Supplementary Materials for “Semiparametric Bayesian Hierarchical Models Heterogeneous Population in Nonlinear Mixed Effects Model with Application to Gastric Emptying Studies”

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A Bayesian Analysis of the Multivariate Normal Distribution

This appendix gives detailed derivations for Section 3.2.1. Suppose both the mean μ_i and precision $\lambda_{ei} = \sigma_i^{-2}$ for the measurement error of the i^{th} subject are unknown. We define $y_{ij}^* = y_{ij} - f_{ij}$, $\underline{Y}_i^* = \underline{Y}_i - \underline{f}_i$, where $\underline{Y}_i = (y_{i1}, \dots, y_{ij}, \dots, y_{in_i})^T$ and $\underline{f}_i = (f_{i1}, \dots, f_{ij}, \dots, f_{in_i})^T$. The likelihood of $\underline{Y}_i^*$ can be written as

$$\begin{aligned} p(\underline{Y}_i^* | \mu_i, \lambda_{ei}) &= \frac{1}{(2\pi)^{\frac{n_i}{2}}} \lambda_{ei}^{\frac{n_i}{2}} \exp\left\{-\frac{\lambda_{ei}}{2} \sum_{j=1}^{n_i} (y_{ij}^* - \mu_i)^2\right\} \\ &= \frac{1}{(2\pi)^{\frac{n_i}{2}}} \lambda_{ei}^{\frac{n_i}{2}} \exp\left[-\frac{\lambda_{ei}}{2} \left\{n_i(\mu_i - \overline{Y}_i^*)^2 + \sum_{j=1}^{n_i} (y_{ij}^* - \overline{Y}_i^*)^2\right\}\right]. \end{aligned}$$

A.1 The Marginal Prior for the Measurement Error Mean, μ_i

The conjugate prior for μ_i and λ_{ei} is the Normal-Gamma prior:

$$\begin{aligned} \text{NG}(\mu_i, \lambda_{ei} | \mu_0 = 0, g, v_0, s_0) &= \text{N}\{\mu_i | (g\lambda_{ei})^{-1}\} \text{Gamma}\{\lambda_{ei} | \frac{v_0}{2}, \text{rate} = \frac{v_0 s_0^2}{2}\} \\ &= \frac{1}{Z_{NG}} \lambda_{ei}^{1/2} \exp\left(-\frac{g\lambda_{ei}}{2} \mu_i^2\right) \lambda_{ei}^{v_0/2-1} e^{-\lambda_{ei} v_0 s_0^2/2} \\ &= \frac{1}{Z_{NG}} \lambda_{ei}^{\frac{v_0+1}{2}-1} \exp\left\{-\frac{\lambda_{ei}}{2} (g\mu_i^2 + v_0 s_0^2)\right\}, \end{aligned}$$

where

$$Z_{NG}(\mu_0, g, a_2, b_2) = \frac{\Gamma(\frac{v_0}{2})}{(\frac{v_0 s_0^2}{2})^{\frac{v_0}{2}}} \left(\frac{2\pi}{g}\right)^{1/2}.$$

We can compute the marginal prior on μ_i , by integrating out λ_{ei} , as follows:

$$\begin{aligned} p(\mu_i) &\propto \int p(\mu_i, \lambda_{ei}) d\lambda_{ei} \\ &= \int \lambda_{ei}^{\frac{v_0+1}{2}-1} \exp\left\{-\lambda_{ei} \left(\frac{v_0 s_0^2 + g\mu_i^2}{2}\right)\right\} d\lambda_{ei}. \end{aligned}$$

Since this formula is an unnormalized $\text{Gamma}\{a^* = (v_0 + 1)/2, b^* = (v_0 s_0^2 + g\mu_i^2)/2\}$ distribution, we can write down

$$p(\mu_i) \propto \frac{\Gamma(a^*)}{(b^*)^{a^*}} \propto (b^*)^{-a^*}$$

$$\begin{aligned}
&= \left(\frac{v_0 s_0^2 + g \mu_i^2}{2} \right)^{-(v_0+1)/2} \\
&= \left\{ 1 + \frac{1}{v_0} \frac{g \mu_i^2}{s_0^2} \right\}^{-(v_0+1)/2},
\end{aligned}$$

which is a $T_{v_0}\{\mu_i|\mu_0 = 0, s_0^2/g\}$ distribution.

A.2 The Posterior Distributions for the Measurement Error Parameters, μ_i and λ_{ei}

The posterior can be derived as follows:

$$p(\mu_i, \lambda_{ei} | \underline{Y_i^*}) \propto \text{NG}(\mu_i, \lambda_{ei} | \mu_0, g, v_0, s_0) p(\underline{Y_i^*} | \mu_i, \lambda_{ei}) \quad (1)$$

$$\propto \lambda_{ei}^{1/2} e^{-\frac{g \lambda_{ei} \mu_i^2}{2}} \lambda_{ei}^{\frac{v_0}{2}-1} e^{-\frac{v_0 s_0^2}{2} \lambda_{ei}} \lambda_{ei}^{n_i/2} e^{-\frac{\lambda_{ei}}{2} \sum^{n_i} (y_{ij}^* - \mu_i)^2} \quad (2)$$

$$\propto \lambda_{ei}^{1/2} \lambda_{ei}^{(v_0+n_i)/2-1} e^{-(v_0 s_0^2/2) \lambda_{ei}} e^{-\frac{\lambda_{ei}}{2} \{g(\mu_i)^2 + \sum (y_{ij}^* - \mu_i)^2\}}. \quad (3)$$

From the previous equation 1, we have

$$\sum_{j=1}^{n_i} (y_{ij}^* - \mu_i)^2 = n_i (\mu_i - \overline{Y_i^*})^2 + \sum_{j=1}^{n_i} (y_{ij}^* - \overline{Y_i^*})^2.$$

Then we can show that

$$\begin{aligned}
g \mu_i^2 + \sum (y_{ij}^* - \mu_i)^2 &= g \mu_i^2 + n_i (\mu_i - \overline{Y_i^*})^2 + \sum (y_{ij}^* - \overline{Y_i^*})^2 \\
&= (g + n_i) (\mu_i - \mu_{post})^2 + \frac{g n_i (\overline{Y_i^*})^2}{g + n_i} + \sum (y_{ij}^* - \overline{Y_i^*})^2.
\end{aligned}$$

Finally we obtain:

$$\begin{aligned}
&p(\mu_i, \lambda_{ei} | \underline{Y_i^*}) \\
&\propto \lambda_{ei}^{1/2} e^{-(\lambda_{ei}/2)(g+n_i)(\mu_i - \mu_{post})^2} \times \lambda_{ei}^{(v_0+n_i)/2-1} \\
&\quad e^{-(\lambda_{ei}/2) \sum (y_{ij}^* - \overline{Y_i^*})^2} e^{-(\lambda_{ei}/2) \frac{g n_i (\overline{Y_i^*} - \mu_0)^2}{g+n_i}} e^{-\lambda_{ei} v_0 s_0^2/2} \\
&\propto N\{\mu_i | \mu_{post}, (g_{post} \lambda_{ei})^{-1}\} \times \text{Gamma}(\lambda_{ei} | a_{post}/2, b_{post}/2),
\end{aligned}$$

where $\mu_{post} = (g \mu_0 + n_i \overline{Y_i^*}) / (g + n_i)$, $g_{post} = g + n_i$, $a_{post} = v_0 + n_i$, and $\{b_{post} = v_0 s_0^2 + \sum^{n_i} (y_{ij}^* - \overline{Y_i^*})^2 + g n_i (\overline{Y_i^*})^2\} / (g + n_i)$.

A.3 The Marginal Distribution for $\underline{Y}_i$

Assume $Y_i^* = (y_{i1}, y_{i2}, \dots, y_{in_i})^T$, and $\mu_0 = 0$. Then

$$\begin{aligned} p(\underline{Y}_i^*) &\propto \int \int NG(\mu_i, \lambda_{ei} | \mu_0 = 0, g, v_0, s_0) p(\underline{Y}_i^* | \mu_i, \lambda_{ei}) d\mu_i d\lambda_{ei} \\ &\propto (\beta_n/2)^{-\alpha_n/2} \\ &\propto [1/2\{v_0 s_0^2 + \sum_{j=1}^{n_i} (y_{ij}^* - \underline{Y}_i^*)^2 + \frac{gn_i(\underline{Y}_i^*)^2}{(g+n_i)}\}]^{-(v_0+n_i)/2}. \end{aligned}$$

It can be shown:

$$\begin{aligned} p(\underline{Y}_i^*) &\propto \{b_2 + \sum_{j=1}^{n_i} (y_{ij} - \underline{Y}_i^*)^2 + \frac{gn_i(\underline{Y}_i^*)^2}{g+n_i}\}^{-(v_0+n_i)/2} \\ &\propto (1 + \frac{1}{v_0} \frac{(\underline{Y}_i^*)' C (\underline{Y}_i^*)}{s_0^2})^{-(v_0+n_i)/2}, \end{aligned}$$

where

$$C = \begin{pmatrix} 1 - \frac{1}{g+n_i} & \dots & \frac{-1}{g+n_i} \\ \dots & \dots & \dots \\ \frac{-1}{g+n_i} & \dots & 1 - \frac{1}{g+n_i} \end{pmatrix}.$$

Therefore, we can know that $\underline{Y}_i^*$ is from a multivariate t distribution $T_{v_0}\{0, \Sigma = s_0^2/C\}$. By assuming $\underline{Y}_i^* = \underline{Y}_i - \underline{f}_i$, it is easy to obtain marginal distribution for $\underline{Y}_i$ is from a multivariate t distribution $T_{v_0}\{\underline{f}_i, \Sigma = s_0^2/C\}$.

B Conjugate Bayesian Analysis for the Random Effects

We assume that $\phi_i = (\beta_i, t_{50_i})$ is from a multivariate lognormal distribution, where $\ln(\phi_i)$ is from the normal distribution with the mean, $\ln(\theta_i)$ and the precision, Λ_i . We also assign a prior for $\ln(\theta_i)$, which is a multivariate normal with the mean, θ_0 and the precision, Λ_{00} . The prior for Λ_i is a Wishart distribution with a positive definite matrix, Λ_0/τ and the degree of freedom, τ .

B.1 Likelihood of the Individual Parameter, ϕ_i

We assume ϕ_i is a 2×1 vector, and the likelihood of ϕ_i is

$$p(\phi_i|\theta_i, \Lambda_i) = (2\pi)^{-h/2} \frac{1}{\beta_i t_{50_i}} |\Lambda_i|^{1/2} \exp[-(\ln\phi_i - \ln\theta_i)^T \Lambda_i (\ln\phi_i - \ln\theta_i)]. \quad (4)$$

B.2 Prior of Population Parameters, θ_i and Λ_i

The prior of θ_i and Λ_i can be written as,

$$\begin{aligned} p(\ln\theta_i, \Lambda_i) &= p(\ln\theta_i)p(\Lambda_i) \\ &= N\{\ln\theta_i|\theta_0, (\Lambda_{00})^{-1}\} \text{Wishart}(\Lambda_i|\Lambda_0/\tau, \tau) \\ &= \frac{1}{Z} |\Lambda_{00}|^{1/2} \exp\left\{-\frac{1}{2}(\ln\theta_i - \theta_0)^T \Lambda_{00} (\ln\theta_i - \theta_0)\right\} \\ &\times |\Lambda_i|^{(\tau-h-1)/2} \exp\left[-\frac{1}{2}\text{Tr}\{(\Lambda_0/\tau)^{-1} \Lambda_i\}\right], \end{aligned} \quad (5)$$

where $Z = (\frac{1}{2\pi})^{h/2} |\Lambda_0/\tau|^{\tau/2} 2^{\tau h/2} \Gamma_h(\tau/2)$.

B.3 Posterior of Population Parameters, θ_i and Λ_i

The posterior distributions of θ_i and Λ_i can be written as,

$$p(\ln\theta_i, \Lambda_i|\phi_i) = N\{\ln\theta_i|\theta_{ipost}, \Lambda_{ipost}^{-1}\} \text{Wishart}\{\Lambda_i|\Lambda_i^d/(\tau^d), \tau^d\}, \quad (6)$$

and

$$\begin{aligned} \theta_{ipost} &= (\Lambda_{ipost})^{-1} \{\Lambda_i \ln\phi_i + \Lambda_{00} \theta_0\}, \\ \Lambda_{ipost} &= \Lambda_{00} + \Lambda_i, \\ \Lambda_i^d &= S_i + \tau \Lambda_0^{-1}, \\ \tau^d &= \tau + 1. \end{aligned}$$

B.4 Marginal Likelihood of the Individual Parameter, ϕ_i .

The marginal likelihood of conditional variance parameters, ϕ_i , can be written as a ratio of constants,

$$\begin{aligned}
 p(\phi_i) &= \frac{p(\phi_i|\theta_i, \Lambda_i)p(\theta_i)p(\Lambda_i)}{p(\theta_i, \Lambda_i|\phi_i)} \\
 &= \frac{Z^*}{Z} \frac{1}{(2\pi)^{h/2}\beta_i t_{50_i}} \\
 &= \frac{1}{\pi^{h/2}} \frac{1}{\beta_i t_{50_i}} \frac{\Gamma_h(\tau^d/2)}{\Gamma_h(\tau/2)} \frac{|\Lambda_{ipost}/\tau^d|^{\tau^d/2}}{|\Lambda_{00}/\tau|^{\tau/2}}.
 \end{aligned}$$

Figure 1: The relationship among the data, unknown parameters and their priors for Model 1 are graphically represented.

Figure 2: The relationship among the data, unknown parameters and their priors for Model 2 are graphically represented.

Figure 3: The relationship among the data, unknown parameters and their priors for Model 3 are graphically represented.

Figure 4: The relationship among the data, unknown parameters and their priors for Model 4 are graphically represented.

Figure 5: The procedure for Slice sampling: (a) Draw an auxiliary variable, defining a horizontal “slice”, S_h ; (b) Find an interval, I ; (c) Draw a new point from $S_h \cap I$.

Figure 6: The estimated individual curves obtained from fitting NLME: The estimated individual curves obtained from fitting the nonlinear mixed effects model using 21 horses: the estimated individual curves of 12 horses are displayed in this figure. The gray dash-dot line curves are from the NLME with parametric priors, the black dash-dot line curves are from the NLME with DP errors, and the dashed curves are from the NLME with DP errors. The circles are the observed from individual horses.

Figure 7: The estimated individual curves obtained from fitting NLME: The estimated individual curves obtained from fitting the nonlinear mixed effects model using 21 horses: the estimated individual curves of 9 horses are displayed in this figure. The gray dash-dot line curves are from the NLME with parametric priors, the black dash-dot line curves are from the NLME with DP errors, and the dashed curves are from the NLME with DP errors. The circles are the observed from individual horses.

	Parametric Prior Model (Model 1)			DP Error Model (Model 2)			DP Random Effect Model (Model 4)		
	BCI25	Median	BCI975	BCI25	Median	BCI975	BCI25	Median	BCI975
β_1	0.23	0.73	1.90	0.62	0.89	1.48	0.63	1.03	1.60
β_2	0.97	1.15	1.33	1.03	1.15	1.29	0.98	1.13	1.30
β_3	0.88	1.09	1.25	0.98	1.08	1.17	0.92	1.08	1.27
β_4	1.11	2.15	4.48	1.22	1.87	2.28	0.76	2.08	3.28
β_5	3.43	4.39	5.58	3.62	4.46	5.34	3.29	4.03	5.21
β_6	0.95	1.22	1.50	1.01	1.21	1.43	0.98	1.20	1.48
β_7	1.12	1.28	1.52	1.13	1.29	1.50	1.10	1.26	1.46
β_8	0.84	0.99	1.16	0.90	1.02	1.13	0.90	1.04	1.23
β_9	0.77	1.25	1.99	1.20	1.36	1.55	0.98	1.37	1.97
β_{10}	1.09	1.64	2.74	1.65	1.87	2.12	1.10	1.47	2.00
β_{11}	0.59	0.74	0.91	0.57	0.72	0.87	0.60	0.74	0.88
β_{12}	0.83	1.28	1.03	0.97	1.06	1.15	0.87	1.07	1.28
β_{13}	0.75	0.92	1.11	0.76	0.92	1.19	0.75	0.94	1.12
β_{14}	0.65	1.26	0.91	0.84	0.96	1.05	0.80	1.07	1.36
β_{15}	3.38	3.80	4.30	3.09	3.67	4.45	3.42	3.86	4.36
β_{16}	2.81	3.13	3.44	2.80	3.10	3.44	2.82	3.12	3.46
β_{17}	0.61	0.78	0.97	0.61	0.79	1.00	0.62	0.78	0.97
β_{18}	2.73	3.09	3.46	2.78	3.06	3.43	2.73	3.07	3.42
β_{19}	2.64	3.03	3.46	2.69	3.04	3.38	2.63	3.03	3.55
β_{20}	1.69	1.93	2.27	1.68	1.91	2.27	1.70	1.99	2.27
β_{21}	2.08	2.40	2.72	2.14	2.40	2.69	2.14	2.45	2.79

Table 1: The estimation of the individual shape parameters: β_i , $i = 1, \dots, 21$, represents the shape parameter for the i^{th} individual. The median of β_i from the parametric model, DP error model, and DP random effects model are shown. BCI25 and BIC975 represent the lower and upper bounds of 95% Bayesian credible intervals for β_i .

	Parametric Prior Model (Model 1)			DP Error Model (Model 2)			DP Random Effect Model (Model 4)		
	BCI25	Median	BCI975	BCI25	Median	BCI975	BCI25	Median	BCI975
t_{50_1}	0.18	0.52	0.93	0.36	0.45	0.57	0.36	0.59	1.07
t_{50_2}	1.06	1.19	1.30	1.11	1.20	1.29	1.06	1.19	1.30
t_{50_3}	0.63	0.73	0.81	0.68	0.73	0.78	0.66	0.73	0.81
t_{50_4}	1.80	2.24	2.82	2.02	2.24	2.42	1.61	2.21	2.76
t_{50_5}	2.49	2.61	2.72	2.53	2.61	2.70	2.48	2.61	2.75
t_{50_6}	0.38	0.45	0.51	0.40	0.45	0.49	0.38	0.44	0.52
t_{50_7}	1.50	1.59	1.69	1.50	1.59	1.68	1.50	1.58	1.68
t_{50_8}	0.55	0.63	0.72	0.59	0.64	0.68	0.56	0.65	0.73
t_{50_9}	0.26	0.35	0.44	0.32	0.35	0.37	0.28	0.37	0.45
$t_{50_{10}}$	0.37	0.45	0.53	0.42	0.45	0.47	0.37	0.43	0.52
$t_{50_{11}}$	1.31	1.50	1.67	1.32	1.50	1.67	1.29	1.50	1.68
$t_{50_{12}}$	0.57	0.67	0.78	0.64	0.67	0.71	0.59	0.68	0.78
$t_{50_{13}}$	0.70	0.81	0.91	0.70	0.81	0.92	0.70	0.82	0.90
$t_{50_{14}}$	0.32	0.42	0.53	0.39	0.42	0.46	0.36	0.46	0.54
$t_{50_{15}}$	2.86	2.93	3.01	2.79	2.92	3.10	2.86	2.93	3.02
$t_{50_{16}}$	3.28	3.36	3.44	3.26	3.35	3.44	3.28	3.36	3.44
$t_{50_{17}}$	1.20	1.34	1.49	1.21	1.35	1.52	1.23	1.38	1.52
$t_{50_{18}}$	2.56	2.62	2.69	2.54	2.62	2.69	2.55	2.62	2.69
$t_{50_{19}}$	2.05	2.13	2.20	2.07	2.12	2.18	2.06	2.12	2.21
$t_{50_{20}}$	1.73	1.83	1.94	1.71	1.84	1.93	1.73	1.85	1.97
$t_{50_{21}}$	2.80	2.91	3.01	2.81	2.92	3.02	2.80	2.91	3.02

Table 2: The estimation of the individual half-meal time parameters: t_{50_i} , $i = 1, \dots, 21$, represents the half-meal time parameter for the i^{th} individual. The median of t_{50_i} from the parametric model, DP error model, and DP random effects model are shown. BCI25 and BIC975 represent the lower and upper bounds of 95% Bayesian credible intervals for t_{50_i} .

	Parametric Prior Model (Model 1)		
	BCI25	median	BCI975
β	1.27	1.48	1.69
t_{50}	0.98	1.12	1.28
λ_e	0.31	0.35	0.39

Table 3: The estimation of the population parameters (Model 1): β represents the population shape parameter. The median of β , t_{50} , and λ_e from the parametric model is shown. BCI25 and BIC975 represent the lower and upper bounds of 95% Bayesian credible intervals.

	DP Error Model (Model 2)		
	BCI25	median	BCI975
β	1.30	1.49	1.73
t_{50}	0.98	1.11	1.27
λ_e^1	2.81	8.14	29.72
λ_e^2	0.31	0.63	1.02
λ_e^3	0.63	1.19	2.69
λ_e^4	0.96	2.48	4.14
λ_e^5	0.38	0.80	1.28
λ_e^6	0.29	0.69	1.46
λ_e^7	0.22	0.33	0.45
λ_e^8	0.45	0.97	2.10
λ_e^9	1.65	4.37	14.09
λ_e^{10}	1.28	3.40	8.05
λ_e^{11}	0.26	0.33	0.56
λ_e^{12}	1.26	3.05	5.80
λ_e^{13}	0.27	0.37	0.88
λ_e^{14}	1.50	3.60	8.13
λ_e^{15}	0.07	0.12	0.19
λ_e^{16}	0.17	0.29	0.38
λ_e^{17}	0.17	0.31	0.40
λ_e^{18}	0.23	0.33	0.46
λ_e^{19}	0.32	0.69	1.08
λ_e^{20}	0.19	0.32	0.40
λ_e^{21}	0.27	0.34	0.54

Table 4: The estimation of the population shape parameters (Model 2): β represents the population shape parameter. The median of β from the DP error model is shown. BCI25 and BIC975 represent the lower and upper bounds of 95% Bayesian credible intervals for β .

	DP Random Effects Model (Model 4)		
	BCI25	median	BCI975
β^1	0.77	1.05	1.37
β^2	0.88	1.07	1.74
β^3	0.89	1.05	1.38
β^4	0.94	2.37	3.31
β^5	1.85	3.20	3.31
β^6	0.87	1.05	1.42
β^7	0.84	1.21	2.68
β^8	0.90	1.05	1.37
β^9	0.87	1.05	1.44
β^{10}	0.87	1.07	1.44
β^{11}	0.67	1.05	1.46
β^{12}	0.87	1.05	1.41
β^{13}	0.87	1.05	1.38
β^{14}	0.87	1.05	1.42
β^{15}	1.89	3.20	3.31
β^{16}	1.85	3.20	3.31
β^{17}	0.69	1.05	1.36
β^{18}	1.85	3.20	3.31
β^{19}	1.60	3.20	3.31
β^{20}	1.13	2.36	3.31
β^{21}	1.64	3.01	3.31

Table 5: The estimation of the population shape parameters (Model 4): β^i , $i = 1, \dots, 21$, represents the population shape parameter. The median of β^i from the DP random effects model is shown. BCI25 and BIC975 represent the lower and upper bounds of 95% Bayesian credible intervals for β^i .

	DP Random Effects Model (Model 4)		
	BCI25	median	BCI975
t_{50}^1	0.41	0.74	1.21
t_{50}^2	0.66	0.78	1.81
t_{50}^3	0.54	0.74	1.04
t_{50}^4	0.74	2.27	2.86
t_{50}^5	1.80	2.64	2.98
t_{50}^6	0.41	0.69	0.83
t_{50}^7	0.68	1.36	2.34
t_{50}^8	0.53	0.74	1.04
t_{50}^9	0.41	0.70	0.99
t_{50}^{10}	0.41	0.69	0.90
t_{50}^{11}	0.66	0.85	1.89
t_{50}^{12}	0.53	0.74	1.04
t_{50}^{13}	0.54	0.76	1.22
t_{50}^{14}	0.41	0.69	0.88
t_{50}^{15}	1.80	2.64	3.02
t_{50}^{16}	1.80	2.64	3.06
t_{50}^{17}	0.66	0.78	1.77
t_{50}^{18}	1.80	2.64	2.98
t_{50}^{19}	1.68	2.64	2.95
t_{50}^{20}	1.17	2.18	2.84
t_{50}^{21}	1.80	2.64	3.02

Table 6: The estimation of the population half-meal time parameters (Model 4): t_{50}^i , $i = 1, \dots, 21$, represents the half-meal time parameter for the i^{th} individual. The median of t_{50}^i from the DP random effects model are shown. BCI25 and BIC975 represent the lower and upper bounds of 95% Bayesian credible intervals for t_{50}^i .

	DP Random Effects Model (Model 4)		
	BCI25	median	BCI975
λ_e	0.31	0.35	0.40

Table 7: The estimation of the population measurement precision parameters (Model 4): λ_e represents the measurement precision parameter for the population model. The median of λ_e from the DP random effects model is shown. BCI25 and BIC975 represent the lower and upper bounds of 95% Bayesian credible intervals for λ_e .

	Log-penalized Posterior Likelihood		
	BCI25	Median	BCI975
Parametric Model (Model 1)	-836.55	-821.28	-811.29
DP Measurement Error Model (Model 2)	-744.51	-728.42	-716.16
DP Random Effects Model (Model 4)	-799.34	-786.07	-773.93

Table 8: Logarithmic values for the penalized posterior likelihood: BCI25 and BIC975 represent the lower and upper bounds of 95% Bayesian credible intervals.