Online Supplement for “Quickly Assessing Contributions to Input Uncertainty” by Eunhye Song and Barry L. Nelson

A Variance and covariance expressions for $\mu(\hat{F}_\ell)$ and $\sigma^2(\hat{F}_\ell)$

Recall that $\mu(\hat{F}_\ell^*)$ and $\sigma^2(\hat{F}_\ell^*)$ are the mean and the variance of the ecdf $\hat{F}_\ell^*$ formed from the bootstrap sample $X_{\ell 1}^*, X_{\ell 2}^*, \dots, X_{\ell m_\ell}^*$. Since $\hat{F}_\ell^*$ has the probability mass function (pmf) $\Pr\{X = X_{\ell i}^*\} = 1/m_\ell$, $\mu(\hat{F}_\ell^*)$ and $\sigma^2(\hat{F}_\ell^*)$ can be calculated as

$$\begin{aligned}\mu(\hat{F}_\ell^*) &= \frac{1}{m_\ell} \sum_{i=1}^{m_\ell} X_{\ell i}^* = \bar{X}_\ell^* \\ \sigma^2(\hat{F}_\ell^*) &= \frac{1}{m_\ell} \sum_{i=1}^{m_\ell} (X_{\ell i}^* - \bar{X}_\ell^*)^2 = \frac{m_\ell - 1}{m_\ell} S_\ell^{*2}\end{aligned}$$

where $\bar{X}_\ell^*$ and S_ℓ^{*2} are the sample mean and the sample variance of $X_{\ell 1}^*, X_{\ell 2}^*, \dots, X_{\ell m_\ell}^*$, respectively. Therefore, the variance and the covariance of $\mu(\hat{F}_\ell^*)$ and $\sigma^2(\hat{F}_\ell^*)$ become

$$\begin{aligned}\text{Var}[\mu(\hat{F}_\ell^*)|\hat{F}_\ell] &= \text{Var}[\bar{X}_\ell^*] \\ \text{Var}[\sigma^2(\hat{F}_\ell^*)|\hat{F}_\ell] &= \left(\frac{m_\ell - 1}{m_\ell}\right)^2 \text{Var}[S_\ell^{*2}] \\ \text{Cov}[\mu(\hat{F}_\ell^*), \sigma^2(\hat{F}_\ell^*)|\hat{F}_\ell] &= \frac{m_\ell - 1}{m_\ell} \text{Cov}[\bar{X}_\ell^*, S_\ell^{*2}].\end{aligned}$$

Cho and Cho (2008) provide variance expressions for the sample mean and sample variance and Zhang (2007) provides the covariance expression. These expressions are in terms of the moments of the population distribution, in our case, the moments of $\hat{F}_\ell$. Therefore, it follows from their results that

$$\begin{aligned}\text{Var}[\mu(\hat{F}_\ell^*)|\hat{F}_\ell] &= \frac{M_\ell^2}{m_\ell} \\ \text{Var}[\sigma^2(\hat{F}_\ell^*)|\hat{F}_\ell] &= \frac{(m_\ell - 1)^2}{m_\ell^3} M_\ell^4 - \frac{(m_\ell - 3)(m_\ell - 1)}{m_\ell^3} (M_\ell^2)^2 \approx \frac{M_\ell^4 - (M_\ell^2)^2}{m_\ell} \\ \text{Cov}[\mu(\hat{F}_\ell^*), \sigma^2(\hat{F}_\ell^*)|\hat{F}_\ell] &= \frac{(m_\ell - 1)^2}{m_\ell^3} M_\ell^3 \approx \frac{M_\ell^3}{m_\ell}\end{aligned}$$

where M_ℓ^k is the k th central moment of $\hat{F}_\ell$.

B Optimal weights for combining the nominal and diagnostic experiments

In Section 4 we combined the estimator from the nominal experiment $\bar{Y}(\hat{\mathbf{F}})$ with the average response from the diagnostic experiment $\bar{\bar{Y}}(\hat{\mathbf{F}}) = \sum_{b=1}^B \bar{Y}(\hat{\mathbf{F}}^{*(b)})/B$ into a weighted estimator

$$\tilde{Y} = \alpha \bar{Y}(\hat{\mathbf{F}}) + (1 - \alpha) \bar{\bar{Y}}(\hat{\mathbf{F}}),$$

and we defined α^* as the weight $\alpha \in [0, 1]$ that minimizes $\text{MSE}[\tilde{Y}|\hat{\mathbf{F}}] = \text{Bias}^2[\tilde{Y}|\hat{\mathbf{F}}] + \text{Var}[\tilde{Y}|\hat{\mathbf{F}}]$. The bias term can be written as

$$\begin{aligned}\text{Bias}[\tilde{Y}|\hat{\mathbf{F}}] &= \text{Bias}[\alpha\bar{Y}(\hat{\mathbf{F}}) + (1-\alpha)\bar{\bar{Y}}(\hat{\mathbf{F}})|\hat{\mathbf{F}}] \\ &= \alpha\text{Bias}[\bar{Y}(\hat{\mathbf{F}})|\hat{\mathbf{F}}] + (1-\alpha)\text{Bias}[\bar{\bar{Y}}(\hat{\mathbf{F}})|\hat{\mathbf{F}}] \\ &= (1-\alpha)b^*,\end{aligned}$$

where $b^* = \text{Bias}[\bar{\bar{Y}}(\hat{\mathbf{F}})|\hat{\mathbf{F}}]$ and the last equality follows from $\text{Bias}[\bar{Y}(\hat{\mathbf{F}})|\hat{\mathbf{F}}] = 0$. The variance term is further decomposed as

$$\begin{aligned}\text{Var}[\tilde{Y}|\hat{\mathbf{F}}] &= \text{Var}[\alpha\bar{Y}(\hat{\mathbf{F}}) + (1-\alpha)\bar{\bar{Y}}(\hat{\mathbf{F}})|\hat{\mathbf{F}}] \\ &= \alpha^2\text{Var}[\bar{Y}(\hat{\mathbf{F}})|\hat{\mathbf{F}}] + (1-\alpha)^2\text{Var}[\bar{\bar{Y}}(\hat{\mathbf{F}})|\hat{\mathbf{F}}] + 2\alpha(1-\alpha)\text{Cov}[\bar{Y}(\hat{\mathbf{F}}), \bar{\bar{Y}}(\hat{\mathbf{F}})|\hat{\mathbf{F}}].\end{aligned}\tag{27}$$

From the homoscedasticity assumption made in (1), we can further analyze each component in (27). The first variance component becomes

$$\begin{aligned}\text{Var}[\bar{Y}|\hat{\mathbf{F}}] &= \text{Var}\left[g(\hat{\mathbf{F}}) + \frac{\sum_{j=1}^n \varepsilon_j}{n} \middle| \hat{\mathbf{F}}\right] \\ &= \frac{\sigma^2}{n}.\end{aligned}$$

Notice that the last equality holds since ε_j is independent from $g(\hat{\mathbf{F}})$. The second variance component becomes

$$\begin{aligned}\text{Var}[\bar{\bar{Y}}(\hat{\mathbf{F}})|\hat{\mathbf{F}}] &= \text{Var}\left[\frac{1}{B}\sum_{b=1}^B g(\hat{\mathbf{F}}^{*(b)}) + \frac{\sum_{j=1}^{BR} \varepsilon_j}{BR} \middle| \hat{\mathbf{F}}\right] \\ &= \frac{1}{B}\text{Var}[g(\hat{\mathbf{F}}^*)|\hat{\mathbf{F}}] + \frac{\sigma^2}{BR} \\ &= \frac{\sigma_I^2}{B} + \frac{\sigma^2}{BR}.\end{aligned}$$

The second equality holds because we can view $g(\hat{\mathbf{F}}^{*(b)})$ as an i.i.d. random variable that is a function of $\hat{\mathbf{F}}$. The last equality follows from the assumption we made in (10). The covariance component becomes

$$\begin{aligned}\text{Cov}[\bar{Y}(\hat{\mathbf{F}}), \bar{\bar{Y}}(\hat{\mathbf{F}})|\hat{\mathbf{F}}] &= \text{Cov}\left[g(\hat{\mathbf{F}}) + \frac{\sum_{j=1}^n \varepsilon_j}{n}, \frac{1}{B}\sum_{b=1}^B g(\hat{\mathbf{F}}^{*(b)}) + \frac{\sum_{j=1}^{BR} \varepsilon_j}{BR} \middle| \hat{\mathbf{F}}\right] \\ &= 0,\end{aligned}$$

since all $g(\hat{\mathbf{F}})$, $g(\hat{\mathbf{F}}^{*(b)})$, and ε_j are independent of each other given $\hat{\mathbf{F}}$. Therefore, (27) becomes

$$\text{Var}[\tilde{Y}|\hat{\mathbf{F}}] = \alpha^2 \frac{\sigma^2}{n} + (1-\alpha)^2 \left(\frac{\sigma_I^2}{B} + \frac{\sigma^2}{BR} \right),$$

which gives us the expression for $\text{MSE}[\tilde{Y}|\hat{\mathbf{F}}]$:

$$\text{MSE}[\tilde{Y}|\hat{\mathbf{F}}] = (1-\alpha)^2 \left((b^*)^2 + \frac{\sigma_I^2}{B} + \frac{\sigma^2}{BR} \right) + \alpha^2 \frac{\sigma^2}{n}.$$

C MSE of $\tilde{Y}$

Recall that Equation (17) provided an expression for the $\text{MSE}[\tilde{Y}|\hat{\mathbf{F}}]$ as a function of the weight α with all terms estimable from the diagnostic experiment. Minimizing the conditional MSE is a sensible objective for deriving the optimal weight α^* since the user is stuck with $\hat{\mathbf{F}}$ as the best-guess input model. However, it clearly underestimates the unconditional $\text{MSE}(\tilde{Y})$ which is affected by the variability in $\hat{\mathbf{F}}$.

To obtain a usable expression for $\text{MSE}(\tilde{Y})$, we make two additional simplifying assumptions: $g(\hat{\mathbf{F}})$ has negligible bias for $g(\mathbf{F}^c)$, and the bias term b^* is not a function of $\hat{\mathbf{F}}$. With these assumptions

$$\begin{aligned} \text{MSE}(\tilde{Y}) &= \mathbb{E} \left[\text{Var}(\tilde{Y}|\hat{\mathbf{F}}) \right] + \text{Var} \left[\mathbb{E}(\tilde{Y}|\hat{\mathbf{F}}) \right] + \text{Bias}^2(\tilde{Y}) \\ &= \alpha^2 \frac{\sigma^2}{n} + (1-\alpha)^2 \left(\frac{\sigma_I^2}{B} + \frac{\sigma^2}{RB} \right) + \sigma_I^2 + [\alpha g(\mathbf{F}^c) + (1-\alpha)(g(\mathbf{F}^c) + b^*) - g(\mathbf{F}^c)]^2 \\ &= \underbrace{\frac{\sigma_I^2}{n}}_{(A)} + \underbrace{\alpha^2 \frac{\sigma^2}{n}}_{(B)} + \underbrace{(1-\alpha)^2 \left((b^*)^2 + \frac{\sigma_I^2}{B} + \frac{\sigma^2}{BR} \right)}_{(C)}. \end{aligned} \quad (28)$$

This expression makes intuitive sense because (A) is the unavoidable variance due to using $\hat{\mathbf{F}}$ for $\mathbf{F}^c$, (B) is the variance of $\alpha\tilde{Y}$, while (C) is the MSE of $(1-\alpha)\tilde{Y}$.

We obtain $\widehat{\text{MSE}}$ by inserting estimates for each term in (28) derived from the diagnostic experiment.

D Optimal weights for combining the nominal and follow-up experiments

In Section 4 we combined the estimators from the nominal experiment $\bar{Y}_n(\hat{\mathbf{F}}_{\mathbf{m}})$ and the follow-up experiment $\bar{Y}_{n'}(\hat{\mathbf{F}}_{\mathbf{m}'})$:

$$\hat{Y} = \alpha \bar{Y}_n(\hat{\mathbf{F}}_{\mathbf{m}}) + (1-\alpha) \bar{Y}_{n'}(\hat{\mathbf{F}}_{\mathbf{m}'}).$$

We defined α^* as the $\alpha \in [0, 1]$ that minimizes $\text{MSE}[\hat{Y}] = \text{Bias}^2[\hat{Y}] + \text{Var}[\hat{Y}]$. The bias term is

$$\text{Bias}[\hat{Y}] = \alpha b_{\mathbf{m}} + (1-\alpha) b_{\mathbf{m}'},$$

where $b_{\mathbf{m}} = \text{Bias}[\bar{Y}_n(\hat{\mathbf{F}}_{\mathbf{m}})]$ and $b_{\mathbf{m}'} = \text{Bias}[\bar{Y}_{n'}(\hat{\mathbf{F}}_{\mathbf{m}'})]$. The variance term is further decomposed as

$$\text{Var}[\hat{Y}] = \alpha^2 \text{Var}[\bar{Y}_n(\hat{\mathbf{F}}_{\mathbf{m}})] + (1-\alpha)^2 \text{Var}[\bar{Y}_{n'}(\hat{\mathbf{F}}_{\mathbf{m}'})] - 2\alpha(1-\alpha) \text{Cov}[\bar{Y}_n(\hat{\mathbf{F}}_{\mathbf{m}}), \bar{Y}_{n'}(\hat{\mathbf{F}}_{\mathbf{m}'})].$$

Thus, $\text{MSE}[\hat{Y}]$ can be written as

$$\begin{aligned} \text{MSE}[\hat{Y}] &= \left(\sigma_I^2 + (\sigma'_i)^2 + \frac{\sigma^2}{n} + \frac{\sigma^2}{n'} + (b_{\mathbf{m}'} - b_{\mathbf{m}})^2 - 2\text{Cov}[\bar{Y}_n(\hat{\mathbf{F}}_{\mathbf{m}}), \bar{Y}_{n'}(\hat{\mathbf{F}}_{\mathbf{m}'})] \right) \alpha^2 \\ &\quad - 2 \left((\sigma'_i)^2 + \frac{\sigma^2}{n'} + b_{\mathbf{m}'}(b_{\mathbf{m}'} - b_{\mathbf{m}}) - \text{Cov}[\bar{Y}_n(\hat{\mathbf{F}}_{\mathbf{m}}), \bar{Y}_{n'}(\hat{\mathbf{F}}_{\mathbf{m}'})] \right) \alpha + (\sigma'_i)^2 + \frac{\sigma^2}{n'} + (b_{\mathbf{m}'})^2, \end{aligned}$$

which is minimized at

$$\alpha^* = \frac{(\sigma'_i)^2 + \frac{\sigma^2}{n'} + b_{\mathbf{m}'}(b_{\mathbf{m}'} - b_{\mathbf{m}}) - \text{Cov}[\bar{Y}_n(\hat{\mathbf{F}}_{\mathbf{m}}), \bar{Y}_{n'}(\hat{\mathbf{F}}_{\mathbf{m}'})]}{\sigma_I^2 + (\sigma'_i)^2 + \frac{\sigma^2}{n} + \frac{\sigma^2}{n'} + (b_{\mathbf{m}'} - b_{\mathbf{m}})^2 - 2\text{Cov}[\bar{Y}_n(\hat{\mathbf{F}}_{\mathbf{m}}), \bar{Y}_{n'}(\hat{\mathbf{F}}_{\mathbf{m}'})]}. \quad (29)$$

Under Model (6), $\bar{Y}_n(\hat{\mathbf{F}}_{\mathbf{m}})$ and $\bar{Y}_{n'}(\hat{\mathbf{F}}_{\mathbf{m}'})$ can be rewritten as

$$\begin{aligned}\bar{Y}_n(\hat{\mathbf{F}}_{\mathbf{m}}) &= \mathbf{g}(\hat{\mathbf{F}}_{\mathbf{m}}) + \frac{\sum_{j=1}^n \varepsilon_j}{n} = \beta_0 + \sum_{\ell=1}^L \beta_\ell \mu(\hat{F}_{\mathbf{m}\ell}) + \sum_{\ell=1}^L v_\ell \sigma^2(\hat{F}_{\mathbf{m}\ell}) + \frac{\sum_{j=1}^n \varepsilon_j}{n} \\ \bar{Y}_{n'}(\hat{\mathbf{F}}_{\mathbf{m}'}) &= \mathbf{g}(\hat{\mathbf{F}}_{\mathbf{m}'}) + \frac{\sum_{j=1}^{n'} \varepsilon_j}{n'} = \beta_0 + \sum_{\ell=1}^L \beta_\ell \mu(\hat{F}_{\mathbf{m}'\ell}) + \sum_{\ell=1}^L v_\ell \sigma^2(\hat{F}_{\mathbf{m}'\ell}) + \frac{\sum_{j=1}^{n'} \varepsilon_j}{n'},\end{aligned}$$

where $\hat{F}_{\mathbf{m}\ell}$ and $\hat{F}_{\mathbf{m}'\ell}$ indicate ℓ th input model of $\hat{\mathbf{F}}_{\mathbf{m}}$ and $\hat{\mathbf{F}}_{\mathbf{m}'}$, respectively. Since all ε_j and $\varepsilon_{j'}$ are independent, $\text{Cov}[\bar{Y}_n(\hat{\mathbf{F}}_{\mathbf{m}}), \bar{Y}_{n'}(\hat{\mathbf{F}}_{\mathbf{m}'})] = \text{Cov}[\mathbf{g}(\hat{\mathbf{F}}_{\mathbf{m}}), \mathbf{g}(\hat{\mathbf{F}}_{\mathbf{m}'})]$ and

$$\begin{aligned}\text{Cov}[\mathbf{g}(\hat{\mathbf{F}}_{\mathbf{m}}), \mathbf{g}(\hat{\mathbf{F}}_{\mathbf{m}'})] &= \sum_{\ell=1}^L \left\{ \beta_\ell^2 \text{Cov}[\mu(\hat{F}_{\mathbf{m}\ell}), \mu(\hat{F}_{\mathbf{m}'\ell})] + \beta_\ell v_\ell \text{Cov}[\mu(\hat{F}_{\mathbf{m}\ell}), \sigma^2(\hat{F}_{\mathbf{m}'\ell})] \right. \\ &\quad \left. + \beta_\ell v_\ell \text{Cov}[\sigma^2(\hat{F}_{\mathbf{m}\ell}), \sigma^2(\hat{F}_{\mathbf{m}'\ell})] + v_\ell^2 \text{Cov}[\sigma^2(\hat{F}_{\mathbf{m}\ell}), \sigma^2(\hat{F}_{\mathbf{m}'\ell})] \right\}.\end{aligned}\quad (30)$$

Therefore, to get an expression for $\text{Cov}[\bar{Y}_n(\hat{\mathbf{F}}_{\mathbf{m}}), \bar{Y}_{n'}(\hat{\mathbf{F}}_{\mathbf{m}'})]$, we need the four covariance components in (30), which are not easy to obtain in general. However, if we assume all input models used in the nominal and the follow-up experiments are ecdfs, then we can get expressions for the bias terms as well as the covariance term.

Under this assumption, the first covariance component, $\text{Cov}[\mu(\hat{F}_{\mathbf{m}\ell}), \mu(\hat{F}_{\mathbf{m}'\ell})]$, becomes

$$\begin{aligned}\text{Cov}[\mu(\hat{F}_{\mathbf{m}\ell}), \mu(\hat{F}_{\mathbf{m}'\ell})] &= \text{Cov}\left[\frac{\sum_{i=1}^{m_\ell} X_{\ell i}}{m_\ell}, \frac{\sum_{i=1}^{m'_\ell} X_{\ell i}}{m'_\ell}\right] \\ &= \frac{\sum_{i=1}^{m_\ell} \text{Var}[X_{\ell i}]}{m_\ell m'_\ell} \\ &= \frac{\mu_\ell^2}{m'_\ell},\end{aligned}\quad (31)$$

where $X_{\ell 1}, X_{\ell 2}, \dots, X_{\ell m_\ell}$ is the nominal real-world sample that we observed from ℓ th input process F_ℓ^c for the nominal experiment, $X_{\ell m_\ell+1}, X_{\ell m_\ell+2}, \dots, X_{\ell m'_\ell}$ is the additional sample collected for the follow-up experiment, if any, and μ_ℓ^2 is the second central moment of F_ℓ^c . Notice that the second equality holds since $X_{\ell i}$ and $X_{\ell i'}$ are independent if $i \neq i'$. In a similar fashion, the second and the third covariance components in (30) become

$$\begin{aligned}\text{Cov}[\mu(\hat{F}_{\mathbf{m}\ell}), \sigma^2(\hat{F}_{\mathbf{m}'\ell})] &= \frac{m'_\ell - 1}{(m'_\ell)^2} \mu_\ell^3 \approx \frac{\mu_\ell^3}{m'_\ell} \\ \text{Cov}[\sigma^2(\hat{F}_{\mathbf{m}\ell}), \sigma^2(\hat{F}_{\mathbf{m}'\ell})] &= \frac{m_\ell - 1}{m_\ell m'_\ell} \mu_\ell^3 \approx \frac{\mu_\ell^3}{m'_\ell}.\end{aligned}\quad (32)$$

The approximation in (32) can be justified under the assumption $m'_\ell \gg m_\ell$. The last component is

$$\begin{aligned}
& \text{Cov}[\sigma^2(\widehat{F}_{\mathbf{m}\ell}), \sigma^2(\widehat{F}_{\mathbf{m}'\ell})] \\
&= \frac{(m_\ell - 1)(m'_\ell - 1)}{m_\ell(m'_\ell)^2} \mathbb{E}[X_{\ell i}^4] + \frac{2((m_\ell - 1)(1 - 2m'_\ell))}{m_\ell(m'_\ell)^2} \mathbb{E}[X_{\ell i}^3] \mathbb{E}[X_{\ell i}] \\
&\quad - \frac{4(m_\ell - 1)(1 + m_\ell - 2m'_\ell)}{m_\ell(m'_\ell)^2} \mathbb{E}[X_{\ell i}^2] \mathbb{E}[X_{\ell i}]^2 - \frac{2(m_\ell - 1)(2m'_\ell - 3)}{m_\ell(m'_\ell)^2} \mathbb{E}[X_{\ell i}]^4 - \frac{(m_\ell - 1)(m'_\ell - 3)}{m_\ell(m'_\ell)^2} \mathbb{E}[X_{\ell i}^2]^2 \\
&= \frac{1}{m'_\ell} (\mathbb{E}[X_{\ell i}^4] - 4\mathbb{E}[X_{\ell i}^3] \mathbb{E}[X_{\ell i}] + 8\mathbb{E}[X_{\ell i}^2] \mathbb{E}[X_{\ell i}]^2 - 4\mathbb{E}[X_{\ell i}]^4 - \mathbb{E}[X_{\ell i}^2]^2) + O((m'_\ell)^{-2}) \\
&= \frac{1}{m'_\ell} \{ (\mathbb{E}[X_{\ell i}^4] - 4\mathbb{E}[X_{\ell i}^3] \mathbb{E}[X_{\ell i}] + 6\mathbb{E}[X_{\ell i}^2] \mathbb{E}[X_{\ell i}]^2 - 3\mathbb{E}[X_{\ell i}]^4) - (\mathbb{E}[X_{\ell i}^2] - \mathbb{E}[X_{\ell i}]^2)^2 \} + O((m'_\ell)^{-2}) \\
&\approx \frac{\mu_\ell^4 - (\mu_\ell^2)^2}{m'_\ell}. \tag{33}
\end{aligned}$$

Again, the last approximation holds under the assumption $m'_\ell \gg m_\ell$. By plugging (31), (32), and (33) into (30), $\text{Cov}[g(\widehat{\mathbf{F}}_{\mathbf{m}}), g(\widehat{\mathbf{F}}_{\mathbf{m}'})]$ can be written as

$$\text{Cov}[g(\widehat{\mathbf{F}}_{\mathbf{m}}), g(\widehat{\mathbf{F}}_{\mathbf{m}'})] \approx \sum_{\ell=1}^L \frac{1}{m'_\ell} \{ \beta_\ell^2 \mu_\ell^2 + 2\beta_\ell v_\ell \mu_\ell^3 + v_\ell^2 (\mu_\ell^4 - (\mu_\ell^2)^2) \}. \tag{34}$$

However, this expression is very similar to $(\sigma'_I)^2$. In fact, if we assume m'_ℓ to be large we can approximate $\mu_\ell^k \approx M_\ell^{k'}$, where $M_\ell^{k'}$ is the k th central moment of $\widehat{F}_{m'_\ell \ell}$, which implies $\text{Cov}[g(\widehat{\mathbf{F}}_{\mathbf{m}}), g(\widehat{\mathbf{F}}_{\mathbf{m}'})] \approx \sigma_I'^2$.

Also, $\mathbf{b}_{\mathbf{m}}$ becomes

$$\begin{aligned}
\mathbf{b}_{\mathbf{m}} &= \mathbb{E}[g(\widehat{\mathbf{F}}_{\mathbf{m}})] - g(\mathbf{F}^c) \\
&= \mathbb{E} \left[\sum_{\ell=1}^L \left(\beta_\ell \mu(\widehat{F}_{\mathbf{m}\ell}) + v_\ell \sigma^2(\widehat{F}_{\mathbf{m}\ell}) \right) \right] - \sum_{\ell=1}^L (\beta_\ell \mu(F_\ell^c) + v_\ell \sigma^2(F_\ell^c)) \\
&= \sum_{\ell=1}^L v_\ell (\mathbb{E}[\sigma^2(\widehat{F}_{\mathbf{m}\ell})] - \sigma^2(F_\ell^c)) \\
&= \sum_{\ell=1}^L v_\ell \left(\frac{m_\ell}{m_\ell - 1} \sigma^2(F_\ell^c) - \sigma^2(F_\ell^c) \right) \\
&= \sum_{\ell=1}^L v_\ell \frac{\sigma^2(F_\ell^c)}{m_\ell - 1}. \tag{35}
\end{aligned}$$

By using $\text{Cov}[g(\widehat{\mathbf{F}}_{\mathbf{m}}), g(\widehat{\mathbf{F}}_{\mathbf{m}'})] \approx (\sigma'_I)^2$ and (35) in (29), we get the expression in (20).