Sensitivity Analysis for Averaged Asset Price Dynamics with Gamma Processes

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Abstract

The main purpose of this paper is to derive unbiased Monte Carlo estimators of various sensitivity indices for an averaged asset price dynamics governed by the gamma Lévy process. The key idea is to apply a scaling property of the gamma process with respect to the Esscher density transform parameter. Our framework covers not only the continuous Asian option, but also European, discrete Asian, average strike Asian, weighted average, spread options, and geometric average Asian options. Numerical results are provided to illustrate the effectiveness of our formulas in Monte Carlo simulations, relative to finite difference approximation.

Keywords: Asian options, Esscher transform, gamma process, Greeks, Lévy process, Malliavin calculus.

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1 Introduction

In mathematical finance, it has been widely known that the logarithmic derivatives of the density function of stochastic differential equations correspond essentially to the so-called Greeks, that is, sensitivity indices with respect to various model parameters of asset price dynamics. The pioneer work in this direction is of Fournié et al. [11], whose approach is based upon the integration-by-parts formula developed in the Malliavin calculus on the Wiener space. They provide a systematic approach to the derivation of various Greeks formulas for a relatively general diffusion asset price

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dynamics. Since then, there has long been a natural and nontrivial question whether a similar Malliavin calculus approach is applicable in the case of jump processes.

The investigation of the Malliavin calculus for jump processes is initiated in Bismut [4] based upon the Girsanov density transform with a view towards the study of the existence and the smoothness of the density. Various types of Malliavin calculus and logarithmic derivatives have been studied on the Poisson space or on the Wiener-Poisson space. In particular, the existence of weights for the logarithmic derivatives dates back to the work of Bichteler, Gravereaux and Jacod [3], while the uniqueness and the closed forms of the weight have not been established in the general framework.

For practical considerations, however, it is most important to have closed, but not necessarily unique, weights on hand, in order to design an efficient Monte Carlo evaluation. In this direction, Greeks formulas are obtained in Davis and Johansson [9] and Cass and Friz [8] for jump diffusion processes. Their approach consists of conditioning on the jump component and then performing the Malliavin calculus techniques on the diffusion component. In other words, their models are required to be a superposition of independent jump and diffusion components, where the diffusion one must not be degenerate. On the other hand, El-Khatib and Privault [10] applied the Malliavin calculus focused on the Poisson arrival times due to Carlen and Pardoux [5], while Bally et al. [1] took a unified approach considering the derivatives with respect to both the Poisson arrival times and the amplitude of the jumps. Kawai and Kohatsu-Higa [13] derives formulas for Lévy process models of time-changed Brownian motion type, by employing the Gaussian Malliavin calculus conditionally on the time-changing process. Takeuchi [17] studied the same problem for the stochastic differential equations with jumps via a martingale approach. Kawai and Takeuchi [14] studied the computation of the Greeks of European payoffs for an asset price dynamics defined with gamma processes and with Brownian motions, possibly time-changed by one of the gamma processes. In Kawai and Takeuchi [14], various well known financial models, such as the Black-Scholes model and the variance gamma model of Madan, Carr and Chang [15], are within its framework. The key idea there is to apply a scaling property of the gamma *distribution* with respect to the Esscher transform.

In this paper, we take an approach again based on the scaling property, not of the gamma distribution as in Kawai and Takeuchi [14], but of the gamma Lévy *process*. This, rather simple, extension widens the applicability to the *averaged* asset price dynamics defined with the gamma process. A typical example of interest is the (continuous) Asian option, which looks at the time average of underlying asset price dynamics, unlike only the terminal value in the European options. Meanwhile, many others, such as European, discrete Asian, average strike Asian, weighted average and spread options, can easily be accommodated in our framework. In addition, Asian options of geometric average type are also within our scope. This work can also be thought of as

a considerable generalization of Kawai and Takeuchi [14], in the sense that some of the European Greeks formulas there can be recovered from our results. Finally, our approach taken in this paper can also be applied in various stochastic systems involving gamma processes.

The rest of the paper is organized as follows. Section 2 recalls generalities on the gamma process and introduces our asset price dynamics model with assumptions to be imposed on its characterizing parameters. In Section 3, we derive formulas of the Greeks for discontinuous payoff function of averaged asset price dynamics defined with the gamma process. The derivation of our formulas entails rather lengthy proofs of somewhat routine nature. To avoid overloading the paper, we omit nonessential details in some instances. We close this study with some numerical results to illustrate remarkable improvements in Monte Carlo simulations in terms of estimator variance relative to the finite difference estimation.

2 Preliminaries

Let us begin with general notations which will be used throughout the text. For each positive integer k , ∂_k indicates the partial derivative with respect to k -th argument. We fix $(\Omega, \mathcal{F}, \mathbb{P})$ as our underlying the probability space. We denote by $\mathbb{E}_{\mathbb{P}}$ the expectation under the probability measure $\mathbb{P}$. We denote by $\delta_x(dy)$ the Dirac delta measure on $\mathbb{R}$ with concentration at $x \in \mathbb{R}$, while $B_\varepsilon := (-\varepsilon, +\varepsilon)$ is the open ball around the origin with radius $\varepsilon > 0$. Let $\{Y_t : t \geq 0\}$ be a one-sided pure-jump Lévy process in $[0, +\infty)$ with the Lévy measure

$$\nu(dz) = a \frac{e^{-bz}}{z} dz, \quad z \in (0, +\infty),$$

where $a > 0$ and $b > 0$. The stochastic process $\{Y_t : t \geq 0\}$ is then called a *gamma process* with parameter (a, b) , whose marginal has the gamma distribution with characteristic function

$$\mathbb{E}_{\mathbb{P}}[e^{iyY_t}] = \exp \left[t \int_{(0, +\infty)} (e^{iyz} - 1) \nu(dz) \right] = \left(1 - \frac{iy}{b} \right)^{-at}.$$

Moreover, the marginal density function at time t is given in the form

$$f_t^{\mathbb{P}}(y) = \frac{b^{at}}{\Gamma(at)} y^{at-1} e^{-by}, \quad y \in (0, +\infty), \quad (2.1)$$

where $\Gamma(p)$ is the gamma function of order $p > 0$.

Throughout the paper, we fix $T > 0$. Define $\Lambda := \{\lambda \in \mathbb{R} : \mathbb{E}_{\mathbb{P}}[e^{\lambda Y_1}] < +\infty\} = (-\infty, b)$. For $\lambda \in \Lambda$, define a new probability measure $\mathbb{Q}_\lambda$ by

$$\left. \frac{d\mathbb{Q}_\lambda}{d\mathbb{P}} \right|_{\mathcal{F}_T} := \frac{e^{\lambda Y_T}}{\mathbb{E}_{\mathbb{P}}[e^{\lambda Y_T}]} = \exp [\lambda Y_T - \varphi_T(\lambda)],$$

where $\varphi_T(\lambda) := aT \ln(b/(b-\lambda))$ and where $(\mathcal{F}_t)_{t \in [0, T]}$ is the natural filtration generated by $\{Y_t : t \in [0, T]\}$. This measure change is the simplest form of the Girsanov transformation, often called the Esscher transform in mathematical finance and actuarial science. The following is the key tool for our discussions.

Lemma 2.1. *The laws of the process Y under $\mathbb{Q}_\lambda$ and of the process $Y^{(\lambda)} := bY/(b-\lambda)$ under $\mathbb{P}$ are identical.*

Proof. It is well known that for Lévy processes, the identity of finite dimensional distributions implies the identity of random processes in law. Fix $0 = t_0 \leq t_1 \leq \dots \leq t_n = T$ and $\zeta_1, \dots, \zeta_n \in \mathbb{R}$ arbitrarily. Due to the independence of the increments under the measure $\mathbb{P}$, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_\lambda} \left[\exp \left(i \sum_{k=1}^n \zeta_k Y_{t_k} \right) \right] &= \mathbb{E}_{\mathbb{P}} \left[\exp \left(i \sum_{k=1}^n \zeta_k Y_{t_k} + \lambda Y_T \right) \right] \left(1 - \frac{\lambda}{b} \right)^{aT} \\ &= \prod_{j=1}^n \left(1 - \frac{i \sum_{k=j}^n \zeta_k + \lambda}{b} \right)^{-a(t_j - t_{j-1})} \left(1 - \frac{\lambda}{b} \right)^{a(t_j - t_{j-1})} \\ &= \mathbb{E}_{\mathbb{P}} \left[\exp \left(i \sum_{k=1}^n \zeta_k \frac{b}{b-\lambda} Y_{t_k} \right) \right], \end{aligned}$$

which proves the claim. □

Let us introduce our asset price dynamics model and with some examples of practical interest within our scope. Let $Z := \{Z_t : t \in [0, T]\}$ be a nondecreasing stochastic process in $\mathbb{R}$ with $Z_0 = 0$, $\mathbb{P}$ -a.s., independent of the process $Y := \{Y_t : t \in [0, T]\}$. Our asset price dynamics model $\{S_t : t \in [0, T]\}$ is defined by

$$S_t := S_0 \exp[\theta Y_t + \kappa Z_t + c(\theta, \kappa)t],$$

where $S_0 \in (0, +\infty)$, $\theta \in (0, +\infty)$, $\kappa \in \mathbb{R}$, and $c : (0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ with $\partial_k c(\theta, \kappa)$ being well defined for $k = 1, 2$. Let us impose some assumptions on the parameters θ and b and on the process Z .

Assumption 2.2. *We henceforth assume the following;*

(i) $4\theta < b$,

(ii) *there exists a constant $p_0 > 2$ such that $\mathbb{E}_{\mathbb{P}}[\exp(p_0|\kappa|Z_T)] < +\infty$.*

Remark 2.3. Let us propose some candidates for the process Z , which makes our asset price dynamics model of adequate practical interest.

The first one is the variance gamma model of Madan, Carr and Chang [15] by setting Z also to be a gamma process with a negative κ . The model has attracted the attention of finance practitioners, and have thus appeared often in the computational finance literature, for example, Carr and Madan [7] and Fu [12]. To be precise, set $\theta = 1$ and $\kappa = -1$, and the gamma processes Y and Z should be characterized respectively with $(a, b) = (v^{-1}, (\mu_p v)^{-1})$ and $(a, b) = (v^{-1}, (\mu_n v)^{-1})$ for some $v > 0$, where $\mu_p = \frac{1}{2} \sqrt{\theta^2 + 2\sigma^2/v} + \theta/2$, and $\mu_n = \frac{1}{2} \sqrt{\theta^2 + 2\sigma^2/v} - \theta/2$.

The inverse Gaussian process is another interesting candidate for Z . It is also a subordinator, induced by the inverse Gaussian distribution, whose marginal at time t is identical in law to the first time that a Brownian motion with drift v reaches the positive level t , that is, $\mathbb{E}_{\mathbb{P}}[e^{-yZ_t}] = \exp[-t(\sqrt{2y + v^2} - v)]$. More details on the inverse Gaussian process may be found in Barndorff-Nielsen [2] and Carr et al. [6]. $\square$

3 Sensitivity formulas

Let $\mu(dt)$ be a finite Borel measure over $[0, T]$. For simplicity in notation, we henceforth write

$$\begin{aligned} S_t^{(\lambda)} &:= S_0 \exp \left[\theta Y_t^{(\lambda)} + \kappa Z_t + c(\theta, \kappa) t \right], \quad F_j^{(\lambda)} := \frac{1}{T} \int_0^T S_t^{(\lambda)} (Y_t^{(\lambda)})^j \mu(dt) \quad (j = 0, 1, 2, 3), \\ H_j^{(\lambda)} &:= \frac{1}{T} \int_0^T S_t^{(\lambda)} (Y_t^{(\lambda)})^j Z_t \mu(dt), \quad I_j^{(\lambda)} := \frac{1}{T} \int_0^T S_t^{(\lambda)} (Y_t^{(\lambda)})^j t \mu(dt) \quad (j = 0, 1), \\ J^{(\lambda)} &:= \frac{1}{T} \int_0^T S_t^{(\lambda)} Z_t t \mu(dt), \quad K^{(\lambda)} := \left(bY_T^{(\lambda)} - aT + 1 \right) \frac{F_0^{(\lambda)}}{\theta F_1^{(\lambda)}} + \frac{F_0^{(\lambda)} F_2^{(\lambda)}}{(F_1^{(\lambda)})^2} - 1, \\ L^{(\lambda)} &:= \left(bY_T^{(\lambda)} - aT + 1 \right) \frac{I_0^{(\lambda)}}{\theta F_1^{(\lambda)}} + \frac{I_0^{(\lambda)} F_2^{(\lambda)}}{(F_1^{(\lambda)})^2} - \frac{I_1^{(\lambda)}}{F_1^{(\lambda)}}, \\ M^{(\lambda)} &:= \left(bY_T^{(\lambda)} - aT + 1 \right) \frac{H_0^{(\lambda)}}{\theta F_1^{(\lambda)}} + \frac{H_0^{(\lambda)} F_2^{(\lambda)}}{(F_1^{(\lambda)})^2} - \frac{H_1^{(\lambda)}}{F_1^{(\lambda)}}, \end{aligned}$$

for $\lambda \in \Lambda$ and $t \in [0, T]$. For further simplicity, when $\lambda = 0$, we suppress the superscript (0) of $Y_t^{(0)}, S_t^{(0)}, F_j^{(0)}, H_j^{(0)}, I_j^{(0)}, J^{(0)}, K^{(0)}, L^{(0)}$ and $M^{(0)}$. We begin with our main results.

Theorem 3.1. *Let $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a measurable function such that $\mathbb{E}_{\mathbb{P}}[|\Phi(F_0)|^2]$ is locally uniformly bounded in S_0 , θ and κ . Then, we have*

(i) [Sensitivity with respect to S_0]

$$\frac{\partial}{\partial S_0} \mathbb{E}_{\mathbb{P}}[\Phi(F_0)] = \frac{1}{S_0} \mathbb{E}_{\mathbb{P}}[\Phi(F_0) K].$$

(ii) [Second derivative with respect to S_0]

$$\begin{aligned} \frac{\partial^2}{\partial S_0^2} \mathbb{E}_{\mathbb{P}} [\Phi(F_0)] &= \frac{1}{S_0^2} \mathbb{E}_{\mathbb{P}} \left[\Phi(F_0) \left(3(bY_T - aT + 1) K \frac{F_0}{\theta F_1} \right. \right. \\ &\quad \left. \left. - \left(2(bY_T - aT + 1)^2 + aT - 1 \right) \left(\frac{F_0}{\theta F_1} \right)^2 + 1 - \frac{F_0^2 F_3}{F_1^3} + 3 \frac{F_0^2 F_2^2}{F_1^4} - 3 \frac{F_0 F_2}{F_1^2} \right) \right] - \frac{1}{S_0} \frac{\partial}{\partial S_0} \mathbb{E}_{\mathbb{P}} [\Phi(F_0)]. \end{aligned}$$

(iii) [Sensitivity with respect to θ]

$$\frac{\partial}{\partial \theta} \mathbb{E}_{\mathbb{P}} [\Phi(F_0)] = \mathbb{E}_{\mathbb{P}} \left[\Phi(F_0) \left(\frac{bY_T - aT}{\theta} + \partial_1 c(\theta, \kappa) L \right) \right].$$

(iv) [Sensitivity with respect to κ]

$$\frac{\partial}{\partial \kappa} \mathbb{E}_{\mathbb{P}} [\Phi(F_0)] = \mathbb{E}_{\mathbb{P}} [\Phi(F_0) (M + \partial_2 c(\theta, \kappa) L)].$$

Remark 3.2. In the context of mathematical finance, the sensitivity (i) and (ii) are called *Delta* and *Gamma*, respectively. Both of the formulas (iii) and (iv) correspond best to the *Vega*, that is, the sensitivity with respect to the Black-Scholes volatility parameter.

Our framework covers several payoffs of practical interest. First, setting $\mu(dt) = T \delta_T(dt)$ reduces to the European framework. With this setting, our results indeed recover the European formulas derived in Kawai and Takeuchi [14]. As a matter of course, $\mu(dt) = dt$ corresponds to the Asian option (also called average price option), while in real financial trading, the continuous average is simply impossible to compute and is thus not very interesting. Meanwhile, we can deal with arithmetic average (discrete Asian) type, which is rather the genuine Asian from a practical point of view, by setting $\mu(dt) = TN^{-1} \sum_{k=1}^N \delta_{t_k}(dt)$, to be numerically tested later, or a little more generally, a weighted arithmetic average by $\mu(dt) = T \left(\sum_{k=1}^N w_k \right)^{-1} \sum_{k=1}^N w_k \delta_{t_k}(dt)$ for reals $\{w_k\}_{k \in \mathbb{N}}$. Options of average strike Asian type and of spread type, respectively with $\mu(dt) = dt - T \delta_T(dt)$ and $\mu(dt) = T(-\delta_{T/2}(dt) + \delta_T(dt))$, are also within our framework. $\square$

The proof of Theorem 3.1 is divided into several preliminary results. The first one is concerned with the moments of $Y_t^{(\lambda)}$ and of $S_t^{(\lambda)}$.

Lemma 3.3. Fix $t \in [0, T]$ and $\lambda \in \Lambda$. It holds that for $p \in (-at, +\infty)$,

$$\mathbb{E}_{\mathbb{P}} \left[(Y_t^{(\lambda)})^p \right] = \frac{\Gamma(at + p)}{(b - \lambda)^p \Gamma(at)},$$

while for $p \in (-\infty, p_0 \wedge (b - \lambda)/\theta)$,

$$\mathbb{E}_{\mathbb{P}} \left[(S_t^{(\lambda)})^p \right] = \left(1 - \frac{p\theta}{b - \lambda} \right)^{-at} S_0^p \exp(p c(\theta, \kappa) t) \mathbb{E}_{\mathbb{P}} [\exp(p \kappa Z_t)].$$

Proof. The first assertion is straightforward by the density (2.1), while the second follows from the independence of Y and Z , the exponential integrability of Z in Assumption 2.2, and again the density (2.1). $\square$

We fix $\varepsilon \in (0, b/2)$ in what follows. Note that $B_\varepsilon \subset \Lambda$, and $p_0 \wedge (b - \lambda)/\theta > 2$ under Assumption 2.2. The following lemma later plays an important role for the interchange of the derivative (with respect to λ) and the expectation $\mathbb{E}_\mathbb{P}$.

Lemma 3.4. *The random variables $S_t^{(\lambda)}$ and $Y_t^{(\lambda)}$ are in $\mathbb{L}^{p_0 \wedge (b-\lambda)/\theta}(\Omega)$, uniformly in $t \in [0, T]$ and $\lambda \in B_\varepsilon$.*

Proof. The assertion is the direct consequence of Assumption 2.2 and Lemma 3.3. $\square$

The following lemma is also an important tool.

Lemma 3.5. *For $k = 0, 1, 2$ and for $\lambda \in B_\varepsilon$, it holds that*

$$\frac{\partial}{\partial \lambda} \int_0^T S_t^{(\lambda)} (Y_t^{(\lambda)})^k Z_t \mu(dt) = \int_0^T \frac{\partial}{\partial \lambda} \left(S_t^{(\lambda)} (Y_t^{(\lambda)})^k \right) Z_t \mu(dt), \quad \mathbb{P}\text{-a.s.}$$

Proof. Clearly, for each $t \in [0, T]$, $M_t^{(\lambda)} := S_t^{(\lambda)} (Y_t^{(\lambda)})^k Z_t$ is continuous in λ , so is $(\partial/\partial \lambda) M_t^{(\lambda)} = M_t^{(\lambda)} (\theta Y_t^{(\lambda)} + k)/(b - \lambda)$. Since $Y_t^{(\lambda)}$ is nondecreasing in t , we have for $h \in (0, b - \lambda)$,

$$\begin{aligned} & \int_0^T \int_0^1 \left(\frac{\partial}{\partial \xi} M_t^{(\xi)} \Big|_{\xi=\lambda+\delta h} \right)^2 d\delta \mu(dt) \\ & \leq S_0^2 \exp(2\theta Y_T^{(\xi)} + 2|\kappa|Z_T + 2|c(\theta, \kappa)|T) (Y_T^{(\xi)})^{2k} Z_T^2 \left(\frac{\theta Y_T^{(\xi)} + k}{b - \xi} \right)^2 \mu([0, T]) \Big|_{\xi=\lambda+h}. \end{aligned}$$

By the Taylor theorem, it holds that

$$\begin{aligned} & \frac{1}{h} \left(\int_0^T M_t^{(\lambda+h)} \mu(dt) - \int_0^T M_t^{(\lambda)} \mu(dt) \right) - \int_0^T \frac{\partial}{\partial \lambda} M_t^{(\lambda)} \mu(dt) \\ & = \int_0^T \int_0^1 \left(\frac{\partial}{\partial \xi} M_t^{(\xi)} \Big|_{\xi=\lambda+\delta h} - \frac{\partial}{\partial \lambda} M_t^{(\lambda)} \right) \mu(dt), \end{aligned}$$

which tends to 0 as $h \downarrow 0$. The proof is complete. $\square$

Let us investigate the integrability of the random variable $F_1^{(\lambda)}$ of negative order.

Lemma 3.6. *For $p \in (1, (p_0 \wedge aT)/2)$, it holds that*

$$\sup_{\lambda \in B_\varepsilon} \mathbb{E}_\mathbb{P} \left[(F_1^{(\lambda)})^{-p} \right] < +\infty.$$

Proof. Choose conjugate exponents (q, r) with $q \in (1, aT/p)$ and $r \in (aT/(aT - p), 2)$, and pick $\zeta \in (pq/a, T)$. Observe that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[\left(F_1^{(\lambda)} \right)^{-p} \right] &\leq T^p \mu([\zeta, T])^{-p} \mathbb{E}_{\mathbb{P}} \left[\left(\frac{1}{\mu([\zeta, T])} \int_{\zeta}^T S_t^{(\lambda)} Y_t^{(\lambda)} \mu(dt) \right)^{-p} \right] \\ &\leq T^p \mu([\zeta, T])^{-p-1} \mathbb{E}_{\mathbb{P}} \left[\left(Y_{\zeta}^{(\lambda)} \right)^{-pq} \right]^{1/q} \left(\int_{\zeta}^T \mathbb{E}_{\mathbb{P}} \left[\left(S_t^{(\lambda)} \right)^{-pr} \right] \mu(dt) \right)^{1/r}, \end{aligned}$$

where we have used the nonnegativity of the integrand, the nonnegativity and the nondecreasingness of the process Y , the Jensen inequality, and the Hölder inequality. Finally, the finiteness of the above last line follows from the moment formulas of Lemma 3.3. $\square$

We investigate the integrability of the random variable $F_j^{(\lambda)}$, $H_j^{(\lambda)}$, $I_j^{(\lambda)}$ and $J^{(\lambda)}$ of positive order.

Lemma 3.7. *For $p \in (1, p_0/2 \wedge (b - \varepsilon)/(2\theta))$, it holds that*

$$\sup_{\lambda \in B_{\varepsilon}} \mathbb{E}_{\mathbb{P}} \left[\sum_{j=0}^3 \left(F_j^{(\lambda)} \right)^p + \sum_{j=0}^1 \left(\left(H_j^{(\lambda)} \right)^p + \left(I_j^{(\lambda)} \right)^p \right) + \left(J^{(\lambda)} \right)^p \right] < +\infty.$$

Proof. By choosing conjugate exponents (q, r) and (u, v) with $q \in (1, \sqrt{2})$ and $v \in (1, \sqrt{2})$, we have by the Jensen inequality,

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}} \left[\sum_{j=0}^3 \left(F_j^{(\lambda)} \right)^p + \sum_{j=0}^1 \left(\left(H_j^{(\lambda)} \right)^p + \left(I_j^{(\lambda)} \right)^p \right) + \left(J^{(\lambda)} \right)^p \right] \\ &\leq c_p T^{-p} \mu([0, T])^p \mathbb{E}_{\mathbb{P}} \left[\left(\frac{1}{\mu[0, T]} \int_0^T S_t^{(\lambda)} \left(Y_t^{(\lambda)} \right)^j Z_t^k t^l \mu(dt) \right)^p \right] \\ &\leq c_p T^{lp-p} \mu([0, T])^{p-1} \mathbb{E}_{\mathbb{P}} \left[\left(Y_T^{(\lambda)} \right)^{jpqu} \right]^{\frac{1}{qu}} \mathbb{E}_{\mathbb{P}} \left[e^{prZ_T} \right]^{1/r} \int_0^T \mathbb{E}_{\mathbb{P}} \left[\left(S_t^{(\lambda)} \right)^{pqv} \right]^{\frac{1}{qv}} \mu(dt), \end{aligned}$$

which is well defined by Lemma 3.3 and Assumption 2.2. $\square$

Proof of Theorem 3.1. We start with the smoothness assumption $\Phi \in C_b^2(\mathbb{R}; \mathbb{R})$, which can be removed later by standard density arguments. (For details, we refer to Kawai and Kohatsu-Higa [13] and Kawai and Takeuchi [14].) Throughout the proof, we let $\lambda \in B_{\varepsilon}$.

(i) By Lemma 2.1, it holds that

$$\mathbb{E}_{\mathbb{P}} \left[\Phi(F_0^{(\lambda)}) \frac{F_0^{(\lambda)}}{S_0 \theta F_1^{(\lambda)}} \right] = \mathbb{E}_{\mathbb{P}} \left[\frac{e^{\lambda Y_T}}{\mathbb{E}_{\mathbb{P}}[e^{\lambda Y_T}]} \Phi(F_0) \frac{F_0}{S_0 \theta F_1} \right]. \quad (3.1)$$

Differentiating both sides of (3.1) at $\lambda = 0$ yields the result, provided that the differentiation with respect to λ and the expectation $\mathbb{E}_{\mathbb{P}}$ are commutative on the both hand sides, which we will prove

in what follows. Concerning the left hand side of (3.1), by the Taylor theorem and Lemma 3.5, it holds that

$$\begin{aligned}
& \sup_{|\lambda| \leq \varepsilon} \left| \mathbb{E}_{\mathbb{P}} \left[\frac{1}{\lambda} \left(\Phi(F_0^{(\lambda)}) \frac{F_0^{(\lambda)}}{S_0 \theta F_1^{(\lambda)}} - \Phi(F_0) \frac{F_0}{S_0 \theta F_1} \right) - \left(\Phi'(F_0) \frac{F_0}{S_0 b} + \Phi(F_0) \frac{U^{(0)}}{S_0 b} \right) \right] \right| \\
& \leq \sup_{|\lambda| \leq \varepsilon} \int_0^1 \mathbb{E}_{\mathbb{P}} \left[\left| \Phi'(F_0^{(\delta\lambda)}) \frac{F_0^{(\delta\lambda)}}{S_0(b-\delta\lambda)} + \Phi(F_0^{(\delta\lambda)}) \frac{U^{(\delta\lambda)}}{S_0(b-\delta\lambda)} - \Phi'(F_0) \frac{F_0}{S_0 b} - \Phi(F_0) \frac{U^{(0)}}{S_0 b} \right| \right] d\delta \\
& \leq \int_0^1 \sup_{|\lambda| \leq \varepsilon} \mathbb{E}_{\mathbb{P}} \left[\left| \Phi'(F_0^{(\delta\lambda)}) \frac{F_0^{(\delta\lambda)}}{S_0(b-\delta\lambda)} - \Phi'(F_0) \frac{F_0}{S_0 b} \right| \right] d\delta \\
& \quad + \int_0^1 \sup_{|\lambda| \leq \varepsilon} \mathbb{E}_{\mathbb{P}} \left[\left| \Phi(F_0^{(\delta\lambda)}) \frac{U^{(\delta\lambda)}}{S_0(b-\delta\lambda)} - \Phi(F_0) \frac{U^{(0)}}{S_0 b} \right| \right] d\delta,
\end{aligned}$$

where

$$U^{(\lambda)} := 1 - \frac{F_0^{(\lambda)} F_2^{(\lambda)}}{(F_1^{(\lambda)})^2} - \frac{F_0^{(\lambda)}}{\theta F_1^{(\lambda)}}.$$

The first integral on the right hand side tends to zero as $\varepsilon \downarrow 0$ by the Cauchy-Schwarz inequality with the boundedness of Φ' and Lemma 3.4. For the second integral, again by the Cauchy-Schwarz inequality with $F_2^{(\lambda)} \leq Y_T^{(\lambda)} F_1^{(\lambda)}$, $\mathbb{P}$ -a.s., it remains to show that for $\lambda \in B_\varepsilon$ and for $k = 0, 1$, $\mathbb{E}_{\mathbb{P}}[(F_0^{(\lambda)}(Y_T^{(\lambda)})^k / F_1^{(\lambda)})^2] < +\infty$. By choosing conjugate exponents (p, q, r) with $p \in (1, (p_0 \wedge aT)/4)$ and $q \in (1, p_0/4 \wedge (b - \varepsilon)/(4\theta))$, we have

$$\mathbb{E}_{\mathbb{P}} \left[\left(\frac{F_0^{(\lambda)} (Y_T^{(\lambda)})^k}{F_1^{(\lambda)}} \right)^2 \right] \leq \mathbb{E}_{\mathbb{P}} \left[(F_1^{(\lambda)})^{-2p} \right]^{1/p} \mathbb{E}_{\mathbb{P}} \left[(F_0^{(\lambda)})^{2q} \right]^{1/q} \mathbb{E}_{\mathbb{P}} \left[(Y_T^{(\lambda)})^{2kr} \right]^{1/r},$$

which is uniformly bounded over $|\lambda| \leq \varepsilon$ by Lemma 3.3, Lemma 3.6 and Lemma 3.7. The interchange of the derivative with respect to λ and the expectation $\mathbb{E}_{\mathbb{P}}$ on the right hand side of (3.1) can be shown by the Cauchy-Schwarz inequality with $\mathbb{E}_{\mathbb{P}}[|\Phi(F_0)|^2] < +\infty$ and with

$$\frac{e^{\lambda Y_T - \varphi_T(\lambda)} - 1}{\lambda} \rightarrow Y_T - \frac{aT}{b}$$

in $\mathbb{L}^2(\Omega)$, as $\lambda \downarrow 0$.

(ii) Define $\Psi(x) := \Phi'(x)x$. Observe that

$$\begin{aligned}
\frac{\partial^2}{\partial S_0^2} \mathbb{E}_{\mathbb{P}}[\Phi(F_0)] &= \frac{\partial}{\partial S_0} \mathbb{E}_{\mathbb{P}} \left[\Phi'(F_0) \frac{F_0}{S_0} \right] = \mathbb{E}_{\mathbb{P}} \left[\Phi''(F_0) \left(\frac{F_0}{S_0} \right)^2 \right] \\
&= \frac{1}{S_0^2} \mathbb{E}_{\mathbb{P}} [(\Psi'(F_0) - \Phi'(F_0)) F_0] = \frac{1}{S_0^2} \mathbb{E}_{\mathbb{P}} [(\Psi(F_0) - \Phi(F_0)) K],
\end{aligned}$$

where the first and the fourth equalities holds by the result (i), the second by Lemma 3.5, and the third by $\Phi''(x)x^2 = \Psi'(x)x - \Phi'(x)x$. The desired result can be derived by taking derivative at $\lambda = 0$ of the identity

$$\mathbb{E}_{\mathbb{P}} \left[\Phi(F_0^{(\lambda)}) \frac{bF_0^{(\lambda)}}{\theta F_1^{(\lambda)}} K^{(\lambda)} \right] = \mathbb{E}_{\mathbb{P}} \left[\frac{e^{\lambda Y_T}}{\mathbb{E}_{\mathbb{P}}[e^{\lambda Y_T}]} \Phi(F_0) \frac{bF_0}{\theta F_1} K \right],$$

where the interchange of the differentiation with respect to λ and the expectation $\mathbb{E}_{\mathbb{P}}$ and the existence of the formulas can be justified in a similar manner to the proof of (i).

(iii)-(iv) The desired formulas can be derived by taking derivative at $\lambda = 0$ respectively of

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[\Phi(F_0^{(\lambda)}) \left(1 + \partial_1 c(\theta, \kappa) \frac{I_0^{(\lambda)}}{F_1^{(\lambda)}} \right) \right] &= \mathbb{E}_{\mathbb{P}} \left[\frac{e^{\lambda Y_T}}{\mathbb{E}_{\mathbb{P}}[e^{\lambda Y_T}]} \Phi(F_0) \left(1 + \partial_1 c(\theta, \kappa) \frac{I_0}{F_1} \right) \right], \\ \mathbb{E}_{\mathbb{P}} \left[\Phi(F_0^{(\lambda)}) \frac{H_0^{(\lambda)} + \partial_2 c(\theta, \kappa) J^{(\lambda)}}{F_1^{(\lambda)}} \right] &= \mathbb{E}_{\mathbb{P}} \left[\frac{e^{\lambda Y_T}}{\mathbb{E}_{\mathbb{P}}[e^{\lambda Y_T}]} \Phi(F_0) \frac{H_0 + \partial_2 c(\theta, \kappa) J}{F_1} \right]. \end{aligned}$$

We omit justifying the interchange of the derivative with respect to λ and the expectation $\mathbb{E}_{\mathbb{P}}$ and the proof of the existence of the formulas. $\square$

We next present Greeks formulas for *geometrically* averaged asset price dynamics. Fix $N \in \mathbb{N}$ and $0 < t_1 \leq t_2 \leq \dots \leq t_N \leq T$. Define

$$G^{(\lambda)} := \sqrt[N]{S_{t_1}^{(\lambda)} S_{t_2}^{(\lambda)} \dots S_{t_N}^{(\lambda)}} = S_0 \exp \left[\theta \tilde{Y}^{(\lambda)} + \kappa \tilde{Z} + c(\theta, \kappa) \tilde{t} \right].$$

where $\tilde{Y}^{(\lambda)} := \sum_{k=1}^N Y_{t_k}^{(\lambda)} / N$, $\tilde{Z} := \sum_{k=1}^N Z_{t_k} / N$, $\tilde{t} := \sum_{k=1}^N t_k / N$. Without essential loss of generality, we have ruled out the case $t_1 = 0$ a priori. Notice that the above geometric averaging framework reduces to the European setting by putting $N = 1$ and $t_N = T$. As before, we suppress the superscript (0), that is, $\tilde{Y} := \tilde{Y}^{(0)}$ and $G := G^{(0)}$. Then, Greeks formulas are as follows.

Theorem 3.8. *Let $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a measurable function such that $\mathbb{E}_{\mathbb{P}}[|\Phi(G)|^2]$ is locally uniformly bounded in S_0 , θ and κ . If $at_N > 2$, it holds that*

$$\begin{aligned} \frac{\partial}{\partial S_0} \mathbb{E}_{\mathbb{P}}[\Phi(G)] &= \frac{1}{S_0} \mathbb{E}_{\mathbb{P}} \left[\Phi(G) \frac{bY_T - aT + 1}{\theta \tilde{Y}} \right], \\ \frac{\partial}{\partial \theta} \mathbb{E}_{\mathbb{P}}[\Phi(G)] &= \mathbb{E}_{\mathbb{P}} \left[\Phi(G) \frac{bY_T - aT}{\theta} \right] + S_0 \partial_1 c(\theta, \kappa) \tilde{t} \frac{\partial}{\partial S_0} \mathbb{E}_{\mathbb{P}}[\Phi(G)], \\ \frac{\partial}{\partial \kappa} \mathbb{E}_{\mathbb{P}}[\Phi(G)] &= \mathbb{E}_{\mathbb{P}} \left[\Phi(G) \frac{(bY_T - aT + 1)\tilde{Z}}{\theta \tilde{Y}} \right] + S_0 \partial_2 c(\theta, \kappa) \tilde{t} \frac{\partial}{\partial S_0} \mathbb{E}_{\mathbb{P}}[\Phi(G)]. \end{aligned}$$

If $at_N > 4$, then it holds that

$$\frac{\partial^2}{\partial S_0^2} \mathbb{E}_{\mathbb{P}} [\Phi(G)] = \frac{1}{S_0^2} \mathbb{E}_{\mathbb{P}} \left[\Phi(G) \frac{(bY_T - aT + 1)^2 + aT - 1 + 2\theta^2(1 - \tilde{Y}^2)}{\theta^2 \tilde{Y}^2} \right] - \frac{1}{S_0} \frac{\partial}{\partial S_0} \mathbb{E}_{\mathbb{P}} [\Phi(G)].$$

Proof. To avoid overloading this proof with nonessential details, we only give the proof of the Delta formula at a sketchy level. With the additional regularity assumption $\Phi \in C_b^2(\mathbb{R}; \mathbb{R})$, we get the formula by differentiating the identity

$$\mathbb{E}_{\mathbb{P}} \left[\frac{\Phi(G^{(\lambda)})}{\tilde{Y}^{(\lambda)}} \right] = \mathbb{E}_{\mathbb{P}} \left[\frac{e^{\lambda Y_T}}{\mathbb{E}_{\mathbb{P}}[e^{\lambda Y_T}]} \frac{\Phi(G)}{\tilde{Y}} \right],$$

at $\lambda = 0$. The interchange of the derivative with respect to λ and the expectation $\mathbb{E}_{\mathbb{P}}$ and the removal of the regularity of Φ can be justified as before, with the help of

$$\mathbb{E}_{\mathbb{P}} [Y_T^2 / \tilde{Y}^2] \leq \mathbb{E}_{\mathbb{P}} [Y_T^2 / Y_{t_N}^2] = \mathbb{E}_{\mathbb{P}} [(Y_T - Y_{t_N})^2] \mathbb{E}_{\mathbb{P}} [Y_{t_N}^{-2}] + 2\mathbb{E}_{\mathbb{P}} [Y_T - Y_{t_N}] \mathbb{E}_{\mathbb{P}} [Y_{t_N}^{-1}] + 1 < +\infty,$$

where we have used the independence of the increments. $\square$

Let us close our study with numerical examples to illustrate the performance of our Greeks formulas in Monte Carlo estimations. We consider a digital payoff $\Phi(F_0) = \mathbb{1}(F_0 > C)$, for some $C > 0$, with the weighting measure $\mu(dt) = TN^{-1} \sum_{k=1}^N \delta_{k\frac{T}{N}}(dt)$, so that $\Phi(F_0)$ is an equidistant discrete Asian option. We divide $[0, T]$ into $N = 20$ intervals. Let us emphasize that as discussed in Remark 3.2, our approach is rigorously valid for this discrete Asian options, not as a numerical approximation of continuously averaged Asians. Fix $S_0 = 100$, $C = 110$, $T = 1$, $a = b = 6$ and $\theta = 1$. In our experiments, the process Z is an inverse Gaussian (Lévy) process, which is introduced in Remark 2.3. We set $\kappa = -1$ and $\nu = 30$. Concerning the generation of a inverse Gaussian process, we follow an algorithm proposed in Rydberg [16]. By setting $c(\theta, \kappa) = a \ln(1 - \theta/b) + (\sqrt{-2\kappa + \nu^2} - \nu)$, we can induce the model risk-neutral. For simplicity, we omit the risk-free discounting. Figure 1 presents numerical results of the sensitivities $\partial/\partial S_0$, $\partial^2/\partial S_0^2$, $\partial/\partial \theta$ and $\partial/\partial \kappa$ of the expectation $\mathbb{E}_{\mathbb{P}}[\Phi(F_0)]$. For clear comparisons, the convergence of the finite difference method is also provided. The figures and the variance ratios evidently indicate a faster convergence of our Greeks formulas. Note also that in general, the finite difference approximation yields some bias.

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$\frac{\partial}{\partial S_0} \mathbb{E}_{\mathbb{P}}[\Phi(F_0)]$, vratio= 144 and $\varepsilon = 1e-3$.

$\frac{\partial^2}{\partial S_0^2} \mathbb{E}_{\mathbb{P}}[\Phi(F_0)]$, vratio= 33 and $\varepsilon = 1e-1$.

$\frac{\partial}{\partial \theta} \mathbb{E}_{\mathbb{P}}[\Phi(F_0)]$, vratio= 9 and $\varepsilon = 5e-3$.

$\frac{\partial}{\partial \kappa} \mathbb{E}_{\mathbb{P}}[\Phi(F_0)]$, vratio= 366 and $\varepsilon = 2e-2$.

Figure 1: "vratio" indicates the variance ratio $\frac{\text{Var}(\text{Finite Difference})}{\text{Var}(\text{Malliavin Calculus})}$. The quantity ε is the increment in the finite difference estimation $\frac{X((1+\varepsilon)\theta) - X((1-\varepsilon)\theta)}{2\varepsilon\theta}$ or $\frac{X((1+\varepsilon)\theta) - 2X(\theta) + X((1-\varepsilon)\theta)}{(\varepsilon\theta)^2}$.

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