Robust estimation of mean and covariance for longitudinal data with dropouts

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This file contains supplementary material for this paper. The material consists of the description of the asymptotic normality of $\hat{\beta}_f$, the proof of $E(R_{ij}\psi(r_{ij})) = 0$ and the proof of Theorem 1.

Lemma 1 *Under conditions (C.1) - (C.3), $\sqrt{n}(\hat{\beta}_f - \beta_0)$ converges to $N(0, P^{-1}QP^{-1})$ in distribution.*

Using regular procedure, it is not difficult to derive the result of Lemma 1 which is thus omitted here.

Lemma 2 *Under the missing mechanism of MAR and the assumption that ϵ_i follow elliptical distribution, $E(R_{ij}\psi(r_{ij})) = 0$ where $r_{ij} = (Y_{ij} - D_{ij}^T\theta_0)/b_j$.*

Proof: Note that if ϵ_i follow elliptical distribution with mean zero, then $e_i = C\epsilon_i$ also follow elliptical distribution with mean zero. Denote $\tilde{e}_{ij} = (e_{i1}, \dots, e_{ij-1})^T$ and $\tilde{Y}_{ij} = (Y_{i1}, \dots, Y_{ij-1})^T$. Thus we have

$$\begin{aligned} E(R_{ij}\psi(r_{ij})) &= E\{E(R_{ij}\psi(r_{ij})|e_i, Y_i)\} \\ &= E\{E(R_{ij}|e_i, Y_i)\psi(e_{ij}/b_j)\} \end{aligned}$$

Under MAR, the probability of being observed only depends on the observed values, i.e., its previously observed values in the case of dropouts. Thus, $E(R_{ij}|e_i, Y_i) = E(R_{ij}|\tilde{e}_{ij}, \tilde{Y}_{ij})$. Denote $\pi_{ij}(\tilde{e}_{ij}, \tilde{Y}_{ij}) = E(R_{ij}|\tilde{e}_{ij}, \tilde{Y}_{ij})$. Then we have

$$\begin{aligned} E[E\{(R_{ij}|e_i, Y_i)\psi(e_{ij}/b_j)\}] &= E[\pi_{ij}(\tilde{e}_{ij}, \tilde{Y}_{ij})\psi(e_{ij}/b_j)] \\ &= E[E\{\pi_{ij}(\tilde{e}_{ij}, \tilde{Y}_{ij})\psi(e_{ij}/b_j)|\tilde{e}_{ij}, \tilde{Y}_{ij}\}] \\ &= E[\pi_{ij}(\tilde{e}_{ij}, \tilde{Y}_{ij})E\{\psi(e_{ij}/b_j)|\tilde{e}_{ij}, \tilde{Y}_{ij}\}]. \end{aligned}$$

Since e_i follow elliptical distribution with mean zero and its covariance are identity matrix, according to the property of elliptical distribution, we have $E(e_{ij}|\tilde{e}_{ij}) = 0$. Thus, $E(\psi(e_{ij}/b_j)|\tilde{e}_{ij}, \tilde{Y}_{ij}) = 0$ considering that elliptical distribution is symmetric. Therefore, $E(R_{ij}\psi(r_{ij})) = 0$. We refer to Gómez, E.(2003) for details on the property of elliptical distribution.

Proof of Theorem 1

Let us recall the robust estimating equation

$$\sum_{i=1}^n \hat{D}_i^T V_i^{-1/2} W_i R_i \psi(V_i^{-1/2}(Y_i - \hat{D}_i \theta)) = 0. \quad (1.1)$$

Using the Taylor expansion, we can get

$$n^{1/2}(\hat{\theta} - \theta_0) = \left\{ \frac{1}{n} \sum_{i=1}^n \hat{D}_i^T V_i^{-1/2} W_i R_i \dot{\psi}(\hat{r}_i) \hat{D}_i \right\}^{-1} \left\{ n^{-1/2} \sum_{i=1}^n \hat{D}_i^T V_i^{-1/2} W_i R_i \psi(\hat{r}_i) \right\} + o_p(1),$$

where $\hat{r}_i = V_i^{-1/2}(Y_i - \hat{D}_i \theta_0)$.

By Lemma 1, $\|\hat{\beta}_f - \beta_0\| = O_p(n^{-1/2})$, so $\|\hat{D}_i - D_i\| = O_p(n^{-1/2})$. Then it is not difficult to show that

$$\left\| \frac{1}{n} \sum_{i=1}^n \hat{D}_i^T V_i^{-1/2} W_i R_i \dot{\psi}(\hat{r}_i) \hat{D}_i \right\| = \left\| \frac{1}{n} \sum_{i=1}^n D_i^T V_i^{-1/2} W_i R_i \dot{\psi}(r_i) D_i \right\| + o_p(1).$$

Next, we shall show the asymptotic normality of the term $n^{-1/2} \hat{D}_i^T V_i^{-1/2} W_i R_i \psi(\hat{r}_i)$.

$$\begin{aligned} & n^{-1/2} \sum_{i=1}^n \hat{D}_i^T V_i^{-1/2} W_i R_i \psi(\hat{r}_i) \\ &= n^{-1/2} \sum_{i=1}^n \sum_{j=1}^m \hat{D}_{ij} (w_{ij} R_{ij} / b_j) \psi(\hat{r}_{ij}) \\ &= n^{-1/2} \sum_{i=1}^n \sum_{j=1}^m \{ (\hat{D}_{ij} - D_{ij}) (w_{ij} R_{ij} / b_j) \psi(\hat{r}_{ij}) + D_{ij} (w_{ij} R_{ij} / b_j) \psi(\hat{r}_{ij}) \} \end{aligned}$$

$$\begin{aligned}
&= n^{-1/2} \sum_{i=1}^n \sum_{j=1}^m (\hat{D}_{ij} - D_{ij})(w_{ij}R_{ij}/b_j)\psi(r_{ij}) \\
&\quad + n^{-1/2} \sum_{i=1}^n \sum_{j=1}^m (\hat{D}_{ij} - D_{ij})(w_{ij}R_{ij}/b_j)\dot{\psi}(r_{ij}^*)(\hat{r}_{ij} - r_{ij}) \\
&\quad + n^{-1/2} \sum_{i=1}^n \sum_{j=1}^m D_{ij}(w_{ij}R_{ij}/b_j)\psi(r_{ij}) \\
&\quad + n^{-1/2} \sum_{i=1}^n \sum_{j=1}^m D_{ij}(w_{ij}R_{ij}/b_j)\dot{\psi}(r_{ij}^*)(\hat{r}_{ij} - r_{ij}) \\
&\doteq A_1 + A_2 + A_3 + A_4,
\end{aligned} \tag{1.2}$$

where $r_{ij}^* = (Y_{ij} - D_{ij}^*\theta_0)/b_j$ and D_{ij}^* are evaluated at β^* which is between the line β_0 and $\hat{\beta}_f$.

Since $\|\hat{\beta}_f - \beta_0\| = O_p(n^{-1/2})$ and thus $\hat{\epsilon}_{ij} - \epsilon_{ij} = O_p(n^{-1/2})$, it is easy to verify that $\|A_1\| = o_p(1)$ and $\|A_2\| = o_p(1)$.

We first consider A_4 .

$$\begin{aligned}
A_4 &= n^{-1/2} \sum_{i=1}^n \sum_{j=1}^m D_{ij}(w_{ij}R_{ij}/b_j)\dot{\psi}(r_{ij}^*)(\hat{r}_{ij} - r_{ij}) \\
&= n^{-1/2} \sum_{i=1}^n \sum_{j=1}^m D_{ij}(w_{ij}R_{ij}/b_j)\dot{\psi}(r_{ij}^*)(D_{ij} - \hat{D}_{ij})\theta_0 \\
&= n^{-1/2} \sum_{i=1}^n \sum_{j=1}^m D_{ij}(w_{ij}R_{ij}/b_j)\dot{\psi}(r_{ij})(D_{ij} - \hat{D}_{ij})\theta_0 + o_p(1)
\end{aligned} \tag{1.3}$$

Since $D_{ij} = (X_{ij}^T, F_{ij}^T)^T$, we have $(D_{ij} - \hat{D}_{ij})\theta_0 = -\sum_{k=1}^{j-1} \varsigma_{jk} X_{ik}^T (\hat{\beta}_f - \beta_0)$. Moreover, using regular procedure, it is not difficult to show

$$\hat{\beta}_f - \beta_0 = \left\{ \sum_{i=1}^n X_{i1} w_{i1} \dot{\psi}(r_{i1}) X_{i1}^T \right\}^{-1} \sum_{i=1}^n X_{i1} w_{i1} \psi(r_{i1}) + o_p(n^{-1/2}).$$

Then we have

$$\begin{aligned}
A_4 &= -n^{-1/2} \sum_{i=1}^n \sum_{j=1}^m D_{ij}(w_{ij}R_{ij}/b_j)\dot{\psi}(r_{ij}^*) \sum_{k=1}^{j-1} \varsigma_{jk} X_{ik}^T (\hat{\beta}_f - \beta_0) \\
&= -n^{-1/2} \sum_{i=1}^n \sum_{j=1}^m D_{ij}(w_{ij}R_{ij}/b_j)\dot{\psi}(r_{ij}) \sum_{k=1}^{j-1} \varsigma_{jk} X_{ik}^T \left\{ \sum_{i=1}^n X_{i1} w_{i1} \dot{\psi}(r_{i1}) X_{i1}^T \right\}^{-1} \sum_{i=1}^n X_{i1} w_{i1} \psi(r_{i1}) \\
&\quad + o_p(1) \\
&= -n^{-1/2} \sum_{i=1}^n \left[\sum_{i=1}^n \sum_{j=1}^m D_{ij}(w_{ij}R_{ij}/b_j)\dot{\psi}(r_{ij}) \sum_{k=1}^{j-1} \varsigma_{jk} X_{ik}^T \left\{ \sum_{i=1}^n X_{i1} w_{i1} \dot{\psi}(r_{i1}) X_{i1}^T \right\}^{-1} X_{i1} w_{i1} \psi(r_{i1}) \right]
\end{aligned}$$

$$+o_p(1) \tag{1.4}$$

and

$$\begin{aligned} A_3 + A_4 = & n^{-1/2} \sum_{i=1}^n \left[\left\{ \sum_{j=1}^m D_{ij}(w_{ij}R_{ij}/b_j)\psi(r_{ij}) \right\} \right. \\ & \left. - \left\{ \sum_{i=1}^n \sum_{j=1}^m D_{ij}(w_{ij}R_{ij}/b_j)\dot{\psi}(r_{ij}) \sum_{k=1}^{j-1} \varsigma_{jk} X_{ik}^T \right\} \left\{ \sum_{i=1}^n X_{i1} w_{i1} \dot{\psi}(r_{i1}) X_{i1}^T \right\}^{-1} X_{i1} w_{i1} \psi(r_{i1}) \right] + o_p(1). \end{aligned}$$

Then according to Lemma 2 and condition (C.5), and by central limit theorem, we obtain

$$A_3 + A_4 \rightarrow N(0, K),$$

and in the end,

$$n^{1/2}(\hat{\theta} - \theta_0) \rightarrow N(0, \Omega),$$

where $\Omega = S^{-1}KS^{-1}$.