb{R}_{\pm})$ with Dirichlet boundary conditions at the origin. We set

$$B_0 := \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix} \quad \text{on} \quad L^2(\mathbb{R}) = L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_-)$$

and

$$F_{\lambda} \psi := \frac{1}{i(\sqrt{\lambda} + \sqrt{-\lambda})} (J f_{\lambda}, \psi) f_{\lambda},$$

where $J = \text{sgn}(\cdot)$ and

$$f_{\lambda}(x) := e^{i\sqrt{\lambda}} \mathbf{1}_{x>0} + e^{-i\sqrt{-\lambda}} \mathbf{1}_{x<0}.$$

We use the following Krein type resolvent formula (see [5]):

$$(H_0 - \lambda)^{-1} = (B_0 - \lambda)^{-1} - F_{\lambda}. \quad (3.18)$$

The operator F_{λ} has rank one, and its unique eigenvalue is given by

$$\mu(\lambda) = \frac{1}{2i(\sqrt{\lambda} + \sqrt{-\lambda})} \left(\frac{1}{\Im(\sqrt{\lambda})} - \frac{1}{\Im(\sqrt{-\lambda})} \right).$$

We then have

$$\|F_{\lambda}\| = \|F_{\lambda}\|_{\text{tr}} = |\mu(\lambda)| \leq \frac{1}{|\Im(\lambda)|},$$

Together with the self-adjointness of B_0 and (3.18) this implies that

$$\|(H_0 - \lambda)^{-1}\| \leq \frac{2}{|\Im(\lambda)|}. \quad (3.19)$$

4. LIMITING ABSORPTION TYPE PRINCIPLE

The main role in the proofs of Theorems 1-3 is played by the Birman-Schwinger operator

$$K_{\lambda} := |V|^{1/2} (H_0 - \lambda)^{-1} V_{1/2} \quad \text{with} \quad V_{1/2} := |V|^{1/2} \text{sgn}(V), \quad (4.1)$$

where $\text{sgn} : \mathbb{C} \rightarrow \mathbb{C}$ is the complex signum function defined by $\text{sgn}(z) := z/|z|$ for $z \neq 0$ with the convention $\text{sgn}(0) := 0$. The operator K_{λ} is well defined on its natural domain of the composition of three operators for all $\lambda \in \mathbb{C}$. Furthermore, we have a useful formula for its integral kernel

$$K_{\lambda}(x, y) = |V|^{1/2}(x) G_{\lambda}(x, y) V_{1/2}(y), \quad (4.2)$$

where G_{λ} is the Green's function of $H_0 - \lambda$.

The following lemma can be considered as a one-sided version of the conventional Birman-Schwinger principle extended to possibly embedded eigenvalues. Its proof is heavily inspired by the ones of the analogous results in [25, 30, 32].

Lemma 2. *Assume that $V \in L^1(\mathbb{R})$. Let $\lambda \in \sigma_p(H_V) \cap (\mathbb{C}^+ \cup \mathbb{R})$ and $\psi \in \mathcal{D}(H_V)$ be an associated eigenvector. Then $\phi := |V|^{1/2} \psi \in L^2(\mathbb{R})$ and*

$$\forall \varphi \in L^2(\mathbb{R}), \quad \lim_{\varepsilon \rightarrow 0^+} (\varphi, K_{\lambda+i\varepsilon} \phi) = -(\varphi, \phi). \quad (4.3)$$

In particular,

$$\liminf_{\varepsilon \rightarrow 0^+} \|K_{\lambda+i\varepsilon}\| \geq 1. \quad (4.4)$$

Proof. It readily follows from (1.3) that $\phi \in L^2(\mathbb{R})$. We fix $\varphi \in L^2(\mathbb{R})$. Given any $\lambda \in \sigma_p(H_V)$, we have $z := \lambda + i\varepsilon \in \mathbb{C}^+ \subset \rho(H_0)$ for all $\varepsilon > 0$. Furthermore,

$$\begin{aligned} (\varphi, K_z \phi) &= \iint_{\mathbb{R} \times \mathbb{R}} \overline{\varphi(x)} |V|^{1/2}(x) G_z(x, y) V(y) \psi(y) dx dy \\ &= \int_{\mathbb{R}} \eta_\varepsilon(y) V(y) \psi(y) \operatorname{sgn}(y) dy, \end{aligned} \quad (4.5)$$

where

$$\eta_\varepsilon := \operatorname{sgn}(\cdot) \left(\int_{\mathbb{R}} \overline{\varphi(x)} |V|^{1/2}(x) G_z(x, \cdot) dx \right) = (H_0 - z)^{-1} |V|^{1/2} \overline{\varphi} \operatorname{sgn}.$$

Here the second equality holds due to the (anti-)symmetric property of the Green's function in (3.1): $G_z(x, y) = G_z(y, x)$ if $\operatorname{sgn}(x) = \operatorname{sgn}(y)$ and $G_z(x, y) = -G_z(y, x)$ if $\operatorname{sgn}(x) = -\operatorname{sgn}(y)$. By the Cauchy-Schwarz inequality, $|V|^{1/2} \overline{\varphi} \in L^2(\mathbb{R})$. Since $z \notin \sigma(H_0)$, we have $\eta_\varepsilon \in \mathcal{D}(H_0) \in H^1(\mathbb{R})$, and the weak formulation of the eigenvalue equation $H_V \psi = \lambda \psi$ yields

$$\begin{aligned} \int_{\mathbb{R}} \eta_\varepsilon(y) V(y) \psi(y) \operatorname{sgn}(y) dy &= -(\overline{\eta_\varepsilon}', \psi') + \lambda (\overline{\eta_\varepsilon}, \psi \operatorname{sgn}) \\ &= -(\overline{\psi}', \eta_\varepsilon') + \lambda (\overline{\psi} \operatorname{sgn}, \eta_\varepsilon) \\ &= -(\overline{\psi}', \eta_\varepsilon') + z (\overline{\psi} \operatorname{sgn}, \eta_\varepsilon) - i\varepsilon (\overline{\psi} \operatorname{sgn}, \eta_\varepsilon) \\ &= -(\overline{\psi}, |V|^{1/2} \overline{\varphi}) - i\varepsilon (\overline{\psi} \operatorname{sgn}, \eta_\varepsilon) \\ &= -(\varphi, |V|^{1/2} \psi) - i\varepsilon (\overline{\eta_\varepsilon}, \psi \operatorname{sgn}). \end{aligned} \quad (4.6)$$

Here the penultimate equality follows from the weak formulation of the resolvent equation $(H_0 - z)\eta_\varepsilon = |V|^{1/2} \overline{\varphi} \operatorname{sgn}$. Consequently, (4.5) and (4.6) imply (4.3) after taking the limit $\varepsilon \rightarrow 0^+$, provided that $\varepsilon (\overline{\eta_\varepsilon}, \psi \operatorname{sgn}) \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. To see the latter, we write

$$|(\overline{\eta_\varepsilon}, \psi \operatorname{sgn})| = |(\varphi \operatorname{sgn}, M_\varepsilon \psi \operatorname{sgn})| \leq \|\varphi\| \|M_\varepsilon\| \|\psi\|,$$

where $M_\varepsilon := |V|^{1/2} (H_0 - z)^{-1}$, and it remains to be shown that $\varepsilon \|M_\varepsilon\|$ tends to zero as $\varepsilon \rightarrow 0^+$. To this end, first we notice that

$$\begin{aligned} \sup_{x \geq 0} \int_{\mathbb{R}} |G_z(x, y)|^2 dy &\leq \sup_{x \geq 0} \int_{-\infty}^0 |G_z(x, y)|^2 dy + \sup_{x \geq 0} \int_0^\infty |G_z(x, y)|^2 dy \\ &\leq \frac{1}{2|z|} \int_{-\infty}^0 e^{2y \Re(\sqrt{z})} dy + \frac{1}{|z|} \int_0^\infty e^{-2y \Im(\sqrt{z})} dy \\ &= \frac{1}{4|z|} \left[\frac{1}{\Re(\sqrt{z})} + \frac{2}{\Im(\sqrt{z})} \right] \end{aligned} \quad (4.7)$$

and that

$$\begin{aligned} \sup_{x \leq 0} \int_{\mathbb{R}} |G_z(x, y)|^2 dy &\leq \sup_{x \leq 0} \int_{-\infty}^0 |G_z(x, y)|^2 dy + \sup_{x \leq 0} \int_0^\infty |G_z(x, y)|^2 dy \\ &\leq \frac{1}{|z|} \int_{-\infty}^0 e^{2y \Re(\sqrt{z})} dy + \frac{1}{2|z|} \int_0^\infty e^{-2y \Im(\sqrt{z})} dy \\ &= \frac{1}{4|z|} \left[\frac{2}{\Re(\sqrt{z})} + \frac{1}{\Im(\sqrt{z})} \right]. \end{aligned} \quad (4.8)$$

Therefore, it follows that

$$\begin{aligned}
\|M_\varepsilon\|^2 &\leq \|M_\varepsilon\|_{\text{HS}}^2 = \iint_{\mathbb{R} \times \mathbb{R}} |V(x)| |G_z(x, y)|^2 dx dy \\
&\leq \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |G_z(x, y)|^2 dy \int_{\mathbb{R}} |V(x)| dx \\
&\leq \frac{1}{2|z|} \left[\frac{1}{\Re(\sqrt{z})} + \frac{1}{\Im(\sqrt{z})} \right] \int_{\mathbb{R}} |V(x)| dx.
\end{aligned} \tag{4.9}$$

On the other hand, elementary calculations show that

$$|z| \Re(\sqrt{z}) \sim \begin{cases} \varepsilon^{1/2} & \text{if } \lambda = 0, \\ 1 & \text{otherwise,} \end{cases} \tag{4.10}$$

while

$$|z| \Im(\sqrt{z}) \sim \begin{cases} \varepsilon^{1/2} & \text{if } \lambda = 0, \\ \varepsilon & \text{if } \Re(\lambda) > 0 \text{ \& } \Im(\lambda) = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{4.11}$$

Hence, we have $\|M_\varepsilon\| = \mathcal{O}(|\varepsilon|^{-3/4})$ as $\varepsilon \rightarrow 0^+$, which concludes the proof of the claim in (4.3).

Now applying (4.3) with $\varphi = \phi$ and taking the limit $\varepsilon \rightarrow 0^+$, we obtain obtain

$$|(\phi, \phi)| \leq |(\phi, K_{\lambda+i\varepsilon}\phi)| \leq \liminf_{\varepsilon \rightarrow 0^+} \|K_{\lambda+i\varepsilon}\| \|\phi\|^2. \tag{4.12}$$

Since $\phi = |V|^{1/2}\psi \neq 0$ (if this were not true, then λ would be an eigenvalue for H_0 , unless $\psi = 0$, which is impossible), we thus conclude (4.4). $\square$

5. PROOFS OF THE MAIN RESULTS

Without loss of generality we work with fixed $\lambda \in \mathbb{C}^+ \cup \mathbb{R}$ in the sequel. The key strategy of the proof is to estimate the norm the Birman-Schwinger operator from above and apply Lemma 2.

5.1. Proof of Theorem 1. First, we consider the case $\lambda \in \mathbb{C}^+$. In view of Lemma 1, we can estimate the norm of the Birman-Schwinger operator as follows

$$\begin{aligned}
\|K_\lambda\|^2 &\leq \|K_\lambda\|_{\text{HS}}^2 = \iint_{\mathbb{R} \times \mathbb{R}} |V(x)| |G_\lambda(x, y)|^2 |V(y)| dx dy \\
&\leq \frac{1}{2} \left(\frac{1}{|\lambda|} + \frac{|\Re(\lambda)|}{|\lambda|^2} \right) \|V\|_1^2.
\end{aligned} \tag{5.1}$$

If $\lambda \in \mathbb{R} \setminus \{0\}$, then the same analysis applied for $\lambda + i\varepsilon$ with $\varepsilon > 0$ (instead of λ) yields

$$\begin{aligned}
\liminf_{\varepsilon \rightarrow 0^+} \|K_{\lambda+i\varepsilon}\|^2 &\leq \liminf_{\varepsilon \rightarrow 0^+} \frac{1}{2} \left(\frac{1}{|\lambda + i\varepsilon|} + \frac{|\Re(\lambda)|}{|\lambda + i\varepsilon|^2} \right) \|V\|_1^2 \\
&= \frac{1}{2} \left(\frac{1}{|\lambda|} + \frac{|\Re(\lambda)|}{|\lambda|^2} \right) \|V\|_1^2.
\end{aligned}$$

Hence, Lemma 2 implies that if $\lambda \in (\mathbb{C}^+ \cup \mathbb{R}) \setminus \{0\}$ is an eigenvalue for H_V , then we must have

$$2 \leq \left(\frac{1}{|\lambda|} + \frac{|\Re(\lambda)|}{|\lambda|^2} \right) \|V\|_1^2,$$

i.e. (1.4) must hold. This completes the proof since (1.4) trivially holds for $\lambda = 0$. $\square$

5.2. Optimality of the eigenvalue bound of Theorem 1. Here we demonstrate that the result (1.4) is sharp in the sense that to any non-real boundary point of the spectral enclosure, there exists a delta-potential V so that this boundary point is an eigenvalue of H_V . By standard approximation arguments, it follows that there exists a sequence of L^1 -potentials V_n such that the eigenvalues of H_{V_n} converge to those of H_V . This shows that the boundary curve in (1.4) cannot be improved.

Let us take an arbitrary non-real boundary point λ of the spectral enclosure. Since the boundary curve is symmetric with respect to the real axis, there is no loss of generality in assuming that $\lambda \in \mathbb{C}^+$. Let us denote the positive numbers $\Re(\sqrt{\lambda})$ and $\Im(\sqrt{\lambda})$ by a and b , respectively. Further, for given $Q > 0$, let us consider $\alpha \in \mathbb{C}$ with $|\alpha| = Q$ and the operator H_V with the Dirac delta potential $V(x) = \alpha\delta(x - x_0)$, $x \in \mathbb{R}$, where

$$x_0 = \frac{1}{2a} \arccos\left(\frac{2ab}{a^2 + b^2}\right). \quad (5.2)$$

The operator H_V can be defined rigorously by form methods (see Section 2). In this case, the Birman-Schwinger operator reduces to the multiplication operator with the constant function $\alpha G_\lambda(x_0, x_0)$, where G_λ is the Green's function defined in (3.1), and the inequality (5.1) becomes equality. Furthermore, we have $Q|G_\lambda(x_0, x_0)| = 1$ (see the proof of Lemma 1). Hence, by fixing the phase of α in such a way that $\alpha G_\lambda(x_0, x_0) = 1$, we deduce from the Birman-Schwinger principle that $\lambda \in \sigma(H_V)$. On the other hand, we have $\sigma_{\text{ess}}(H_V) = \mathbb{R}$ (since the perturbation is a point interaction) and $\sigma_r(H_V) = \emptyset$ (since H_V is J -self-adjoint, where J is the complex conjugation operator). Therefore, $\lambda \in \mathbb{C}^+$ must be a discrete eigenvalue for H_V . $\square$

5.3. Proof of Theorem 2. The proof uses complex interpolation and the result of Theorem 1. Observe that (1.6) holds trivially for $\lambda \in \mathbb{R}$. For $\lambda \in \mathbb{C}^+$, let us consider the operator family

$$T_z := |V|^{zp/2}(H_0 - \lambda)^{-1}|V|^{zp/2},$$

for $z \in \mathbb{C}$ with $0 \leq \Re z \leq 1$. First, we note that T_1 is a bounded operator under our hypothesis on V . Indeed, $|V|^{p/2}$ maps $L^2(\mathbb{R})$ to $H^{-1}(\mathbb{R})$ by duality and $(H_0 - \lambda)^{-1}$ is an isomorphism between $H^{-1}(\mathbb{R})$ and $H^1(\mathbb{R})$, while the latter space is mapped by $|V|^{p/2}$ back to $L^2(\mathbb{R})$. Further, we note that T_z is continuous in the closed strip $0 \leq \Re z \leq 1$, analytic in its interior and we have

$$\sup_{0 \leq \Re z \leq 1} \|T_z\| \leq \max\left\{\frac{2}{|\Im(\lambda)|}, \|T_1\|\right\},$$

see (3.19). In particular, T_z is uniformly bounded for $0 \leq \Re z \leq 1$. Since $V \in L^p(\mathbb{R})$ by the hypothesis, we can proceed as in the proof of Theorem 1 and conclude

$$\|T_{1+iy}\| \leq \| |V|^{p/2}(H_0 - \lambda)^{-1}|V|^{p/2} \| \leq \frac{\sqrt{|\lambda| + |\Re(\lambda)|}}{\sqrt{2}|\lambda|} \|V\|_p^p,$$

for any $y \in \mathbb{R}$. Moreover, for all $y \in \mathbb{R}$, we have also the trivial estimate

$$\|T_{iy}\| \leq \frac{2}{|\Im(\lambda)|},$$

see (3.19). Thus, Stein's complex interpolation theorem (see *e.g.* [43, Thm. V.4.1]) yields the following bound for the Birman-Schwinger operator

$$\|K(\lambda)\| \leq \|T_{1/p}\| \leq \frac{(|\lambda| + |\Re(\lambda)|)^{\frac{1}{2p}}}{2^{\frac{3}{2p}-1}|\lambda|^{\frac{1}{p}}} \frac{\|V\|_p}{|\Im(\lambda)|^{1-\frac{1}{p}}}. \quad (5.3)$$

If λ is an eigenvalue for H_V , then the standard Birman-Schwinger principle implies that the expression on the right-hand-side of (5.3) cannot be strictly less than 1, thus yielding the estimate (1.6). $\square$

5.4. Proofs of Corollaries 1 and 2. Let $1 \leq p < \infty$. For $\lambda \in \mathbb{C}^+$, let $x = \Re(\lambda)$ and $y = \Im(\lambda)$. Then (1.6) (resp. (1.4)) can be written, equivalently, as

$$2^{3-2p}(x^2 + y^2)y^{2p-2} \leq (\sqrt{x^2 + y^2} + |x|)\|V\|_p^{2p}. \quad (5.4)$$

Since the region corresponding to (5.4) is symmetric with respect to the imaginary axis, there is no loss of generality in assuming that $x \geq 0$. Further with the change of the variables $x = \|V\|_p^{\frac{2p}{2p-1}} t$, $y = \|V\|_p^{\frac{2p}{2p-1}} s$, (5.4) reads as

$$2^{3-2p}(t^2 + s^2)s^{2p-2} \leq \sqrt{t^2 + s^2} + t. \quad (5.5)$$

Next, let us consider the function

$$f(x) = \frac{x}{1+x^2} + \frac{x}{\sqrt{1+x^2}} \quad (5.6)$$

for $x \geq 0$. Letting $x = \tan(\alpha)$ for $\alpha \in [0, \frac{\pi}{2})$, we easily get $f(\tan(\alpha)) = \sin(\alpha) + \frac{1}{2} \sin(2\alpha)$ which attains its global maximum at $\alpha = \frac{\pi}{3}$. Consequently, we have

$$\sup_{x \geq 0} f(x) = \sup_{\alpha \in [0, \frac{\pi}{2}]} f(\tan(\alpha)) = \frac{3\sqrt{3}}{4} = f(\sqrt{3}).$$

Hence, it follows from (5.5) that $2^{3-2p}s^{2p-1} \leq f(s/t) \leq 3\sqrt{3}/4$ for all $s \geq 0$ and $t > 0$. Therefore,

$$s \leq 2 \left(\frac{3\sqrt{3}}{16} \right)^{\frac{1}{2p-1}} \quad (5.7)$$

and (5.5) becomes equality if and only if

$$(s, t) = \left(2 \left(\frac{3\sqrt{3}}{16} \right)^{\frac{1}{2p-1}}, \frac{2}{\sqrt{3}} \left(\frac{3\sqrt{3}}{16} \right)^{\frac{1}{2p-1}} \right),$$

proving the claim. $\square$

5.5. Proof of Theorem 3. Let $\lambda \in \mathbb{C}^+$ and denote by Ω the support of V . For arbitrary weight function $\rho > 0$, the Schur test yields

$$\|K(\lambda)\| \leq \left(\sup_{x \in \Omega} \int_{\Omega} |K_{\lambda}(x, y)| \frac{dy}{\rho(x, y)} \right)^{1/2} \left(\sup_{y \in \Omega} \int_{\Omega} |K_{\lambda}(x, y)| \rho(x, y) dx \right)^{1/2}.$$

By choosing the weight function

$$\rho(x, y) := |V(x)|^{1/2} |V(y)|^{-1/2}, \quad x, y \in \Omega,$$

and using (3.2), we obtain

$$\|K(\lambda)\| \leq \sup_{x \in \Omega} \int_{\Omega} |G_{\lambda}(x, y)| |V(y)| dy \leq \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} |G_{\lambda}(x, y)| |V(y)| dy. \quad (5.8)$$

On the other hand, in view of (3.1), we have

$$\begin{aligned} \sup_{x \geq 0} \int_{\mathbb{R}} |G_{\lambda}(x, y)| |V(y)| dy &\leq \sup_{x \geq 0} \int_{-\infty}^0 |G_{\lambda}(x, y)| |V(y)| dy + \sup_{x \geq 0} \int_0^{\infty} |G_{\lambda}(x, y)| |V(y)| dy \\ &\leq \frac{1}{\sqrt{2|\lambda|}} \int_{-\infty}^0 e^{\Re(\sqrt{\lambda})y} |V(y)| dy + \frac{1}{\sqrt{|\lambda|}} \int_0^{\infty} e^{-\Im(\sqrt{\lambda})y} |V(y)| dy. \end{aligned}$$

By the Hölder inequality,

$$\int_{-\infty}^0 e^{\Re(\sqrt{\lambda})y} |V(y)| dy \leq \|V\|_{p,-} \left(\int_{-\infty}^0 e^{q\Re(\sqrt{\lambda})y} dy \right)^{1/q} = \frac{\|V\|_{p,-}}{\sqrt[q]{q\Re(\sqrt{\lambda})}}$$

and, similarly,

$$\int_0^{\infty} e^{-\Im(\sqrt{\lambda})y} |V(y)| dy \leq \frac{\|V\|_{p,+}}{\sqrt[q]{q\Im(\sqrt{\lambda})}}.$$

Therefore, we have

$$\sup_{x \geq 0} \int_{\mathbb{R}} |G_{\lambda}(x, y)| |V(y)| dy \leq \frac{1}{\sqrt[q]{q} \sqrt{|\lambda|}} \left(\frac{\|V\|_{p,-}}{\sqrt{2} \sqrt[q]{\Re(\sqrt{\lambda})}} + \frac{\|V\|_{p,+}}{\sqrt{2} \sqrt[q]{\Im(\sqrt{\lambda})}} \right) \quad (5.9)$$

and, analogously,

$$\sup_{x \leq 0} \int_{\mathbb{R}} |G_{\lambda}(x, y)| |V(y)| dy \leq \frac{1}{\sqrt[q]{q} \sqrt{|\lambda|}} \left(\frac{\|V\|_{p,-}}{\sqrt{2} \sqrt[q]{\Re(\sqrt{\lambda})}} + \frac{\|V\|_{p,+}}{\sqrt{2} \sqrt[q]{\Im(\sqrt{\lambda})}} \right). \quad (5.10)$$

Recalling (5.8), we conclude that the maximum of the quantities on the right-hand-sides of (5.9) and (5.10) dominates the operator norm of the Birman-Schwinger operator $K(\lambda)$ and the result immediately follows from the Birman-Schwinger principle as in the proof of Theorem 2. $\square$

5.6. Proof of Theorem 4. First we establish an auxiliary lemma. The proof is a straightforward calculation and is omitted.

Lemma 3. *Let $\lambda > 0$ be arbitrary. Then the function*

$$u(x) = e^{\sqrt{\lambda}x} \mathbf{1}_{x \leq 0} + \sqrt{2} \sin(\sqrt{\lambda}x + \pi/4) \mathbf{1}_{x \geq 0} \quad (5.11)$$

satisfies $H_0 u = \lambda u$ in the weak sense. Moreover, $u \in L^2(\mathbb{R}_-)$ and $u \notin L^2(\mathbb{R}_+)$.

Proof of Theorem 4. We first discuss the case $n = 1$. Set $\psi(x) = u(x)\chi(x)$, with u given by (5.11) and with $\chi \in C^\infty(\mathbb{R})$ to be chosen later. Then

$$(H_0 - \lambda)\psi(x) = -\operatorname{sgn}(x)(2u'(x)\chi'(x) + u(x)\chi''(x)).$$

Selecting

$$V = 2 \frac{u'\chi'}{u\chi} + \frac{\chi''}{\chi}, \quad (5.12)$$

the equation $H_V \psi = \lambda \psi$ is satisfied by definition, provided that V is well-defined. The issue is of course that u has zeros on $\mathbb{R}_+$. To cancel these we first set

$$g(x) = \int_0^x \sin^2(\sqrt{\lambda}t + \pi/4) dt, \quad x \in \mathbb{R},$$

and choose

$$\chi(x) = (1 + g(x)^2)^{-1}, \quad x \in \mathbb{R}. \quad (5.13)$$

We then get

$$V = \frac{8g^2 g'^2}{(1 + g^2)^2} - \frac{2(g'^2 + gg'')}{1 + g^2} - \frac{4gg'u'}{u(1 + g^2)}.$$

Since g'/u vanishes on the zero set of u and $g(x) \geq c|x|$ for some $c > 0$ and all sufficiently large x , we see that V satisfies (1.12) for $n = 1$. For arbitrary $n \in \mathbb{N}$ we replace χ in (5.13) by $(n^2 + g(x)^2)^{-1}$. $\square$

5.7. Proof of Theorem 5. By scaling (1.10) with $\rho = \sqrt{\mu}$ we may assume that $\mu = 1$. We make the following Ansatz for the wavefunction ψ :

$$\psi(x) = \begin{cases} e^{\sqrt{\lambda}x} & x \leq 0, \\ Ae^{ikx} + Be^{-ikx} & 0 \leq x \leq R, \\ Ce^{i\sqrt{\lambda}x} & x \geq R, \end{cases}$$

where

$$k^2 + V_0 = \lambda, \quad \Im(k) > 0, \quad \Im(\sqrt{\lambda}) > 0, \quad \Re(\sqrt{\lambda}) > 0 \quad (5.14)$$

and $R = R(\varepsilon), k = k(\varepsilon)$ will be chosen later. Taking

$$A = A(k, \lambda) = \frac{1}{2} + \frac{\sqrt{\lambda}}{2ik}, \quad B = B(k, \lambda) = \frac{1}{2} - \frac{\sqrt{\lambda}}{2ik},$$

it follows that ψ, ψ' are continuous at $x = 0$. It is easy to see that there exists $C \in \mathbb{C}$ such that ψ, ψ' are continuous at $x = R$ if and only if

$$\sqrt{\lambda} = -k \frac{B - Ae^{2ikR}}{B + Ae^{2ikR}}. \quad (5.15)$$

We set

$$k = k(\varepsilon) = -1 + i\varepsilon, \quad R = R(\varepsilon) = \frac{|\ln \varepsilon|}{2\varepsilon} + \theta \quad (5.16)$$

with $\theta = \theta(\varepsilon) \in [0, \pi]$ to be chosen later. Changing variables from λ to ω v.i.z. $\sqrt{\lambda} = 1 + i\varepsilon + \omega$ and setting

$$f_\varepsilon(\omega) = 1 + i\varepsilon + \omega + k \frac{B - Ae^{2ikR}}{B + Ae^{2ikR}}, \quad \omega \in B(0, C\varepsilon^2),$$

with C independent of ε to be chosen sufficiently large, we see that (5.15) is equivalent to $f_\varepsilon(\omega) = 0$. Since $e^{-2i\Im k R} = \varepsilon(1 + \mathcal{O}(\varepsilon))$ and $A/B = (-1 + i)/(1 + i) + \mathcal{O}(\varepsilon)$ as $\varepsilon \rightarrow 0^+$, we obtain, by choosing $\theta(\varepsilon)$ such that $(-1 + i)/(1 + i)e^{-2iR(\varepsilon)} = -1$,

$$\begin{aligned} |f_\varepsilon(\omega) - \omega| &= |1 + i\varepsilon + (-1 + i\varepsilon)(1 - 2A/Be^{2ikR} + \mathcal{O}(\varepsilon^2))| \\ &= |2i\varepsilon + 2A/Be^{2ikR} + \mathcal{O}(\varepsilon^2)| = \mathcal{O}(\varepsilon^2) \end{aligned}$$

as $\varepsilon \rightarrow 0^+$. It follows that, for ε sufficiently small and C sufficiently large,

$$|f_\varepsilon(\omega) - \omega| < |\omega|, \quad \omega \in \partial B(0, C\varepsilon^2).$$

By Rouché's theorem f_ε has exactly one zero in $B(0, C\varepsilon^2)$. This implies the existence of an eigenvalue λ with the claimed properties. Recalling the relation between k and V_0 in (5.14) and using (5.16) we get the estimate

$$\|V(\varepsilon)\|_p \approx \varepsilon R(\varepsilon)^{1/p} = \mathcal{O}(\varepsilon^{1-1/p} |\ln \varepsilon|^{1/p}) \quad (5.17)$$

as $\varepsilon \rightarrow 0^+$. This proves (1.13). $\square$

Sharpness of exponents in (1.6): Since $|\lambda| \approx |\Re(\lambda)| \approx 1$ in the above example, in equality (1.6) says that

$$\varepsilon^{1-1/p} \leq C \|V(\varepsilon)\|_p.$$

Comparing this to (5.17) we see that this is essentially sharp (up to logarithms). In particular, the exponent $1 - 1/p$ of $\varepsilon \approx \Im(\lambda)$ cannot be made smaller. $\square$

5.8. Proof of Theorem 6. As usual, we split the Birman-Schwinger operator into a singular and a regular part (as $\lambda \rightarrow 0$),

$$K_\lambda = L_\lambda + M_\lambda.$$

We first ignore M_λ and concentrate on L_λ ,

$$\begin{aligned} L_\lambda \psi &= \frac{1}{2\alpha\sqrt{\lambda}} ((g^+, \psi)f^+ - (g^-, \psi)f^+ + (g^+, \psi)f^- - (g^-, \psi)f^-) \\ &= \frac{1}{2\alpha\sqrt{\lambda}} (g^+ - g^-, \psi) |V|^{1/2}, \end{aligned} \quad (5.18)$$

where we set

$$f^\pm(x) = \mathbf{1}_\pm(x) |V(x)|^{1/2}, \quad g^\pm(x) = \mathbf{1}_\pm(x) V_{1/2}(x), \quad \mathbf{1}_\pm = \mathbf{1}_{\mathbb{R}_\pm}.$$

From the second equality in (5.18) we see that L_λ has rank 1; then

$$\det(I + \varepsilon L_\lambda) = 1 + \varepsilon \operatorname{Tr} L_\lambda = 1 + \frac{\varepsilon}{2\alpha\sqrt{\lambda}} v_{\text{sgn}}$$

Hence,

$$0 \in \sigma(I + \varepsilon L_\lambda) \iff \det(I + \varepsilon L_\lambda) = 0 \iff -\sqrt{\lambda} = \frac{\varepsilon}{2\alpha} v_{\text{sgn}}.$$

Since we are assuming that $\Im(\sqrt{\lambda}) > 0$ the rightmost equality can only hold if $\Re(v_{\text{sgn}}) + \Im(v_{\text{sgn}}) < 0$. We also see that

$$\lambda \in \mathbb{C}^+ \iff \Re(\sqrt{\lambda}) > 0 \iff \Re(v_{\text{sgn}}) < \Im(v_{\text{sgn}}).$$

We now repeat the argument, but this time taking M_λ into account. The key observation is that the function $(x, y) \mapsto |\lambda| |M_\lambda(x, y)|^2$ is bounded from above, up to a constant, by the L^1 -majorant $(x, y) \mapsto |V(x)| |V(y)|$. Clearly,

$$\lim_{\lambda \rightarrow 0} |\lambda| |M_\lambda(x, y)|^2 = 0.$$

Hence, by dominated convergence,

$$\|M_\lambda\|_{\text{HS}} = o(|\lambda|^{-1/2}), \quad (\lambda \rightarrow 0). \quad (5.19)$$

Since we expect that $c^{-1}\varepsilon \leq |\lambda|^{1/2} \leq c\varepsilon$ for $\lambda = \lambda(\varepsilon)$ and some $c > 0$, we assume this from now on; we will see later that this assumption is indeed justified. Since then $\varepsilon \|M_\lambda\| < 1$ for sufficiently small ε , it follows that $(I + \varepsilon M_\lambda)$ is invertible and

$$0 \in \sigma(I + \varepsilon K_\lambda) \iff \det(I + \varepsilon(I + \varepsilon M_\lambda)^{-1} L_\lambda) = 0.$$

Similarly as before,

$$\begin{aligned} \det(I + \varepsilon(I + \varepsilon M_\lambda)^{-1} L_\lambda) &= 1 + \varepsilon \operatorname{Tr}(I + \varepsilon M_\lambda)^{-1} L_\lambda \\ &= 1 + \frac{\varepsilon}{2\alpha\sqrt{\lambda}} v_{\text{sgn}} + r(\varepsilon, \lambda), \end{aligned}$$

where $r(\varepsilon, \lambda) = O(\varepsilon \|M_\lambda\|)$. We now change variables from λ to $z = \sqrt{\lambda}$. Clearly, $f_\varepsilon(z) := \det(I + \varepsilon(I + \varepsilon M_\lambda)^{-1} L_\lambda)$ is an analytic function for $z \in B(-\frac{\varepsilon}{2\alpha} v_{\text{sgn}}, \rho(\varepsilon))$, provided $\rho(\varepsilon) = o(\varepsilon)$; we may choose $\rho(\varepsilon) = \max_{c^{-1}\varepsilon \leq |\lambda|^{1/2} \leq c\varepsilon} |r(\varepsilon, \lambda)|$. Then we have

$$\left| f_\varepsilon(z) - \left(1 + \frac{\varepsilon}{2\alpha z} v_{\text{sgn}} \right) \right| \leq \rho(\varepsilon), \quad z \in \partial B\left(-\frac{\varepsilon}{2\alpha} v_{\text{sgn}}, \rho(\varepsilon)\right).$$

On the other hand, since $v_{\text{sgn}} \neq 0$, we have

$$\left|1 + \frac{\varepsilon}{2\alpha z} v_{\text{sgn}}\right| \geq \frac{2|\alpha|\rho(\varepsilon)}{\varepsilon|v_{\text{sgn}}| + 2|\alpha|\rho(\varepsilon)} \geq \frac{\rho(\varepsilon)}{\varepsilon|v_{\text{sgn}}|}, \quad z \in \partial B\left(-\frac{\varepsilon}{2\alpha} v_{\text{sgn}}, \rho(\varepsilon)\right).$$

Hence, for $\varepsilon < 1/|v_{\text{sgn}}|$, we have

$$\left|f_\varepsilon(z) - \left(1 + \frac{\varepsilon}{2\alpha z} v_{\text{sgn}}\right)\right| < \left|1 + \frac{\varepsilon}{2\alpha z} v_{\text{sgn}}\right|, \quad z \in \partial B\left(-\frac{\varepsilon}{2\alpha} v_{\text{sgn}}, \rho(\varepsilon)\right).$$

By Rouché's theorem, f_ε has exactly one zero in $B(-\frac{\varepsilon}{2\alpha} v_{\text{sgn}}, \rho(\varepsilon))$. Since $\rho(\varepsilon) = o(\varepsilon)$ this proves the theorem. $\square$

Sharpness of (1.4): If we assume that V is real-valued and $v_{\text{sgn}} < 0$, then the assumptions of Theorem 6 are satisfied. If we assume in addition that V is supported either on $\mathbb{R}_+$ or on $\mathbb{R}_-$, then $v_{\text{sgn}} = -\|V\|_1$. Since $1/\alpha^2 = 2i$ we see that

$$\lambda(\varepsilon) = \frac{i}{2} \|\varepsilon V\|_1^2 + o(\varepsilon^2), \quad \Re(\lambda(\varepsilon)) = o(\varepsilon^2).$$

This shows that inequality (1.4) is saturated in the limit $\varepsilon \rightarrow 0$. $\square$

5.9. Proof of Theorem 7. We start with the Schatten bound

$$\|W_1(H_0 - \lambda)^{-1}W_2\|_{\mathfrak{S}^{2p}} \leq C_p |\Im(\lambda)|^{1-\frac{1}{p}} |\lambda|^{-\frac{1}{2p}} \|W_1\|_{2p} \|W_2\|_{2p}, \quad 1 \leq p \leq \infty, \quad (5.20)$$

which is a consequence of (3.19) and the pointwise bound for the resolvent kernel. Here and in the following we use the notation

$$\|A\|_{\mathfrak{S}^p}^p := \sum_j s_j(A)^p$$

where $s_j(A)$ are the singular numbers of the compact operator A . We make the conformal transformation $z = \lambda^2 \in \mathbb{C} \setminus [0, \infty)$ and apply Theorem 3.1 in [27] to the analytic family $z \mapsto K(\lambda(z))$, which by (5.20) satisfies the bound

$$\|K(\lambda(z))\|_{\mathfrak{S}^{2p}} \leq C_p |\Im(\sqrt{z})|^{1-\frac{1}{p}} |z|^{-\frac{1}{4p}} \|V\|_p.$$

In particular, for $p = 1$, we have

$$\|K(\lambda(z))\|_{\mathfrak{S}^2} \leq C |z|^{-\frac{1}{4}} \|V\|_p.$$

By [27, Theorem 3.1], for any $\varepsilon > 0$, there exists C such that

$$\sum_j \delta(z_j) |z_j|^{-\frac{1}{2} + \frac{1}{2}(-\frac{1}{2} + \varepsilon)_+} \leq C \|V\|_1^{2(1+(-\frac{1}{2} + \varepsilon)_+)}.$$

Taking $\varepsilon = 1/2$ and using the distortion bound $|\Im(\lambda)| \approx \delta(z_j) |z_j|^{-\frac{1}{2}}$ yields the claim. $\square$

5.10. Proof of Theorem 8. We write $\lambda = k^2$ for the spectral parameter, where $k \in \mathbb{C} \setminus \{0\}$. This amounts to a double covering of the punctured complex λ -plane, with $\Im(k) < 0$ corresponding to the second (unphysical) sheet. From (3.4)–(3.5) we have that

$$|G(\lambda(k))|^2 \leq \frac{1}{|\lambda|} e^{2\Im(k) - (|x| + |y|)},$$

which implies

$$\|K(\lambda(k))\|_{\mathfrak{S}^2} \leq \frac{1}{|k|} \int_{\mathbb{R}} e^{2\Im(k) - |x|} |V(x)| dx.$$

This is analogous to [29, Proposition 4.1], and the proof follows from the same arguments as in [29]. $\square$

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