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Recent results in tilt control design and assessment of high-speed railway vehicles

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Abstract: Active tilt control is a well-established technology in modern railway vehicles, for which currently used control approaches have evolved in an intuitive manner. This paper presents work on a set of novel strategies for achieving local tilt control, i.e. applied independently for each vehicle rather than the whole train precedence approach that is commonly used. A linearized dynamic model is developed for a modern tilting railway vehicle with a tilt mechanism (tilting bolster) providing tilt below the secondary suspension. It addresses the fundamental problems associated with straightforward feedback control, and briefly discusses the current industry norm, which employs command-driven with precedence strategy. Two new advanced schemes are proposed, a model-based estimation approach, and an optimal LQG-based approach, and compared to the command-driven with precedence. The performance of the control schemes is assessed through simulation using a new proposed assessment method.

Keywords: tilting railway vehicles, tilt control, tilt performance assessment, Kalman–Bucy filter, linear quadratic regulator

1 INTRODUCTION

Active tilting has become well established in modern railway vehicle technology, with most new high-speed trains in Europe now fitted with tilt and an increasing interest for regional express trains. There have been a number of evolutionary changes: refinements to the mechanical schemes that provide the tilting capability, a progressive change from hydraulic actuation towards the use of electro-mechanical actuation, and work on the electronic system design to take advantage of digital technology and provide more effective fault-tolerance and condition monitoring [1].

Early tilt systems used controllers based only upon local vehicle measurements, although it proved impossible at the time to get an appropriate combination of straight track and curve transition

performance, and in Europe, most systems now use the precedence control schemes [2] devised in the early 1980s as part of the Advanced Passenger Train development [3]. In this scheme, a bogie-mounted accelerometer from the vehicle in front is used to provide ‘precedence’: the signal measures the cant deficiency, and therefore provides a tilt command, but requires filtering to remove the high frequency components caused by track irregularities. The controller is carefully designed so that the delay introduced by the filter compensates for the preview time corresponding to a vehicle length. There has been some development of the concept, including the use of additional sensors such as gyroscopes measuring bogie roll rate, but the overall principles remain the same. In addition, there are various developments based upon using curving information from track databases, either using direct train location data or by synchronizing the database information on the basis of the tilt sensor signals.

Nevertheless, achieving a satisfactory local tilt control strategy is still an important research

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target because of the system simplifications and more straightforward failure detection, and this paper describes work from a research project that is investigating advanced control techniques, with the particular objective of identifying effective strategies that can be applied to each vehicle independently, i.e. without using precedence control.

2 CONCEPT OF TILT

When a train traverses a curve at a high speed, the passengers experience a centrifugal force, and similar forces act on the body and the bogies. The track is usually canted by up to typically 6° to reduce the effect of curving such that the acceleration becomes $(v^2/R - g\theta_0)$. The passengers are also affected by the amount of roll into or out of the curve by the vehicle determined by the suspension geometry and the forward speed of the vehicle. As the speed increases the lateral acceleration rises rapidly, for example, doubling the vehicle speed will more than quadruple the acceleration.

The amount of lateral acceleration experienced by the passengers can be reduced by tilting (leaning inwards) the vehicle body. Figure 1(a) illustrates the forces acting on a non-tilting vehicle traversing a curve, and Fig. 1(b) presents the situation for a tilting vehicle on the same curve (θ_v is the body roll, θ_0 is the track cant angle, R the curve radius, and v the forward speed).

As the train passes from straight to curved track there is a transition, during which the cant and curvature change (magnitudes increasing or decreasing linearly). This of course has an impact on the forces acting on the train and thus on the levels of lateral acceleration perceived by the passengers. Note that the duration of curve transitions depends both upon the track layout and the operating vehicle speed.

Fig. 1 Curving forces applied on a railway vehicle

Figure 2(a) illustrates the passenger acceleration for a non-tilting (conventional) vehicle running at nominal speed for a given track. At higher speeds, and for the same track, the transition becomes more severe (slope on transition is sharper) due to the smaller duration time, and also the level of steady-state lateral acceleration felt by the passengers increases (Fig. 2(b)). The introduction of tilt action will therefore allow the vehicle operation at speeds higher than those acceptable to passengers in a non-tilting vehicle, Fig. 2(b). Note that some modern high-speed lines (i.e. in France and Germany) are designed to have curved tracks with longer transitions and larger curvatures to allow for the operation of non-tilting high-speed trains.

Although tilt action could be used to provide an increase in passenger comfort at conventional vehicle speeds, the main commercial benefit from the use of tilting vehicles is the reduction of journey times on conventional tracks without degrading passenger comfort levels. A deciding factor for the reduction in journey time is the frequency of curves appearing in the particular route, i.e. the more curvaceous the route, the greater is the benefit of incorporating tilt action.

Fig. 2 Passenger perceived curving acceleration

3 REQUIREMENTS AND ASSESSMENT

3.1 Tilt control objectives

The performance of the tilt control system on the curve transitions is critical, but if it acts too quickly the passenger ride comfort provided by the tilting vehicle may be degraded compared with the non-tilting vehicle speeds. The main objective of a tilt control system is to provide an acceptably fast response to changes relative to track cant and curvature (deterministic features) while not reacting significantly to track irregularities (stochastic features), which represents a fundamental trade-off. Moreover, any tilt control system directly controls the secondary suspension roll angle and not the vehicle lateral acceleration. Incorporating an excessively fast controller may provide high roll rates and also jerk levels that are unacceptable. On the other hand, a slow controller will provide low roll rates and probably jerk levels, thus giving an unacceptable increase of the lateral acceleration during the curve transition before compensating by tilting the vehicle body.

The assessment of tilting train curve transition performance based upon the P_{CT} factors arose from the difficulties with tilting trains and relies upon a comprehensive experimental/empirical study undertaken in the 1980s [4], which derived factors indicating the percentage of passengers feeling uncomfortable on the transition, known as P_{CT} (see Appendix 2). This approach is now accepted as a European standard [5] and details of the equations that provide the values are given in Appendix 2, from which it can be seen that the passenger comfort is affected by three variables: lateral acceleration, lateral jerk, and roll velocity. The expressions are derived empirically and provide the percentage of passengers who feel uncomfortable during the curve transition, both standing and seated, hence providing a realistic and objective measure. The P_{CT} evaluation formula applies for the transition entry on curves and reverse transitions, having a time duration of at least 2 s.

The lateral acceleration experienced on the vehicle body during a curved track consists of: (a) a

component due to the deterministic track features (cant and curvature) combined with the body tilt angle; (b) a component due to the suspension dynamic response (lower sway oscillations) to both deterministic and stochastic track features. The main performance requirements for the tilt control system are summarized in Table 1.

From a control point of view the objectives of the tilt control system can be translated in terms of the tilt control loop's frequency response: increase the response of the system at low frequencies (deterministic track features) and attenuate the high frequency system response (stochastic track features) while maintaining stability.

3.2 Tilt controller performance assessment method

Although active tilting has become a standard technology incorporated into the railway industry, a number of issues remain, which need to be resolved for determining the performance of tilting trains. Qualitatively, a good tilt control system will respond principally to the deterministic track inputs, while ignoring as much as possible any random track irregularities. In order to assess different tilt control approaches in an objective manner, it is essential to define appropriate criteria and conditions.

3.2.1 Part I: curve transition performance – deterministic criterion

The assessment of tilt controllers is based upon work in reference [6], which proposes a rigorous overall approach for accessing the deterministic performance of tilt control systems. The procedure is organized as follows.

The curve transition response is separated into two aspects.

1. Investigation of the fundamental tilting response based upon the P_{CT} factor.
2. Investigation of the transitional dynamic suspension effects based upon comparison with the 'ideal tilting' response.

Table 1 Tilt control system performance requirements

Deterministic (steady state)	Deterministic (transition)	Stochastic
(a) To reduce the lateral acceleration perceived by the passengers on curves (this paper utilises 75 per cent tilt compensation, for the control designs, i.e. passenger acceleration is one-fourth of its non-tilting counterpart)	(b) To provide a comfortable response during curve transitions (tilting trains are designed to operate at higher speeds and the curve transition time therefore decreases) based upon the P_{CT} factors and the ideal tilting criterion	(c) To maintain the straight track performance within acceptable limits (specified as not more than 7.5 per cent deterioration compared to the passive suspension system at the same speed)

The fundamental tilting response, provided by the P_{CT} factors, must be as good as a passive vehicle at lower (non-tilting) speed, otherwise the passenger comfort will inevitably be diminished, regardless of the effectiveness of the tilt control system. It is possible therefore to introduce the idea of ideal tilting, i.e. where the tilt action follows the specified tilt compensation in an ideal manner, defined on the basis of the maximum tilt angle and cant deficiency compensation factor. This combination of parameters can be optimized using the P_{CT} factor approach for deterministic inputs in order to choose a basic operating condition, and this will give 'ideal' P_{CT} values (one for standing and one for sitting).

Moreover, it is necessary to quantify the additional dynamic effects that are caused by the suspension/controller dynamics as the transitions to and from the curves are encountered, essentially the deviations from the ideal response mentioned in the previous paragraph. These deviations relate to both the lateral acceleration and roll velocity, although the former is the main consideration. The aim is to minimize the resultant deviations, and the values derived for a normal passive suspension can be used as a guide for their acceptable size. The calculation of the deviations is defined as follows:

- (a) $|\ddot{y}_{mm} - \ddot{y}_{mmi}|$, the deviation of the actual lateral acceleration $\ddot{y}_{mm}$ from the ideal lateral acceleration $\ddot{y}_{mmi}$, in the time interval between 1 s before the start of the curve transition and 3.6 s after the end of the transition (Fig. 3);

Fig. 3 Calculations for deviations of actual versus ideal tilt responses for both acceleration and roll velocity

- (b) $|\dot{\theta}_{mm} - \dot{\theta}_{mmi}|$, the deviation of the actual absolute roll velocity $\dot{\theta}_{mm}$ from the ideal absolute roll velocity $\dot{\theta}_{mmi}$, in the time interval between 1 s before the start of the curve transition and 3.6 s after the end of the transition (Fig. 3).

The analysis is based upon a perfectly-aligned track, in which the cant and curvature rise linearly with time/distance, while the tilting action is applied in a similarly synchronized manner. The following example provides an insight into the proposed assessment method [6].

3.2.2 Example of deterministic tilt performance

Consider the curved track input for the non-tilting condition given in Table 2.

Regarding the tilting case two things need to be specified, the cant deficiency compensation factor and the speed-up factor, i.e. the ratio of tilting to non-tilting speeds. The right hand diagram of Fig. 4 illustrates the ideal values for a typical tilting condition – 30 per cent increase in speed and 60 per cent cant deficiency compensation (note that the compensation factor used here is for illustrative purposes, the main control designs in this paper use 75 per cent tilt compensation as mentioned earlier in the section). The comparison of the two diagrams illustrates that, although the lateral acceleration is reduced, the jerk and roll rates are increased compared to the passive case.

The next step is to evaluate the P_{CT} factors for the tilting vehicle and compare with those for the non-tilting train; the required tilt angle also emerges from the calculation process. Figure 5 shows the two P_{CT} factors and the maximum tilt angle for speed-up factors of 15–35 per cent with compensation factors varying from 40–80 per cent. From these, it can be seen that to satisfy the requirement for seated passengers a 30 per cent increase in speed is possible with a compensation factor of 0.63 and a tilt angle of 9° ; for standing passengers the corresponding values are 0.69 and 10° . It is clear therefore that, given the industry maximum of around 8° – 9° , 30 per cent speed-up cannot be achieved without deteriorating the passengers' comfort during curve transitions. For a 25 per cent increase in speed the values are: for seated 0.57° and 6.6° ; for standing 0.61° and 7.8° .

Table 2 Sample track input set – non-tilting

Curve radius $R = 1000$ m; track cant $\theta_{cant} = 6^\circ$; transition length = 145 m
Cant deficiency = 6° ; passive vehicle speed $v_o = 159$ (km/h)
Passive roll-out (assumed) = 0.6° ; $P_{CT}^{(standing)} = 29.0\%$; $P_{CT}^{(seated)} = 7.9\%$

Fig. 4 Acceleration, jerk, and roll rate time history for ideal tilting response

Note that the transition length used is relatively long (more than 3 s at non-tilting speed). Use of a shorter transition would further increase the tilting P_{CT} , effectively reducing the speed-up potential.

Two sample control strategies have been used for comparison based upon the proposed method: a command-driven with precedence type and a model-based estimation scheme used for local-per-vehicle control (more details on this scheme can be seen later in the paper). The assessment involves 30 per cent

speed increase and 60 per cent tilt compensation. Since the passive (non-tilting) case obviously provides a useful baseline for the size of the deviations, this has been included, while Figs 6(a) and (b) show the time histories. The ideal acceleration and roll rates are also shown on the graphs for the dynamic deviations caused by the suspension and/or controller dynamics to be clearly illustrated. Table 3 provides a qualitative comparison, presenting the r.m.s. values of deviations during the curve transition.

Fig. 5 P_{CT} factors: (a) seated, (b) standing, and (c) max tilt angle

Fig. 6 Vehicle responses (tilt vehicle with active anti-roll bar)

3.3 Straight track performance (stochastic criterion)

The analysis of the performance of the tilting suspension in the stochastic case relies upon the calculation of precise values for the ride quality in response to the effects of the track misalignments.

The criterion for straight track performance is to allow the degradation of the lateral ride quality by no more than a specified margin compared with the non-tilting vehicle, a typical value being 7.5 per cent, which is used throughout in this research work. For the assessment of the tilt controller performance this comparison should be made at the higher speed (note however that the passive vehicle is used only for comparison, and in reality it will not run at excess speeds). Naturally, a comparison of ride quality with a lower speed vehicle would be also needed, although achieving an acceptable ride quality at elevated speeds will involve either improved overall suspensions or better quality track, i.e. not a function of the tilt controller [6].

It should be noted that although some control concepts have suggested trying to disable the tilt action on straight track, in practice, it is very difficult to detect the start of a curve transition and re-enable

the action quickly enough, and the authors believe this is not an effective approach.

4 MODELLING

4.1 Vehicle model

The mathematical model of the system is based upon the end view of a railway vehicle as shown in Fig. 7(a), incorporating both the lateral and roll degrees of freedom for both the body and the bogie structures. A representation of a pair of airsprings is used to model the roll effect of the secondary vertical suspension, however, the actual vertical degrees of freedom are neglected. For further simplicity, wheelset dynamics are not taken into account, but the associated effect is included in the model using an appropriate low-pass filter to characterize the bogie dynamic response. Pairs of parallel spring/damper combinations were used to model the primary and secondary suspensions (vertical and lateral), with additional damper end-stiffness included in the case of lateral secondary suspensions. The stiffness of an anti-roll bar connected between the body and the bogie/bolster is also incorporated in the model (including roll damping).

Table 3 Comparison of r.m.s. deviations for the example

	Passive at 45 (m/s)	Passive at 58.5 (m/s)	Precedence at 58.5 (m/s)	Model-based estimation 58.5 (m/s)
Deviations roll rate (°/s)	0.009	0.012	0.017	0.016
Deviations acceleration (%g)	0.90	1.775	1.05	1.40

Fig. 7 Tilting vehicle with tilt mechanism below secondary

Active tilting is provided via a tilting mechanism, called a 'tilting bolster', mounted on the vehicle bogie – this mechanical arrangement is known as 'tilt below secondary'. In this arrangement the secondary suspension acts between the tilting bolster and the vehicle body, and as a consequence the lateral suspension does not have to react to the increased curving forces, which reduces the suspension deflections. There are still issues with bogie weight and complexity as well as with increased actuator force, albeit available technology can overcome such problems (though such structures are more expensive compared to simpler forms of tilt). Note that the inclined swing links imply that the effective tilt centre is still above the vehicle body floor level even if the tilt action is applied below the vehicle body. The tilt mechanism can provide tilt action up to 10° compared to the simpler active anti-roll bar schemes, which provide only restricted amounts of tilt [7].

A simplified representation of the actuation system, which represents a position servo in series with the mechanism, is depicted in Fig. 7(b). The parameters were chosen so that they provide 3.5 Hz bandwidth and 50 per cent damping for the closed-loop position servomechanism. The vehicle model is characterized by the following set of linearized equations (which correspond to local reference axis).

For the vehicle body (lateral and roll respectively)

$$m_v \ddot{y}_v = -2k_{sy}y_v + 2k_{sy}h_1\theta_v + 2(k_{sy} + k_{csy})y_b + 2(k_{sy} + k_{csy})h_2\theta_b - 2k_{csy}y_{es} - [2h_{mt}(k_{sy} + k_{csy}) - m_v g]\theta_m + m_v g\theta_o - \frac{m_v v^2}{R} - m_v h_{g1}\ddot{\theta}_o \quad (1)$$

$$i_{vr} \ddot{\theta}_v = (2h_1k_{sy} + m_v g)y_v - [k_{vr} + 2h_1^2k_{sy} + 2d_1^2(k_{az} + k_{sz})]\theta_v - [2h_1(k_{sy} + k_{csy}) + m_v g]y_b + [k_{vr} + 2d_1^2k_{az} - 2h_1h_2(k_{sy} + k_{csy})]\theta_b - c_{vr}\dot{\theta}_v + c_{vr}\dot{\theta}_b + 2k_{sz}d_1^2\theta_r + 2h_1k_{csy}y_{es} + [k_{vr} + 2d_1^2k_{az} + 2(k_{sy} + k_{csy})h_1h_{mt}]\theta_m + c_{vr}\dot{\theta}_m - i_{vr}\ddot{\theta}_o \quad (2)$$

for the vehicle bogie (lateral and roll respectively)

$$m_b \ddot{y}_b = 2k_{sy}y_v - 2h_1k_{sy}\theta_v - 2[(k_{sy} + k_{csy}) + k_{py}]y_b - 2[h_2(k_{sy} + k_{csy}) - h_3k_{py}]\theta_b - 2c_{py}\dot{y}_b + 2h_3c_{py}\dot{\theta}_b + 2k_{csy}y_{es} + 2k_{py}y_w + 2c_{py}\dot{y}_w + 2h_{mt}(k_{sy} + k_{csy})\theta_m + m_b g\theta_o - \frac{m_b v^2}{R} - m_b h_{g2}\ddot{\theta}_o \quad (3)$$

$$\begin{aligned}
i_{br}\ddot{\theta}_b = & 2h_2k_{sy}y_v + [k_{vr} - 2h_2h_1k_{sy} + 2d_1^2(k_{az} + k_{sz})]\theta_v \\
& - 2[h_2(k_{sy} + k_{csy}) - h_3k_{py}]y_b \\
& - [k_{vr} + 2h_2^2(k_{sy} + k_{csy}) + 2h_3^2k_{py} + 2d_2^2k_{pz} \\
& + 2d_1^2k_{az}]\theta_b + c_{vr}\dot{\theta}_v + 2h_3c_{py}\dot{y}_b \\
& - (c_{vr} + 2d_2^2c_{pz} + 2h_3^2c_{py})\dot{\theta}_b - 2k_{sz}d_1^2\theta_r \\
& + 2h_2k_{csy}y_{es} - 2h_3k_{py}y_w - 2h_3c_{py}\dot{y}_w \\
& - [k_{vr} + 2d_1^2k_{az} - 2(k_{sy} + k_{csy})h_2h_{mt}]\theta_m \\
& - c_{vr}\dot{\theta}_m - i_{br}\ddot{\theta}_o
\end{aligned} \quad (4)$$

Additionally, for the tilt actuator (servomechanism) and the bogie dynamic response (wheelset lateral 'filtering' effect) respectively

$$\begin{aligned}
\frac{\theta_m}{\theta_{mi}}(s) &= \frac{483.6}{s^2 + 22s + 483.6} \\
\frac{y_w}{y_o}(s) &= \frac{987}{s^2 + 12.57s + 987}
\end{aligned} \quad (5)$$

and also for the damper end-stiffness state

$$\begin{aligned}
\dot{y}_{es} = & c_{sy}^{-1}(k_{csy}y_b + h_2k_{csy}\theta_b + c_{sy}\dot{y}_v \\
& - c_{sy}h_1\dot{\theta}_v - k_{csy}h_{mt}\theta_m - k_{csy}y_{es})
\end{aligned} \quad (6)$$

The model parameter values are listed in Appendix 3. Note that equation (2) includes an 'end-moment'

effect: $m_v g(y_v - y_b)$, which models the roll effect of the body weight due to the lateral displacement of its centre of gravity (c.o.g.) (this effect is neglected in the case of the bogie mass (4) owing to the high stiffness of the primary suspensions). Moreover, both the translation and rotation of the reference axes associated with curves are allowed for in the equations for completeness. The system is dynamically complex with strong coupling between the lateral and roll mode, resulting in upper and lower sway modes (Fig. 8).

4.2 Track profile and tilt compensation

Both deterministic and stochastic track features were incorporated in the simulation for studying the behaviour of the vehicle model. The deterministic track input used consists of a curved section with a radius of 1200 m superimposed by a maximum track cant of 150 mm (5.84°). A non-tilting vehicle curving speed of 162 km/h was also assumed. At each end of the curve there are transition sections of 145 m during which the curvature and cant increase steadily. The stochastic track inputs represent the irregularities in the track alignment on both straight track and curves, and these were characterized by an approximate spatial spectrum equal to $\Omega_l v^2 / f_s^3$ (m²/cycle/m) with a lateral track roughness $\{\Omega_l\}$ of 0.33×10^{-8} m [8].

Fig. 8 Tilting vehicle sway modes and illustration of their virtual movement

The resulting steady-state acceleration of the tilting vehicle depends upon the degree of compensation provided by the active tilt system; for this study a compensation factor of 75 per cent is used, which results in a steady-state lateral acceleration of 0.45 m/s compared with 2.1 m/s at 209 km/h (30 per cent higher than the non-tilting train). This level of compensation is a typical value used by tilting train manufacturers.

5 LINEAR ANALYSIS

This section investigates the structure of the linearized model prior to control design. The equations of motion derived in the previous section can be arranged in a state space description for system analysis and control design as follows

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} + \mathbf{\Gamma w} \quad (7)$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} + \mathbf{Hw} \quad (8)$$

where

$$\dot{\mathbf{x}} = [\dot{y}_v \quad \dot{\theta}_v \quad \dot{y}_b \quad \dot{\theta}_b \quad \dot{y}_w \quad \dot{\theta}_w \quad \dot{y}_m \quad \dot{\theta}_m]^T \quad (9)$$

$$\mathbf{u} = [\theta_m], \quad \mathbf{w} = \begin{bmatrix} R^{-1} & \dot{R}^{-1} & \theta_o \\ \dot{\theta}_o & \ddot{\theta}_o & y_o \end{bmatrix}^T \quad (10)$$

The equations described result in a 14th-order state space system, although this dimension can change depending upon the design procedure. For example, model reduction [9, 10] or minimal realization [11] for chosen input–output channels results in lower order systems, or in case disturbances need to be included as states for estimation purposes the order increases accordingly.

The eigenstructure of the model is analysed via straightforward modal analysis starting from the non-diagonal state matrix and, using similarity transformations, getting a diagonal form, in which the diagonal elements are the eigenvalues of the system. By describing the system as separately decoupled modes yields useful information on mode state participation, controllability, and observability. Thus, a new set of states $\mathbf{z}$ related to the original set of states $\mathbf{x}$ is defined by

$$\mathbf{z} = \mathbf{T}^{-1}\mathbf{x} \quad (11)$$

giving the following state space expression in the new coordinates

$$\dot{\mathbf{z}} = \mathbf{T}^{-1}\mathbf{ATz} + \mathbf{T}^{-1}\mathbf{Bu} \quad (12)$$

$$\mathbf{y} = \mathbf{CTz} \quad (13)$$

In the above expression, all disturbance inputs have been set to zero and the control input has no effect on the output. The dynamic modes of the model can be seen in Table 4. The modes of interest are listed in boldface. Note that the bogie kinematic states $y_w, \dot{y}_w$ participate only in the filtering of lateral track irregularities (and also not required for control purposes).

$\mathbf{T}$ is the modal matrix with each column representing the motion along the coordinate axes of the state vector for a particular mode, thus providing useful information on the participation of states for each of the system modes. This can be seen in Fig. 9, which illustrates the state participation for the body upper and lower sway and the bogie modes. It is based on taking the absolute value of each element of the normalized column vectors of $\mathbf{T}$. Note that the loss of phase information resulting from taking the absolute values is considered to be of less importance than the magnitude of the motion along a particular component. The vertical scale of 0–100 per cent reflects the relative participation of each state, with the mode and the associated eigenvalues listed above the graph.

The main points to note are the roll contributions for the body upper and bogie roll modes and the lateral contributions for the body lower and bogie lateral modes. The actuator states do not participate because the system analysis currently concerns open loop structure without control action.

6 CONVENTIONAL TILT CONTROL APPROACHES

The approaches which can be utilized for controlling tilt systems have mainly evolved in an intuitive manner since the early days of tilt [2]. This section presents the basic schemes for tilt control.

6.1 Nulling control

Nulling control was the early control approach, attempting to drive the measured lateral body acceleration to zero on a steady curve (full tilt compensation, Fig. 10(a)), soon replaced by partial tilt compensation (or ‘partial-nulling’ as it is better known) for reasons explained earlier in the paper.

Table 4 Modes of vehicle with tilt mechanism

Mode	Damping (%)	Frequency (Hz)
Lower sway	21.8	0.48
Upper sway	20.9	1.35
Bogie lateral	10.5	16.7
Bogie roll	28.3	7.26
Airspring mode	100.0	3.72
Servo (actuator)	50.0	3.50
Secondary end-stiffness	100.0	68.0

Fig. 9 State participation for the main vehicle modes

The feedback signal is provided from a body-mounted accelerometer. The clear advantage is that the body-mounted sensor does not encounter the large effects of track irregularities due to the action of the secondary suspension as a mechanical filter.

(a) Nulling tilt:-

$k_2 = 0, k_1 = 1$ for full compensation;

$k_1 + k_2 = 1$ with $k_2 \neq 0$ for partial compensation)

(b) Command-driven with precedence generalised scheme (*preview = delay*)

Fig. 10 Basic control strategies

The primary drawback with this strategy is that the sensor mounted on the tilting vehicle body is within the control loop, thus causing interactions between suspension and controller dynamics, which leads to control effort limitations and potential stability problems. The term ‘nulling’ will refer to partial-nulling throughout this paper without loss of generality.

6.2 Command driven with precedence control

It derives the tilt command signal from the preceding vehicle bogie with a noise-reduction filter designed in such a way that the delay will be compensated by the precedence effect, Fig. 10(b). There has been some development of the concept as mentioned earlier in the paper, i.e. by using additional bogie roll gyroscopes to further optimize the system response, but the overall principles remain the same. Normally a single command signal would be generated from the first vehicle and transmitted digitally with appropriate time delays down the train. Consequently the velocity and the direction of travel are important factors for the correct operation of the tilt system. This strategy proved to be successful and it is nowadays the industrial norm used by most European tilting train manufacturers. However, it is a more complex direction-sensitive scheme, with signal connections between vehicles, and the tilt system parameters need to be optimized for each specific route. Moreover, leading vehicles have inferior performance due to lack of precedence. As mentioned earlier, various improvements are

ongoing based upon fusing the control action with information taken from track databases, but such effects are not modelled here.

7 ADVANCED LOCAL TILT CONTROL DESIGN

The nulling type classical control approach, although straightforward and simple, proved difficult to solve the tilt control problem in an effective manner. The very nature of this type of control means that it is difficult to improve tilt performance while maintaining acceptable ride quality. This section presents control schemes, via modern control methods, with the aim to sustain the simplicity of nulling control, i.e. applying independent control to each vehicle, while providing tilt performance comparable to the precedence strategy [12].

7.1 Estimator-based tilt controller

In conventional nulling control the suspension dynamic interactions in the body cant deficiency measurement constrains the controller design. This section proposes an alternative way to obtain the body cant deficiency largely unaffected by suspension dynamic interactions, with the aim to provide a more effective feedback signal for vehicle tilt control. The new signal is defined as the 'true' cant deficiency (14), i.e. the cant deficiency largely unaffected by the suspension interaction. This forms a pseudo-reference to drive the tilt mechanism and provide the necessary amount of tilt. Note that the tilt actuator resembles a servomechanism system that closely follows the command signal

$$\tilde{r} = \frac{v^2}{gR} - (\hat{\theta}_o) \quad (14)$$

The estimation of this true cant deficiency signal utilizes the well-established Kalman filter (in particular its continuous-time version the Kalman–Bucy filter) KBF, the essentials of which are included in Appendix 4 (KBF basics) [13, 14].

7.1.1 Estimation of pseudo-reference cant deficiency

The important factor to notice in this application is that the curving acceleration feedback is associated with some signals of the disturbance vector $\mathbf{w}$ in equation (8). These signals are related to the track on which the vehicle is travelling, for which no prior knowledge is assumed to be available and also is not practical to measure such track parameters directly. Hence, the system state space should be reformulated for the design of the KBF in

order to treat parts of $\mathbf{w}$ as states rather than disturbance inputs shown in equation (16), [15, 16]

$$\dot{\mathbf{x}}_k = \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u} + \mathbf{\Gamma}_k \mathbf{w}_k \quad (15)$$

where

$$\mathbf{x}_k = [\mathbf{x} \quad \tilde{\mathbf{w}}]^T \quad (16)$$

The output equation for the sensors is then given by

$$\mathbf{y}_k = \mathbf{C}_k \mathbf{x}_k + \mathbf{D}_k \mathbf{u} + \mathbf{v} \quad (17)$$

where $\mathbf{C}_k$ and $\mathbf{D}_k$ are based upon the relative rows of $\mathbf{A}_k$ and $\mathbf{B}_k$.

The selection of the extra states $\tilde{\mathbf{w}}$ depends on the application, the required feedback signals and the selected output measurements. In the current case of 'true' cant deficiency estimate $\tilde{r}$, the application is mainly connected with the performance on design track, thus signals θ_o , R^{-1} should be incorporated as extra states. It has been found that only three body measurements were necessary for the Kalman filter design: (a) body lateral accelerometer (for cant deficiency information); (b) body roll gyroscope (cant information); and (c) yaw gyroscope (required only for extra information on the curvature R^{-1}). The body roll gyroscope measures absolute roll rate $(\dot{\theta}_v + \dot{\theta}_o)$, thus θ_o must also be included in the state estimates, making a total number of three extra states $\tilde{\mathbf{w}} = [\theta_o \dot{\theta}_o R^{-1}]^T$. The reformulated state space system is given in Appendix 4 (KBF basics).

Although the Kalman filter is principally a stochastic device, here the approach has been to develop the filter based upon the deterministic criteria. Any stochastic track inputs, i.e. signals related to track irregularities, were neglected in the filter design, so it is expected that the filter would reject their effects. Ultimately the filter design should be effective for both deterministic and stochastic inputs.

The KBF can be now designed offline using equations (16) and (17), while the state estimates will be calculated in real-time by solving the following differential equation

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}_k \hat{\mathbf{x}} + \mathbf{B}_k \mathbf{u} + \mathbf{K}_f (\mathbf{y}_k - \mathbf{C}_k \hat{\mathbf{x}} - \mathbf{D}_k \mathbf{u}) \quad (18)$$

where $\hat{\mathbf{x}}$ is the vector of the reformulated state estimates and $\mathbf{K}_f$ is the KBF gain matrix, which is designed offline [14, 17]. The performance of the KBF can be thoroughly assessed by tuning the process and measurement noise covariances, as explained further in Appendix 4 (KBF design).

Figure 11 shows the estimator performance for cant deficiency $(v^2/R) - g\theta_o$ estimate, at a speed of 58 m/s based on the tuning values given in

Fig. 11 Estimator performance compared with conventional measurements of cant deficiency (58 m/s)

Appendix 4 (KBF design). For illustration purposes, the non-tilting vehicle response has been selected and the deterministic (design track) and stochastic (straight track) cases are presented separately. The cant deficiency signal provided by the estimator is compared with the measurement of cant deficiency given by body- and bogie-mounted lateral accelerometers. Note that sensor noise has been included only in the estimator case.

The body and bogie accelerometer measurements are significantly affected by the suspension dynamics in both cases, deviating substantially from the expected cant deficiency values. It can be clearly seen that the estimator performs exceptionally well in both deterministic and stochastic cases, providing an estimate of cant deficiency close to the ideal value regardless the sensor noise included. This can be an effective feedback signal for subsequent tilt control design. The straight track case r.m.s. values can be seen in Table 5.

7.1.2 Estimator-based classical oriented control design

The control design inclusive of the KBF comprises two loops: (a) the main loop (pseudo-reference loop), which provides the estimate of the cant deficiency pseudo-reference, for 75 per cent tilt compensation, via a filter $F(s)$ and (b) a secondary loop

(roll rate loop) to control the body roll rate via a compensator $K_2(s)$. The feedback signals $\tilde{y}_1$ and $\tilde{y}_2$ are the estimates of cant deficiency and relative body roll rate respectively, i.e.

$$\tilde{y}_1 = \frac{v^2}{\hat{R}} - g\hat{\theta}_o \quad (19)$$

$$\tilde{y}_2 = \hat{\dot{\theta}}_v \quad (20)$$

obtained from the estimator via a selector matrix $\tilde{C}$. Note that, as mentioned earlier in the paper, the actuator closely follows the command signal (in this case the pseudo-reference) up to the bandwidth of 3.5 Hz. The secondary loop (of body roll rate) improves both the transient behaviour of the system and the response on straight track (track irregularities) via damping improvement.

The estimator-based controller, including $F(s)$, $K_2(s)$, and the KBF structure, will be referred to as $K_E(s)$ (Fig. 12).

The design of both $F(s)$ and $K_2(s)$ can be performed based on the plant, without the estimator, i.e. ideal case assuming the required signals can be measured. However, when $K_E(s)$ is fully implemented, including the estimator, the following should be also considered:

- $K_E(s)$ is stable, i.e. $\text{eig}(K_E(s)) \leq 0$ (it can include integrators if necessary);
- $K_E(s)$ should not introduce any poorly damped modes (that might cause large oscillations) in the closed loop system, especially within the frequency range of interest;
- check input sensitivity [18], i.e. sensitivity at point '1' in Fig. 12, with $K_E(s)$ included in the loop.

In the case of square plants, number of inputs equals the number of outputs, it is possible to

Table 5 Estimator performance on straight track for cant deficiency

Signal	r.m.s. value (%)
Estimator-based	0.31
Body lateral accelerometer	3.20
Bogie lateral accelerometer	22.43

Fig. 12 Estimated cant deficiency pseudo-reference with roll rate control

check the sensitivity at both point '1' and '2'. Further discussion on 'squaring' plants, by adding dummy inputs or outputs, in order to recover stability margins based on the usual LTR procedure can be found in references [19] and [17] mainly related to LQ or state feedback designs.

$F(s)$ resembles a prefilter as in the case of proper reference signals. The cant deficiency pseudo-reference is generated from the estimator including some high frequency noise components, $F(s)$ is chosen a first-order low-pass filter (25 rad/s cut-off frequency) given by

$$F(s) = \frac{0.75}{g} \cdot \frac{25}{s + 25} \quad (21)$$

Note the DC gain, $F(0)$, set to convert the estimated signal in radians (angle) and provide 75 per cent tilt compensation on steady curve as required. A lower cut-off frequency can be specified as far as no large delays are introduced in the pseudo-reference feedback $\tilde{r}$. Moreover, $K_2(s)$ is a phase-lead compensator designed to improve the transient behaviour, i.e. stability margins, of the body roll rate $\dot{\theta}_v$ and is given by

$$K_2(s) = 0.248 \cdot \frac{(s/25) + 1}{(s/51) + 1} \quad (22)$$

effectively adding damping in the system. Figure 13 illustrates the frequency response of $F(s)$, $K_2(s)$, and the input sensitivity of the designed system at point

Fig. 13 Designed system frequency responses

'1', respectively. Note the peak of the input sensitivity at around 4.54 dB, i.e. 1.69 in linear terms, indicating sufficient robustness based on the rule of thumb proposed in reference [18] (large peaks indicate poor robustness while a value less than 2 (6 dB) is a typical requirement for good robustness). The time domain results on design track can be seen in Fig. 14, illustrating the effectiveness of the scheme. Few high frequency components can be seen on the lateral acceleration profile (Fig. 14(a)) and roll gyroscope (Fig. 14(d)), note that sensor noise is included only in the active tilt case, due to the estimation process mainly contributed from the bogie states. Figure 14(c) presents the relation between the acceleration and roll (tilt) angle profile. The slight delay of tilt angle response, note that all signals are estimated locally, causes the lateral acceleration initially to follow the uncompensated profile soon after to be correctly compensated as it approaches the steady curve.

7.2 Linear optimal nulling-tilt control

The design of the tilt controller in this section is based on linear optimal control theory by feeding back all system states via a gain matrix. Linear optimal control generally possesses a number of advantages compared to other forms of optimal control. Many engineering plants can in fact be considered to be linear, and implementing linear controllers physically is a simple task. Also, the majority of linear optimal control problems have readily computable solutions, which often can be carried over to non-linear optimal control problems. However, it is difficult and usually impractical to measure all states (especially in the case of large-scale or complex systems). Thus, the Kalman–Bucy estimator, designed in the previous section, is included together with the linear quadratic regulator (LQR) to provide the required estimates of the original states (the overall controller can be also referred to as linear

(a) Passenger acceleration

(b) Body roll angle (Tilt angle)

(c) Passenger acceleration and tilt angle profiles

(d) Body roll gyroscope ($\dot{\theta}_v + \dot{\theta}_o$) profile

Fig. 14 Estimator-based tilt control time domain results

quadratic Gaussian LQG-type). The available literature on the optimal control problem is extensive [17, 19, 20], while details on the preliminaries of LQR can be found in Appendix 5.

7.3 LQR with integral control for steady curve tilt compensation

For correct steady curve tilt compensation and consequently good disturbance rejection, referring back to classical control theory, a new state should be defined and this is the integral of the effective cant deficiency θ'_{dmm} . This approach produces an optimal $P+I$ controller [20] rather than a proportional state feedback controller. Hence, the system is augmented to include $\int \theta'_{dmm}$ as a state for the design procedure

$$\theta'_{dmm} = -k_1 \frac{\ddot{y}_{vm}}{g} - k_2 \theta_m \quad (23)$$

where $k_1 = 0.75$, $k_2 = 0.25$, and $\ddot{y}_{vm} = (v^2/R) - g(\theta_o + \theta_v) + \ddot{y}_v$. The augmented system is given in Appendix 5.

The critical design issue relates to choosing suitable values for the output (which also relates to the states) and control weighting matrices ($\mathbf{Q}_o$, $\mathbf{R}_k$ in Appendix 5), and the approach adopted has been to initially set values equal to the square of the inverse of the expected value of each of the weighted signals, for each parameter of interest. The controller can be then designed by varying (tuning) the above weights until a satisfactory result arises. The weights for the best design were found to be

$$\mathbf{Q}_o = \begin{pmatrix} \frac{1}{0.5^2} & 0 \\ 0 & \frac{1}{0.05^2} \end{pmatrix}, \quad \mathbf{R}_k = \frac{1}{0.14^2}$$

making the corresponding performance index to be minimized equal to

$$J = \lim_{T \rightarrow \infty} \frac{1}{T} E \left\{ \int_0^T \left(\frac{1}{0.5^2} (\dot{\theta}_v - \dot{\theta}_m)^2 + \frac{1}{0.05^2} \left(\int \theta'_{dmm} \right)^2 + \frac{1}{0.14^2} u^2 \right) d\tau \right\} \quad (24)$$

and finally the optimal gain matrix for the above setup was found to be ($\mathbf{K}_r = [K_p \ K_i]$)

$$\mathbf{K}_r = \begin{bmatrix} 0.94 & -1.8 & 0.25 & -0.07 & 0.43 & 0.075 & 0.006 \\ 0.015 & 0.13 & -0.16 & 0 & 0 & 2.53 & 0.26 & -2.8 \end{bmatrix} \quad (25)$$

which corresponds to the following state vector, which will be in fact estimated by the KBF, $[y_v \ \theta_v \ y_b \ \theta_b \ \dot{y}_v \ \dot{\theta}_v \ \dot{y}_b \ \dot{\theta}_b \ \theta_r \ y_{es} \ y_w \ \dot{y}_w \ \theta_m \ \dot{\theta}_m \ \int \theta'_{dmm}]^T$. After introducing the Kalman–Bucy estimator (i.e. forming the LQG-type controller, Fig. 15), the feedback control law is based on the state estimates and is given by

$$u = -(K_p \ K_i) \begin{pmatrix} \hat{x} \\ \hat{x}' \end{pmatrix} \quad (26)$$

Note that the gain matrix $\mathbf{K}_r$ entries for $y_w, \dot{y}_w$ are zero as these states do not participate in the control (removed in the minimal realization of the original plant, i.e. they can be removed from the estimator).

The input sensitivity of the system can be seen in Fig. 16, with good robustness properties (low peak), although the response is, as expected, slightly slower compared to the estimator-based scheme because of the slightly reduced bandwidth. The time domain results on deterministic track can be seen in Fig. 17, with sufficiently good performance. The choice of the relative body roll-mechanism roll rates proved effective in minimizing unwanted oscillations. This scheme successfully rejects all noise components on the lateral acceleration and roll gyroscope profiles as a result of integral action.

8 DISCUSSION

Both the proposed estimator-based and optimal control schemes performed well on design and straight track, with the estimator-based controller having an extra advantage due to the use of cant deficiency pseudo-reference and additional body roll rate feedback. The optimal controller, although based on the measured cant deficiency, due to its integral action can handle noise components better, which is clearly seen in Fig. 17. This can be also confirmed by checking the input sensitivity plots in Figs 16 and 13(b), where the estimator-based controller has a larger range of operation (up to 10 rad/s), albeit the optimal controller can still

Fig. 15 Optimal LQG-type nulling control structure

Fig. 16 LQG-type input sensitivity at point '1'

Fig. 18 Lateral acceleration profiles comparison with different controllers

(a) Passenger acceleration

(b) Body roll angle (Tilt angle)

(c) Mechanism Roll

(d) Body roll gyroscope ($\dot{\theta}_v + \dot{\theta}_o$) profile

Fig. 17 LQG-type tilt control time domain results

Table 6 Performance assessment results

	Conventional-PI	LQG-type	Estimation-based	Preceding
Deterministic				
Lateral accelerometer				
Steady state (%g)	n/a	4.6	4.6	4.6
R.m.s deviation error (%g)	4.7	2.33	1.7	0.73
Peak value (%g)	13.0	9.0	7.24	5.0
Roll gyroscope				
R.m.s. deviation error (rad/s)	0.03	0.024	0.02	0.015
Peak value (rad/s)	0.093	0.1	0.1	0.11
P_{CT} factors				
Max jerk level (%g/s)	6.95	5.03	5.17	3.02
Standing (% of passengers)	47.9	35.0	29.85	20.6
Seated (% of passengers)	14.0	8.95	7.5	3.7
Stochastic ^a				
Ride quality				
Active tilt(%g)	3.39	3.15	2.76	2.29
Ride quality degradation (%)	5.94	-1.56	-13.75	-28.6

^aStochastic passive vehicle ride quality = 3.2 %g at 58 m/s.

reject disturbances up to approximately 35 rad/s. Moreover, both schemes exhibit sufficient robustness properties as indicated by the low sensitivity peaks.

The estimator-based control scheme with the pseudo-reference feedback mimics a precedence-type strategy, while the optimal control scheme is clearly of nulling-type. Note that the use of pseudo-reference and roll rate feedback can be extended, with the feedback gain for the latter designed via optimal control theory rather than a straightforward classical compensator.

Figure 18 compares the lateral acceleration profile of the proposed advanced local tilt schemes with two sample conventional tilt schemes, an early classical nulling-type and a currently used precedence-type. Moreover, the performance of the schemes, based on the assessment described in section 3.2, is listed in Table 6. Note that only the estimator-based and LQG-type scheme simulation were subject to sensor noise. The results clearly illustrate the progress, in terms of improved performance, from the conventional early classical nulling-type, to its optimal LQG-type extension, the estimator-based controller and finally the industrial-norm precedence-type scheme. There is no doubt that the precedence type has the significant advantage of preview information, whereas the proposed advanced schemes and especially the estimator-based strategy following very closely in performance although based upon local information only (i.e. no precedence).

9 CONCLUSIONS

Two novel control schemes have been proposed for improving the performance of local tilt control, an

estimator-based controller and an optimal LQG-type controller. The designed estimator was based on practical body-sensors, namely lateral accelerometer, roll, and yaw gyroscopes, providing efficient estimates of all required states and feedback signals. The LQG-type controller is a straightforward extension of the conventional nulling scheme in an optimal control framework, providing substantial improvement in local tilt performance. The estimator-based scheme improves the overall tilt response further, as it mimics a 'precedence-type' scheme with the difference that the tilt command is estimated locally. A pseudo-reference tilt command signal is generated for the tilt mechanism to follow, while the secondary feedback of body roll rate injects damping. Note that both the advanced local tilt schemes ease the trade-off between design and straight track, which has been difficult to solve in the early conventional nulling control cases. The frequency domain and time domain results based on an appropriate tilt control assessment clearly illustrate the effectiveness of the proposed schemes. Overall, the main contributions of the paper are twofold:

- a rigorous assessment approach that both enables the essential tilt characteristics and the detailed control design to be optimized;
- two novel model-based control schemes that enable performance close to that provided by precedence control to be achieved using local vehicle-based sensors alone.

Future work is mainly concentrated on the extension of the estimator-based scheme in an optimal control framework reformulation, and a rigorous investigation on controller reduction and robustness analysis.

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APPENDIX 1

Notation

A, B, C, D	state space realization of a system
A^T	transpose of a matrix or vector
$E\{\cdot\}, E[\cdot]$	expected value of a given quantity
J	performance index for optimal tilt control
K_r, K_f	optimal regulator (LQR) and estimator gain matrices
P_c, P_f	covariance matrices solutions of the optimal control and estimator riccati equations
P_{CT}	index for evaluating passenger comfort on curved track defines the percentage of both standing and seated passengers feeling uncomfortable
Q, R_k	state and control weighting matrices in the performance index J for optimal control
Q_{kf}, R_{kf}	process noise and measurement noise weighting matrices in the estimator design
Q_o	output weighting matrix (optimal control)
v	vehicle forward speed (m/s)
y_v, y_b, y_w, y_o	lateral displacement of body, bogie, wheelset, and track (m)
θ	body roll angle with respect to horizontal frame (absolute) (rad)
θ_{m_i}, θ_m	ideal and actual (applied) mechanism tilt angle (rad)
θ_o, R	track cant and track curve radius (rad m)
$\theta_v, \theta_b, \theta_r$	roll displacement of body, bogie, and airspring reservoir (rad)

APPENDIX 2

 P_{CT} factor calculation

$$P_{CT} = (A\ddot{y} + B\ddot{\ddot{y}} - C)_{\geq 0} + D(\dot{\theta})^E \quad (27)$$

where A, B, C, D , and E are constants defined below:

Condition	A	B	C	D	E
Standing passengers	2.80	2.03	11.1	0.185	2.283
Seated passengers	0.88	0.95	5.9	0.120	1.626

- P_{CT} = passenger comfort index on curve transition, representing the percentage of passengers that will feel discomfort
- $\ddot{y}$ = maximum vehicle body lateral acceleration, in the time interval between the beginning of the curve transition and 1.6 s after the end of the transition (expressed in per centage of g), g denotes gravity (Fig. 19)
- $\ddot{y}''$ = maximum lateral jerk level, calculated as the maximum difference between two subsequent values of $\ddot{y}$ no closer than 1 s, in the time interval between 1 s before the start of the curve transition and the end of the transition (expressed in per cent age of g per second) (Fig. 19)
- $\dot{\theta}$ = maximum absolute value of vehicle body roll speed, in the time interval between the beginning of the curve transition to the end of the curve transition (expressed in degrees per second), dot denotes the derivative with respect to time t (Fig. 19).

APPENDIX 3

Parameter values

- m_v, i_{vr} half body: mass, 16 000 (kg), roll inertia, 20 000 (kg m²)
- m_b, i_{br} bogie: mass, 3680 (kg), roll inertia, 2500 (kg m²)
- Values per bogie side

Fig. 19 Calculation of quantities $\ddot{y}$, $\ddot{y}''$, and $\dot{\theta}$ for P_{CT} factor evaluation

- k_{az}, k_{sz} airspring area and series stiffness, 201 500 (N/m) and 300 000 (N/m)
- k_{rz} airspring reservoir stiffness, 201 000 (N/m)
- c_{rz} airspring reservoir damping, 20 000 (Ns/m)
- k_{sy}, c_{sy} secondary lateral stiffness and damping, 100 000 (N/m) and 18 000 (Ns/m)
- $k_{c_{sy}}$ secondary lateral damper end-stiffness, 8 000 000 (N/m)
- k_{vr} anti-roll bar stiffness per bogie, 1 500 000 (Nm/rad)
- c_{vr} anti-roll bar damping per bogie, 18 200 (Nsm/rad)
- k_{pz}, c_{pz} primary vertical stiffness and damping, 1 600 000 (N/m) and 20 000 (Ns/m)
- k_{py}, c_{py} primary lateral stiffness and damping, 18 600 000 (N/m) and 20 000 (Ns/m)
- d_1 airspring semi-spacing, 0.835 (m)
- d_2 primary vertical suspension semi-spacing, 1.00 (m)
- h_1 secondary lateral suspension spacing (body c.o.g.), 0.844 (m)
- h_2 secondary lateral suspension spacing (bogie c.o.g.), 0.252 (m)
- h_3 primary lateral suspension spacing (bogie c.o.g.), 0.194 (m)
- h_{mt} mechanism c.o.g. vertical separation from effective tilt centre, 0.6 (m)
- h_{g1}, h_{g2} height ARL of body c.o.g. and bogie c.o.g., 1.696(m) and 0.6 (m)

APPENDIX 4

Kalman–Bucy Filter

KBF basics

The KBF design is based on the following dynamic process

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} + \mathbf{\Gamma}w \quad (28)$$

with known input u and output measurements given by

$$\mathbf{y} = \mathbf{Cx} + \mathbf{v} \quad (29)$$

where w (process noise) and v (measurement noise) are usually assumed to be uncorrelated white noise processes having known constant spectral density matrices $\mathbf{Q}_{kf}$ and $\mathbf{R}_{kf}$, respectively. Their covariances

are expressed by

$$\mathbf{E}\{\mathbf{w}(t)\mathbf{w}(\tau)^T\} = \mathbf{Q}_{kf}\delta(t - \tau) \quad (30)$$

$$\mathbf{E}\{\mathbf{v}(t)\mathbf{v}(\tau)^T\} = \mathbf{R}_{kf}\delta(t - \tau) \quad (31)$$

$$\mathbf{E}\{\mathbf{w}(t)\mathbf{v}(\tau)^T\} = 0, \mathbf{E}\{\mathbf{v}(t)\mathbf{w}(\tau)^T\} = 0 \quad (32)$$

The KBF [13, 14] effectively has the structure of an ordinary observer (state-estimator) and is given by the following state space expression

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{K}_f(\mathbf{y} - \hat{\mathbf{y}}) \quad (\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}}) \\ &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{K}_f(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \end{aligned} \quad (33)$$

$$\hat{\mathbf{y}} = \mathbf{C}\hat{\mathbf{x}} \quad (34)$$

where $\mathbf{K}_f$ is the optimally derived observer gain matrix, minimizing $\mathbf{E}\{[\mathbf{x} - \hat{\mathbf{x}}]^T[\mathbf{x} - \hat{\mathbf{x}}]\}$, and given by

$$\mathbf{K}_f = \mathbf{P}_f \mathbf{C}^T \mathbf{R}_{kf}^{-1} \quad (35)$$

where $\mathbf{P}_f$ is the unique positive semi-definite, $\mathbf{P}_f = \mathbf{P}_f^T \geq 0$, of the following are

$$\mathbf{P}_f \mathbf{A}^T + \mathbf{A} \mathbf{P}_f - \mathbf{P}_f \mathbf{C}^T \mathbf{R}_{kf}^{-1} \mathbf{C} \mathbf{P}_f + \mathbf{I} \mathbf{Q}_{kf} \mathbf{I}^T = 0 \quad (36)$$

subject to $(\mathbf{C}, \mathbf{A})$ being detectable, $\mathbf{R}_{kf} > 0$, $\mathbf{Q}_{kf} \geq 0$ and $(\mathbf{A}, \mathbf{I} \mathbf{Q}_{kf} \mathbf{I}^T)$ has no uncontrollable modes on the imaginary axis. In fact, the optimum estimation problem is dual to the deterministic optimum control problem [14, 17].

Reformulated state space equation

The reformulated state space system for the true cant deficiency estimation is given by

$$\mathbf{x}_k = [\mathbf{x} \ \tilde{\mathbf{w}}]^T \quad (37)$$

$$= [\gamma_v \ \theta_v \ \gamma_b \ \theta_b \ \dot{\gamma}_v \ \dot{\theta}_v \ \dot{\gamma}_b \ \dot{\theta}_b \ \theta_r \ \gamma_{es} \ \gamma_w \ \dot{\gamma}_w \ \theta_m \ \dot{\theta}_m \ \theta_o \ \dot{\theta}_o \ R^{-1}]^T \quad (38)$$

and the process noise vector exciting the system is

$$\mathbf{w}_k = [\dot{R}^{-1} \ \ddot{\theta}_o]^T \quad (39)$$

The associated state, input and process noise matrices are

$$\mathbf{A}_k = \begin{bmatrix} \mathbf{A} & \tilde{\mathbf{I}} \\ \mathbf{0}_{3 \times 14} & \mathbf{A}_w \end{bmatrix} \quad \mathbf{B}_k = [\mathbf{B}^T \ \mathbf{0}_{1 \times 3}]^T \quad (40)$$

$$\mathbf{I}_k = [\mathbf{I}_{w_k}^T \ (0 \ 0)^T \ (0 \ 1)^T \ (1 \ 0)^T]^T \quad (41)$$

Note that the eigenvalues of $\mathbf{A}_w$ are at the origin, introducing three pure integrators in $\mathbf{A}_k$. However,

to make the pair $(\mathbf{C}_k, \mathbf{A}_k)$ detectable, for the design procedure, the eigenvalues of $\mathbf{A}_w$ must be moved slightly to the left of the origin (with the distance been much smaller than the required bandwidth). The perturbed $\mathbf{A}_w$ was chosen to have three ‘small’ distinct eigenvalues $\epsilon_1, \epsilon_2, \epsilon_3$ set to 10^{-3} , 1.25×10^{-3} , 1.5×10^{-3} , respectively (repeated eigenvalues can be also used, although distinct eigenvalues are preferred for computational issues in similarity transformations, balancing, and model reduction). Thus

$$\tilde{\mathbf{A}}_w = \begin{bmatrix} 0 & 1 & 0 \\ -\epsilon_1 \epsilon_2 & -(\epsilon_1 + \epsilon_2) & 0 \\ 0 & 0 & -\epsilon_3 \end{bmatrix} \quad (42)$$

defining, $\hat{\theta}_o = (1/(s + \epsilon_1)(s + \epsilon_2))\ddot{\theta}_o$ and $\hat{R}^{-1} = (1/(s + \epsilon_3))\dot{R}^{-1}$. Note that in the implementation stage of the estimator or estimator-based controllers the eigenvalues of $\mathbf{A}_w$ should be moved back to zero in order to obtain true integration.

KBF design

The design aim is mainly connected to the deterministic performance of the tilt controller and the following, an ‘application-oriented’ procedure for choosing the process and noise covariances $\mathbf{Q}_{kf}$, $\mathbf{R}_{kf}$, is utilized.

In this design, $\mathbf{R}_{kf}$ is a 3×3 diagonal matrix (cross-correlation terms are set to zero)

$$\mathbf{R}_{kf} = \begin{bmatrix} R_{kf}(1, 1) & 0 & 0 \\ 0 & R_{kf}(2, 2) & 0 \\ 0 & 0 & R_{kf}(3, 3) \end{bmatrix} \quad (43)$$

where $R_{kf}(1, 1)$ is the covariance of the body lateral accelerometer sensor, $R_{kf}(2, 2)$ the covariance of the body roll gyroscope sensor, and $R_{kf}(3, 3)$ the covariance of the body yaw gyroscope sensor. The value for each of the covariances is set to the square of 1 per cent of the expected maximum value taken as, 3 times the true r.m.s. value of the sensor output signal on straight track with irregularities plus the peak value on the pure curved track. This corresponds to high quality (realistic) sensors currently used in tilting trains. In reality, any detailed design would have a real sensor with actual noise values from the manufacturer, but this paper uses a typical (sensible) level in the absence of detailed design information. Reducing the measurement noise will obviously improve the accuracy of the estimator.

The values for matrix $\mathbf{Q}_{kf}$ were chosen as the square of a typical standard deviation value of the excitation (or process) signals $\dot{R}^{-1}$, $\ddot{\theta}_o$. Note that for

the design procedure $\mathbf{R}_{kf}$ remained fixed, with varying $\mathbf{Q}_{kf}$ (the best values were found via appropriate tuning). Thus

$$\mathbf{R}_{kf} = \begin{bmatrix} 1.1 \times 10^{-3} & 0 & 0 \\ 0 & 1.42 \times 10^{-6} & 0 \\ 0 & 0 & 1 \times 10^{-6} \end{bmatrix},$$

$$\mathbf{Q}_{kf} = \begin{bmatrix} 7.2 \times 10^{-6} & 0 \\ 0 & 1.6 \times 10^{-3} \end{bmatrix} \quad (44)$$

The Kalman gain obtained for the above configuration was

$$\mathbf{K}_f = \begin{bmatrix} -0.32 & -0.12 & -0.01 & -0.05 & -1.87 \\ -4.83 & -3.75 & -0.28 & -1.45 & -9.11 \\ -0.70 & 0.08 & -0.26 & -0.1 & -53.51 \\ -0.68 & 0.02 & -0.27 & -0.09 & -0.03 \\ -7.27 & -0.87 & -3.53 & -2.63 & -0.64 \\ -0.8 & -13.38 & -7.67 & -0.06 & -0.36 \\ 0 & 0 & 0 & 0 & 0.09 & 0.75 & 0.0012 \\ 0 & 0 & 0 & 0 & 5.37 & 26.43 & -0.016 \\ 0 & 0 & 0 & 0 & -0.02 & -0.52 & 0.85 \end{bmatrix}^T \quad (45)$$

The estimator gain matrix $\mathbf{K}_f$ has zero entries for $y_w, \dot{y}_w, \theta_m, \dot{\theta}_m$, thus these four states could be removed from the estimation procedure. This is true for $y_w, \dot{y}_w$ as they are purely connected with the stochastic track irregularities and also do not participate in the control design (pole/zero cancellations). However, $\theta_m, \dot{\theta}_m$ should be kept in the design as they are important for control purposes in closed-loop (and also any high frequency noise is filtered out from the KBF). Thus, for the remaining part of the paper, the states $y_w, \dot{y}_w$ are removed from the estimator.

APPENDIX 5

Linear Quadratic Regulator

LQR basics

The standard description of the plant and output is given by the following equations (external disturbances or reference inputs not included)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (46)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \quad (47)$$

with n number of states $\mathbf{x}$, m number of inputs $\mathbf{u}$, and q number of outputs $\mathbf{y}$. The assumptions made are

that the system is linear, time-invariant (for simplicity), and controllable. Full state feedback is considered, and it is desired to find a suitable linear control law

$$\mathbf{u} = -\mathbf{K}_r\mathbf{x} \quad (48)$$

where $\mathbf{K}_r$ is a gain matrix, which minimizes the following general form quadratic index

$$J = \lim_{T \rightarrow \infty} \frac{1}{T} E \left\{ \int_0^T [\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R}_k \mathbf{u}] d\tau \right\} \quad (49)$$

The weighting matrices $\mathbf{Q}$ (state weighting matrix) and $\mathbf{R}_k$ (control weighting matrix) must be symmetric (because J is a scalar), i.e. $\mathbf{Q}^T = \mathbf{Q}$ and $\mathbf{R}_k^T = \mathbf{R}_k$. There is no specific restriction about the form in which $\mathbf{Q}$ and $\mathbf{R}_k$ should appear, but in most cases they are diagonal matrices. If, instead of the states, the output $\mathbf{y}$ is to be controlled then the quadratic performance index needs to be arranged into

$$J = \lim_{T \rightarrow \infty} \frac{1}{T} E \left\{ \int_0^T [\mathbf{y}^T \mathbf{Q}_o \mathbf{y} + \mathbf{u}^T \mathbf{R}_k \mathbf{u}] d\tau \right\} \quad (50)$$

where $\mathbf{Q}_o$ is the output weighting matrix, and it can be easily shown that $\mathbf{Q} = \mathbf{C}^T \mathbf{Q}_o \mathbf{C}$ by setting $\mathbf{y} = \mathbf{C}\mathbf{x}$ for a strictly proper system.

The gain matrix $\mathbf{K}_r$ is the solution of the following general form matrix Riccati differential equation

$$\mathbf{A}^T \mathbf{P}_c + \mathbf{P}_c \mathbf{A} + \dot{\mathbf{P}}_c + \mathbf{Q} = \mathbf{P}_c \mathbf{B} \mathbf{R}_k^{-1} \mathbf{B}^T \mathbf{P}_c \quad (51)$$

subject to given $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{Q}$, and $\mathbf{R}_k$. Restricting ourselves in the time-invariant case, $\mathbf{P}_c$ should be constant, which states that $\dot{\mathbf{P}}_c = \mathbf{0}$. The Riccati equation is then simplified to

$$\mathbf{A}^T \mathbf{P}_c + \mathbf{P}_c \mathbf{A} + \mathbf{Q} - \mathbf{P}_c \mathbf{B} \mathbf{R}_k^{-1} \mathbf{B}^T \mathbf{P}_c = \mathbf{0} \quad (52)$$

and the solution of the gain matrix is given by

$$\mathbf{K}_r = \mathbf{R}_k^{-1} \mathbf{B}^T \mathbf{P}_c \quad (53)$$

subject to $(\mathbf{A}, \mathbf{B})$ being stabilizable, $\mathbf{R}_k > \mathbf{0}$ (positive definite, for finite control energy), $\mathbf{Q} \geq \mathbf{0}$ (positive semi-definite), and that $(\mathbf{Q}, \mathbf{A})$ has no unobservable modes on the imaginary axis [17].

Augmented state space system with integral action

The augmented state space system with integral action is given by

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}' \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C}' & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \end{pmatrix} + \begin{pmatrix} \mathbf{B} \\ \mathbf{0} \end{pmatrix} \mathbf{u} \quad (54)$$

where $x' = \int \theta'_{\text{dmm}}$ and C' is the selector matrix for integral action and is found from $\theta'_{\text{dmm}} = C'x$. Note that the state vector x now includes the following vehicle states (for the new model after minimal realization)

$$[y_v \ \theta_v \ y_b \ \theta_b \ \dot{y}_v \ \dot{\theta}_v \ \dot{y}_b \ \dot{\theta}_b \ \theta_r \ y_{\text{es}} \ \theta_m \ \dot{\theta}_m]^T$$

and also $u = [\theta_{m_i}]$. The control signal is

$$u = -(K_p \ K_i) \begin{pmatrix} x \\ x' \end{pmatrix} \quad (55)$$

and the quadratic performance index for output regulation is

$$J = \lim_{T \rightarrow \infty} \frac{1}{T} E \left\{ \int_0^T [y^T Q_o y + u^T R_k u] d\tau \right\} \quad (56)$$

where $y = [(\dot{\theta}_v - \dot{\theta}_m), \int \theta'_{\text{dmm}}]$ and $u = \theta_{m_i}$. The weight on $\int \theta'_{\text{dmm}}$ emphasizes the speed of response, while the weight on $(\dot{\theta}_v - \dot{\theta}_m)$ limits the oscillations between the vehicle body and the tilt mechanism (the secondary suspensions are situated on top of the tilting mechanism/bolster).