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V.V. Krylov

**PROPAGATION OF LOCALIZED VIBRATION MODES ALONG
EDGES OF IMMERSSED WEDGE-LIKE STRUCTURES:
GEOMETRICAL-ACOUSTICS APPROACH**

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The theory of antisymmetric localized elastic modes propagating along edges of immersed wedge-like structures is developed using the geometrical-acoustics approach to the description of flexural waves in elastic plates of variable thickness. The velocities of these modes, often called wedge acoustic waves, are calculated using solutions of the dispersion equation of Bohr - Sommerfeld type following from the geometrical-acoustics description of localized wedge modes. In a subsonic regime of wave propagation, i.e., for wedge modes being slower than sound in liquid, the influence of liquid loading results in significant decrease of wedge wave velocities in comparison with their values in vacuum. This decrease is a nonlinear function of a wedge apex angle θ and is more pronounced for small values of θ . In a supersonic regime of wedge wave propagation, a smaller decrease in velocities takes place and the waves travel with the attenuation due to radiation of sound into surrounding liquid. The comparison is given with the recent experimental investigations of wedge waves carried out by independent researchers.

1. Introduction

The existence of antisymmetric localized vibration modes propagating along the tips of wedge-like solid structures has been predicted theoretically in 1972 by Lagasse¹ and Maradudin et al.². Using numerical calculations, it has been shown^{1,2} that such waves, often called wedge acoustic waves, are characterized by low propagation velocity (generally much lower than that of Rayleigh waves), and their elastic energy is concentrated in the area of about one wavelength near the wedge tips (Fig.1). Since then, wedge acoustic waves were investigated in a number of papers with regard to their possible applications to acoustoelectronic signal processing devices and to nondestructive testing of special engineering constructions (see, e.g., Refs. 3-8 and references there).

All of the above mentioned papers considered propagation of wedge waves in contact with vacuum. The first experimental work studying their interaction with liquids has been published by Chamuel⁹ in 1993. In the year 1994 the existence of localized flexural elastic waves on the edges of wedge-like immersed structures has been described theoretically by the present author¹⁰. The Ref. 10 was followed by subsequent experimental investigations of wedge waves in

immersed structures which considered samples made of different materials and having different values of the wedge apex angle. Thus, measurements of wedge wave velocities for slender Plexiglas wedges with apex angles varying from about 8 to 13 degrees were carried out by Chamuel¹¹. The results have shown excellent agreement with the approximate analytical theory developed by the present author for slender immersed wedges in subsonic regime of wave propagation¹⁰. De Billy¹² carried out measurements of the velocities of modes propagating along tips of Plexiglas and brass wedges with the apex angle varying from 20 to 90 degrees. For Plexiglas samples, for which a subsonic regime of wedge wave propagation takes place, the experiments showed a noticeable decrease in velocities due to water loading, in agreement with the predictions of the theory¹⁰. The results for brass samples, which provide both subsonic and supersonic propagation regimes, remained unexplained since no theory existed to deal with this case. Recently, finite element calculations were carried out by Hladky-Hennion et al.¹³ for Plexiglas and brass wedges with the values of apex angle in the range from 20 to 90 degrees. Although for Plexiglas wedges a good quantitative agreement of the calculations¹³ with the experiments¹² has been achieved, in the case of brass wedges only a subsonic regime has been calculated.

The above mentioned theoretical calculations of wedge elastic waves in immersed solid structures and the experimental confirmation of their existence represents new fundamental knowledge which may become a key to the explanation of many as yet poorly understood phenomena in related fields of structural dynamics, physics and environmental acoustics and result in many useful practical applications. In particular, it is expected that these waves play an important role in the dynamics of wedge-shaped civil engineering structures immersed in water (such as piers, dams, wave-breakers, etc.), and in the formation of vibration patterns and resonance frequencies of propellers, turbine blades, disks, cutting tools and airfoils. They may be responsible for specific mechanisms of helicopter noise, wind turbine noise and ship propeller noise. Promising mechanical engineering applications of wedge elastic waves include non-destructive evaluation of specific engineering constructions (with edges), measurements of cutting edge sharpness, environmentally friendly water pumps and domestic ventilators utilizing wave-generated flows. Another possible and very promising application earlier suggested by the present author¹⁰ may be the use of wedge waves for in-water propulsion of ships and submarines, the main principle of which being similar to that used in nature by fish of the ray family. As everyone knows, these fish apply a wave-like motion of their wings for moving forward. The unique benefit of using wedge elastic waves for wave-like propulsion of man-inhabited vessels (in comparison with all other waves used for this purpose in the previous attempts, e.g., flexural waves in plates) is that the main body of the vessel could be virtually quiet because of the wedge wave energy being concentrated near the wedge tip.

Obviously, there may appear even more novel applications of these new waves which might arise from their more detailed investigations. From the point of view of the historical development of science, the attention to wedge waves reflects the current tendency of transferring the focus of research from investigating mechanical, physical, and chemical phenomena in the bulk of materials to studying their specific behavior on the surfaces and (as in the present case) on the intersections of surfaces, i.e., on edges or lines.

In the present paper we give an overview of the geometrical acoustics approach to the theory of localized elastic waves in wedge-like solid structures and generalize the earlier developed

approximate analytical theory of localized vibration modes in immersed wedges¹⁰ to describe the effect of heavier wedge material and a supersonic regime of wave propagation. The results are then compared with the existing experiments^{11,12}.

2. Geometrical-Acoustics Approach

The approximate theory of localized elastic waves in immersed solid wedges initially developed in Ref. 10 is based on the geometrical acoustics approach considering a slender wedge as a plate with a local variable thickness $d(x) = x\Theta$, where Θ is the wedge apex angle and x is the distance from the wedge tip measured in the middle plane (Fig.1).

We start from a general two-dimensional equation for bending motions of a thin plate in vacuum, the plate thickness d being nonuniform along the x direction:

$$\begin{aligned} \frac{\partial^2}{\partial x^2} [D(x) (\frac{\partial^2 w}{\partial x^2} + \sigma \frac{\partial^2 w}{\partial y^2})] + 2(1 - \sigma) \frac{\partial^2}{\partial x \partial y} [D(x) \frac{\partial^2 w}{\partial x \partial y}] + \\ \frac{\partial^2}{\partial y^2} [D(x) (\frac{\partial^2 w}{\partial y^2} + \sigma \frac{\partial^2 w}{\partial x^2})] - \omega^2 \rho h(x) w = 0. \end{aligned} \quad (2.1)$$

Here w is the normal displacement of the median plane of the plate, $D(x) = Ed^3(x)/12(1 - \sigma^2) = (\rho c_p^2/12)d^3(x)$ is the local bending stiffness, E and σ are the Young's modulus and Poisson's ratio of the plate material, ρ is the mass density of the plate material, $c_p = 2c_t(1 - c_t^2/c_l^2)^{1/2}$ is the velocity of longitudinal waves in thin plates (or plate wave velocity), and ω is the angular frequency.

We seek a solution of Eq. (2.1) in the usual geometrical-acoustics (or optics) representation^{6,7}:

$$w = A(x) \exp[ik_p S(x, y)], \quad (2.2)$$

where $A(x)$ and $S(x, y) = S'(x) + (\beta/k_p)y$ are the slowly varying amplitude and eiconal of the quasi-plane wave respectively, β is the conserved projection of the wave vector of this wave onto the y axis.

Substituting (2.2) into (2.1) and equating the real part to zero, we derive, after retaining only the leading terms, the following 'eiconal equation':

$$|\nabla S(x, y)|^4 = k^4(x)/k_p^4 = n^4(x), \quad (2.3)$$

where $k(x) = 12^{1/4} k_p^{1/2} (\Theta x)^{-1/2}$ is the local wave number of the bending wave in the plate, $k_p = \omega/c_p$, and $n(x)$ is the corresponding refractive index.

A solution of Eq. (2.3) corresponding to wave propagation in the positive direction along x axis is

$$S'(x) = (1/k_p) \int [k^2(x) - \beta^2]^{1/2} dx. \quad (2.4)$$

Equating the imaginary part to zero, we derive the so-called ‘transport equation’ for the case considered:

$$2 \frac{d}{dx} (\ln A) \frac{dS'}{dx} + 3 \frac{d^2 S'}{dx^2} \frac{[1 + (\beta^2 / 3k_p^2)(dS'/dx)^{-2}]}{[1 + (\beta^2 / k_p^2)(dS'/dx)^{-2}]} + 3 \frac{1}{x} \frac{dS'}{dx} = 0. \quad (2.5)$$

Solving Eq. (2.5) and taking into account (2.4), we obtain

$$A(x) = \frac{G}{x[k^2(x) - \beta^2]^{1/4}}, \quad (2.6)$$

where G is an arbitrary constant. For normal incidence of the wave ($\beta=0$) it follows from (2.6) that $A(x) = G/xk^{1/2}$, i.e., $A(x)$ is proportional to $x^{-3/4}$. It is simple to verify that Eq. (2.6) satisfies the law of energy conservation of the bending waves propagating through various cross sections of the plate of variable thickness⁷.

An estimate of the terms which have been discarded in deriving Eqs. (2.3) and (2.5) shows that the above described geometrical-acoustics solution (2.2), (2.4), (2.6) is valid under the condition

$$k_p x / \Theta \gg 1. \quad (2.7)$$

In other words, this solution breaks down in the immediate vicinity of the wedge edge (at small values of $k_p x$) and/or at large wedge angles Θ . Furthermore, the well known limitation of ray-tracing theory follows directly from (2.6): the observation point must be far from the ray turning points (determined from the condition that the term in square brackets vanishes) or far from caustic regions. In the case under consideration, there are caustics for rays which propagate away from the edge in the positive x direction, i.e., out of a region of lower local bending-wave velocity into a region with a higher one. We should of course not forget that far from the wedge tip, specifically, under the condition $h(x) = \Theta x > 2\pi k_p$, the solution in the form (2.2), (2.4), (2.6) again becomes meaningless, since in this case the wedge becomes thick enough, and the equation for the bending waves of a thin plate, i.e., Eq. (2.1), is no longer valid.

To apply the above mentioned geometrical-acoustics approach to develop a theory of antisymmetric vibrations of a solid wedge which are localized near an edge, we can use the ray-tracing solution (2.2), (2.4), (2.6) and assume that the quantity β in this solution is the wave number which we are seeking for the wedge modes. Using the well known procedure for the geometrical acoustics calculation of waveguide modes in a near-surface acoustic channel (see, e.g., Ref. 14), noting that the phase shift of a bending wave upon its reflection from the edge is

$\pi/2$ and the phase shift upon reflection from a simple caustic is $-\pi/2$, we can immediately write the following Bohr - Sommerfeld type equation for the velocities c of the localized wedge modes propagating in y -direction⁶⁻⁸

$$\int_0^{x_t} [k^2(x) - \beta^2]^{1/2} dx = \pi n, \quad (2.8)$$

where $\beta = \omega/c$ is yet unknown wavenumber of a wedge mode, $n = 1, 2, 3, \dots$ is the mode number, and x_t is the so called ray turning point being determined from the equation $k^2(x) - \beta^2 = 0$.

As we mentioned before, in the case of wedge in vacuum $k(x) = 12^{1/4} k_p^{1/2} (\theta x)^{-1/2}$, and $x_t = 2\sqrt[3]{k_p/\Theta\beta^2}$, where $k_p = \omega/c_p$, so that the solution of Eq. (2.8) yields the extremely simple analytical expression for wedge wave velocities⁶⁻⁸:

$$c = c_p n \Theta / \sqrt[3]{\beta}. \quad (2.9)$$

The expression (2.9) agrees well with the other theories¹⁻⁵ and with the experimental results³. Note that, although, according to the above mentioned limitations, the geometrical acoustics approach is not valid for the lowest order wedge mode ($n = 1$) (see Ref. 7), in practice it provides quite accurate results for wedge wave velocities in this case as well. The analytical expressions for amplitude distributions of wedge modes have the following form⁶⁻⁸:

$$w(x, y, t) = \frac{W_0}{(k_p x / \Theta)^{1/2} [L(x)]^{1/2}} \cos[L(x) k_p x / \Theta - n \sin^{-1} M(x) - \pi/4] \exp[i(\sqrt[3]{k_p} y / n \Theta - \omega t)], \quad (2.10)$$

where $L(x) = [\sqrt[3]{2\Theta/k_p x} - \sqrt[3]{n^2}]^{1/2}$ and $M(x) = (1 - \sqrt[3]{k_p x / n^2 \Theta})$. Note that in the geometrical-acoustics approximation the wave field does not penetrate into the region $x > x_t$, where $x_t = 2\sqrt[3]{k_p/\Theta\beta^2}$ determines the ray-turning points or the location of caustics.

3. Effect of Liquid Loading

To calculate the velocities of wedge modes in a wedge embedded in liquid one has to make use of the expression for a plate wave local wavenumber $k(x)$ which takes into account the effect of liquid loading¹⁰. The starting point to derive $k(x)$ for this case is the well known dispersion equation for the lowest order flexural mode in a plate embedded in liquid:

$$\frac{1}{2} \frac{\omega^4}{c_t^4 k^4} - \frac{1}{6} (c_l^2 - c_t^2) \frac{d^2}{c_l^2 c_t^2} \omega^2 + \frac{\rho_f}{\rho_s} \frac{\omega^4}{c_t^4 k^4} \frac{1}{d \sqrt{k^2 - \omega^2/c_f^2}} = 0. \quad (3.1)$$

Here d is the plate thickness, $k = \omega/c$ is the wavenumber of propagating flexural mode, c is its phase velocity, and ω is circular frequency, $k_l = \omega/c_l$ and $k_t = \omega/c_t$ are respectively the wavenumbers of longitudinal and shear acoustic waves in plate material, c_l and c_t are their propagation velocities, ρ_s and ρ_f are respectively the mass densities of solid and liquid. Note that, using the notation for flexural rigidity $D = (\rho_s c_p^2 / 12) d^3$, where $c_p = 2c_t(1 - c_t^2/c_l^2)^{1/2}$ is the plate wave velocity, one can easily transform Eq. (3.1) to the more familiar form which is often used in the literature for flexural waves in thin plates faced to liquid at both sides^{15,16}. In the absence of liquid ($\rho_f = 0$) Eq. (3.1) reduces to the well known dispersion relation for a flexural wave in a thin plate in vacuum. In further consideration it is convenient to distinguish two characteristic cases of the relation between the mass densities of liquid and wedge material¹⁷: $\rho_f/\rho_s \approx 1$ and $\rho_f/\rho_s \ll 1$.

3.1 Wedge wave velocities for $\rho_f/\rho_s \approx 1$

Let us first consider the case $\rho_f/\rho_s \approx 1$ typical for light solid materials in water and let us limit our analysis by a subsonic regime of wave propagation ($k > \omega/c_f$). Moreover, for simplicity, we impose even more severe restriction on wave velocities considering very slow propagating plate flexural modes ($k \gg \omega/c_f$). In such a case one can use the approximation of incompressible liquid, i.e., neglect the ω/c_f term in (3.1), and the solution of Eq. (3.1) versus ω yields

$$\omega = \frac{1}{\sqrt{3}} \frac{c_t}{c_l} k^{5/2} \sqrt{\rho_s} \frac{d^{3/2}}{\sqrt{\rho_s k d + 2\rho_f}} \sqrt{c_l^2 - c_t^2}. \quad (3.2)$$

For $kd \ll 1$ typical for thin plates one can also neglect the term $\rho_s k d$ in Eq. (3.2), keeping in mind that $\rho_f/\rho_s \approx 1$. Then, taking into account that for a plate of variable thickness representing a wedge $d = d(x) = x\Theta$ and solving (3.2) versus k , one obtains the following expression for the local flexural wavenumber $k(x)$ describing the effect of liquid loading:

$$k(x) = \left[\sqrt{6} \frac{c_l}{c_s} \frac{1}{\sqrt{c_l^2 - c_t^2}} \frac{\sqrt{\rho_f}}{\sqrt{\rho_s}} \frac{\omega}{(x\Theta)^{3/2}} \right]^{2/5}. \quad (3.3)$$

Substituting (3.3) into (2.8) and introducing the nondimensional notations $\eta = c/c_t$ and $z = dk_t = x\Theta\omega/c_t$, one can derive the following equation versus η (see also Refs. 10, 17):

$$\int_0^{(Ay)^{5/3}} \left[\frac{A^2}{z^{6/5}} - \frac{1}{\eta^2} \right]^{1/2} dz = \pi \theta, \quad (3.4)$$

where

$$A = 6^{1/5} (\rho_l/\rho_s)^{1/5} (1 - c_l^2/c_l^2)^{-1/5} = 6^{1/5} (\rho_l/\rho_s)^{1/5} [2(1-\sigma)]^{1/5} \quad (3.5)$$

is a nondimensional parameter which depends on the relation between the mass densities of liquid and solid ρ_l/ρ_s and on the Poisson ratio σ .

Eq. (3.4) can be easily solved numerically using any appropriate algorithm, e.g., standard Mathcad package¹⁰. Note, however, that by change of variable $z = A^{5/3} \eta^{5/3} x$, it may be further simplified as

$$A^{5/3} \eta^{2/3} \int_0^1 (x^{-6/5} - 1)^{1/2} dx = \pi \theta. \quad (3.6)$$

[The author's attention to this way of simplification of Eq. (3.4) has been drawn by A.N. Norris.] After numerical calculation of the integral in Eq. (3.6), one can easily derive the explicit analytical expression for wedge wave velocities¹⁷:

$$c = c_l A^{-5/2} D^{-3/2} (\pi)^{3/2} \Theta^{3/2}, \quad (3.7)$$

where $D = \int_0^1 (x^{-6/5} - 1)^{1/2} dx = 2.102$. According to Eq. (3.7), the dependence of c on Θ for immersed slender wedges is proportional to $\Theta^{3/2}$. This agrees well with the empirical power law $\Theta^{1.522}$ established by Chamuel¹¹ by matching the numerical solution of Eq. (3.4).

3.2 Wedge wave velocities for $\rho_l/\rho_s \ll 1$

In the case $\rho_l/\rho_s \ll 1$ typical for heavy solid materials in water and for any kind of solid material in gas, one can solve Eq. (3.1) approximately, to the first perturbation order versus ρ_l/ρ_s , seeking a solution for k in the form

$$k = k_{(0)} + k_{(1)}. \quad (3.8)$$

Here $k_{(0)}$ is the solution of Eq. (3.1) for a plate in vacuum ($\rho_l/\rho_s = 0$) and $k_{(1)}$ is yet unknown small correction term which takes into account the effect of liquid loading. Substituting (3.8)

into (3.1) and retaining terms of only the first order versus ρ_f/ρ_s , one can derive the following approximate expression for $k(x)$:

$$k(x) = \left(\frac{\sqrt{3}\omega c_l}{c_t \sqrt{c_l^2 - c_t^2} x \theta} \right)^{1/2} \left[1 + \frac{\rho_f}{2\rho_s x \theta} \left(\frac{\sqrt{3}\omega c_l}{c_t \sqrt{c_l^2 - c_t^2} x \theta} - \frac{\omega^2}{c_f^2} \right)^{-1/2} \right]. \quad (3.9)$$

Obviously, for $\rho_f/\rho_s = 0$ Eq. (3.9) goes over to the well known expression for a wavenumber $k_{(0)}$ of a flexural plate wave in vacuum. The second term, proportional to ρ_f/ρ_s , gives a small correction $k_{(1)}$ describing liquid loading. It is seen that this term becomes imaginary (and hence a plate wave becomes leaky) in supersonic regime of wave propagation, when a local wavenumber of a plate wave in vacuum exceeds the velocity of sound in liquid. Note that Eq. (3.9) is not valid for very small values of $x\theta$ and for the values of $x\theta$ corresponding to the flexural wave velocity being equal to the velocity of sound in liquid. In both these cases the contribution of the second term in square brackets of (3.9) is not small and the applied perturbation solution is no longer valid. Regarding wedge waves, this means that the above considered perturbation approach is not applicable for wedges with very small apex angles θ and for velocities of wedge waves approaching the velocity of sound in liquid.

To apply the Bohr - Sommerfeld type equation for calculating wedge wave velocities in the case considered, one should follow the well known way of generalization of geometrical acoustics for complex wavenumbers and substitute a real part of (3.9) into Eq. (2.8). Doing so and using the nondimensional notations $\eta = c/c_t$ and $z = dk_t = x\theta\omega/c_t$, one can derive the following equation versus η :

$$\int_0^{B/2} \left[\frac{B}{z} \operatorname{Re} \left(1 + \frac{\rho_f}{\rho_s} \frac{1}{z \sqrt{(B/z) - (c_t^2/c_f^2)}} \right) - \frac{1}{\eta^2} \right]^{1/2} dz = \pi n \theta, \quad (3.10)$$

where $B = \sqrt{3}c_l / \sqrt{c_l^2 - c_t^2}$.

Note that the above described geometrical acoustics approach also allows calculation of the wedge wave energy loss factor 2γ due to the radiation of sound into liquid, so that the wavenumber of attenuated wedge waves can be written in the form $\beta' = \beta(1+i\gamma) = (\omega/c)[1+i\gamma]$. To determine the attenuation coefficient γ one should integrate the term $\operatorname{Im}[k(x)]$ over the ray trajectory for the corresponding wedge mode taken between any two coordinates y_1 and y_2 characterizing the points of ray reflection from the wedge tip and to relate the result to the corresponding distance along y-axis: $y_2 - y_1$. However, calculation of γ is out of the scope of this paper.

4. Numerical Results and Their Comparison with Experiments

To demonstrate that waves in immersed wedges can propagate at very low velocities the numerical calculations of Eqs. (3.4) or (3.7) have been performed for wedges made of rubber ($\rho_s = 930 \text{ kg/m}^3$, $\rho_f = 1000 \text{ kg/m}^3$, $c_f = 1478 \text{ m/s}$, $c_l = 1040 \text{ m/s}$, $c_t = 30 \text{ m/s}$, $\sigma = 0.499$; these yields $A = 1.452$). The calculated values of η_1 and η_2 corresponding to the two lowest order flexural wedge modes ($n = 1, 2$) as functions of the wedge angle Θ are shown in Fig. 2. Straight lines represent the corresponding solutions $H1$ and $H2$ for wedges in vacuum described by Eq. (2.9). One can see that the influence of liquid results in noticeable decrease of wedge wave velocities in comparison with their values in vacuum. This decrease is a nonlinear function of a wedge apex angle θ and is more pronounced for small values of θ . For example, for the lowest order wedge mode propagating in a sharp elastic wedge ($\theta = 0.1 \text{ rad}$) embedded in water, the wedge wave velocity is more than 5 times smaller than for the same wedge in vacuum. This implies that for wedges made of rubber-like materials, the wave velocities may be as low as few meters per second. Wedge waves with such low propagation velocities can be suitable for use as a new source of in-water propulsion¹⁰.

The values of η_1 for Plexiglas wedges in water ($\rho_s = 1180 \text{ kg/m}^3$, $\rho_f = 1000 \text{ kg/m}^3$, $c_f = 1478 \text{ m/s}$, $c_l = 2732 \text{ m/s}$, $c_t = 1363 \text{ m/s}$, $\sigma = 0.334$; these yields $A = 1.466$) have been calculated according to Eqs. (3.4) or (3.7) for small apex angles. The results are shown in Fig. 3 together with the corresponding experimental data obtained in Ref. 11. It is seen that the agreement of the theory with the experiment is very good (note that earlier a similar comparison has been made in Ref. 11, where the author solved our Eq. (3.4) himself and presented the results in terms of absolute values of wedge wave velocity).

Although the above described geometrical acoustics theory has been developed for slender elastic wedges, it can be used successfully for description of the effect of liquid loading also for wedges with large values of the wedge apex angle. In this case one should apply it for calculating relative values of wedge wave velocity, in comparison with those for wedges in vacuum. Following Ref. 12 presenting its experimental results as the ratio c_{wat}/c_{vac} between the velocities of the first order localized modes in immersed Plexiglas wedges and in the same wedges in vacuum, we give the solution of Eqs. (3.4) or (3.7) in the same form for wedges with apex angles varying from 20 to 90 degrees. The results are shown in Fig.4 (solid curve) together with the corresponding experimental points¹². It is clearly seen that the agreement between the theory and the experiment is remarkably good, although for values of the wedge apex angle larger than 30 degrees the geometrical acoustics theory is not expected to be accurate^{6,7}. The reason for such a good agreement is that, because of the presentation of the results in terms of relative values of wedge wave velocity (in comparison with wedges in vacuum), the corresponding systematic errors caused by the limits of applicability of thin plate theories to plates of relatively high local thickness typical for large apex angles Θ are the same for immersed wedges and for wedges in vacuum. Therefore, they cancel each other.

For comparison with the experiments for brass wedges in water ($\rho_s = 8600 \text{ kg/m}^3$, $\rho_f = 1000 \text{ kg/m}^3$, $c_f = 1478 \text{ m/s}$, $c_l = 4350 \text{ m/s}$, $c_t = 2127 \text{ m/s}$, $\sigma = 0.343$; these yields $B = 1.986$), it was Eq. (3.10) that was numerically solved in the similar way to the above (if not mention the introduction of small artificial damping under square root in the second term of Eq. (3.10) - to

avoid singularity when the value of local flexural wave velocity becomes equal to the velocity of sound in liquid). The results are shown in Fig. 5 as the ratio c_{wat}/c_{vac} between the velocities of the first order wedge mode in immersed wedges and in the same wedges in vacuum (solid curve).

Note that a small local minimum around $\theta = 47$ degrees corresponding to the damped singularity indicates the values of the wedge apex angle where the above theory is not accurate. The corresponding experimental results¹² are displayed in the same picture. One can see that the agreement between the theory and the experiment is good enough for wedge angles from 20 to 90 degrees used in the experiment.

5. Conclusions

It has been demonstrated that the approximate analytical theory of localized vibration modes in immersed solid wedges based on geometrical acoustics approach provides clear understanding of the phenomena involved and gives results which are in good agreement with the existing experiments for both subsonic and supersonic regimes of wave propagation.

In a subsonic regime of wave propagation, i.e., for wedge modes being slower than sound in liquid, the influence of liquid loading results in significant decrease in wedge wave velocities in comparison with their values in vacuum. This decrease is a nonlinear function of a wedge apex angle θ and is more pronounced for small values of θ . In a supersonic regime of wedge wave propagation, a smaller decrease in velocities takes place and the waves travel with the attenuation due to radiation of sound into surrounding liquid.

In particular, it has been shown that for relative values of wedge wave velocity, as compared with those for wedges in vacuum, the theory provides good quantitative agreement with the experiments on Plexiglas samples for all values of the wedge apex angle. A generalization of the theory to describe the effect of heavier wedge material and a supersonic regime of wave propagation explains the existing wedge wave velocity measurements for immersed brass wedges.

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Figure captions

Fig. 1 Antisymmetric wedge waves.

Fig. 2 Relative velocities of two wedge modes for rubber wedges in water (η_1, η_2) and in vacuum (H_1, H_2).

Fig. 3 Relative velocity η_1 of the first order mode for slender Plexiglas wedges in water as a function of the apex angle (solid curve); points indicate the corresponding experimental results¹¹.

Fig. 4 Theoretically calculated ratio c_{wat}/c_{vac} for Plexiglas wedges in water as a function of the apex angle (solid curve); points indicate the corresponding experimental results¹².

Fig. 5 Theoretically calculated ratio c_{wat}/c_{vac} for brass wedges in water as a function of the apex angle (solid curve); points indicate the corresponding experimental results¹².

Figures

Fig. 1 Antisymmetric wedge waves.

Fig. 2 Relative velocities of two wedge modes for rubber wedges in water (η_1, η_2) and in vacuum (H_1, H_2).

Fig. 3 Relative velocity ηl of the first order mode for slender Plexiglas wedges in water as a function of the apex angle (solid curve); points indicate the corresponding experimental results¹¹.

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