Predictive Ratio Cusum (PRC): A Bayesian Approach in Online Change Point Detection of Short Runs

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Supplementary Material

Appendix A: Technical details regarding the derivation of the log ratio of the predictive OOC over IC models, $log(L_{n+1})$, for all PRC scenarios presented in Table 1, is available as supplementary material.

A1: PRC for the rate of a Poisson likelihood.

Assume $X_i|\theta \sim P(\theta \cdot s_i)$, where s_i is the known number of events for the i^{th} observation, while for the rate (per event) unknown parameter we assume $\theta \sim G(c, d)$. Then, the resulting IC posterior is $\theta|\boldsymbol{\tau}_n \sim G\left(\hat{c}_n, \hat{d}_n\right)$, while the corresponding predictive is $f\left(X_{n+1}|\boldsymbol{X}_n\right) = NBin\left(\hat{c}_n, s_{n+1}/\left(\hat{d}_n + s_{n+1}\right)\right)$, where $\hat{c}_n = c + \sum_{j=1}^{N_D} w_j d_j$ and $\hat{d}_n = d + \sum_{j=1}^{N_D} w_j s_j$. Thus, the vector of IC posterior parameters, the predictive's sufficient statistic and $K\left(\boldsymbol{\tau_n}\right)$, needed in PRC are

$$\boldsymbol{\tau}_n = \left(\hat{d}_n, \hat{c}_n - 1\right), \ \boldsymbol{t}_f(X_{n+1}) = (s_{n+1}, x_{n+1}) \quad \text{and} \quad K\left(\boldsymbol{\tau_n}\right) = \frac{\Gamma(\tau_{n,1}+1)}{\left(\tau_{n,0}\right)^{\tau_{n,1}+1}}$$

For the OOC scenario we introduce the shift to the unknown rate parameter θ by multiplying it by k (i.e. the OOC parameter is $k \cdot \theta$), which corresponds to a $(k-1)\cdot 100\%$ rate increase if $k > 1$, or to a $(1-k)\cdot 100\%$ decrease when $k < 1$. Since Gamma is a scale family it follows that the OOC posterior will be $\theta|\boldsymbol{\tau}_n' \sim G\left(\hat{c}_n, \hat{d}_n/k\right)$, resulting the predictive $f'\left(X_{n+1}|\boldsymbol{X}_n\right) = NBin\left(\hat{c}_n, s_{n+1}/\left(\hat{d}_n/k + s_{n+1}\right)\right)$. Therefore, the vector of intervened posterior parameters will be $\boldsymbol{\tau}_n' = \left(\hat{d}_n/k, \hat{c}_n - 1\right)$. Finally, the score function $log\left(L_{n+1}\right)$ will be given by

$$
\begin{aligned}
log\left(L_{n+1}\right) &= log\frac{f'\left(X_{n+1}|\boldsymbol{X}_n\right)}{f\left(X_{n+1}|\boldsymbol{X}_n\right)} = log\frac{K\left(\boldsymbol{\tau}_n' + \boldsymbol{t}_f(X_{n+1})\right)\cdot K\left(\boldsymbol{\tau}_n\right)}{K\left(\boldsymbol{\tau}_n + \boldsymbol{t}_f(X_{n+1})\right)\cdot K\left(\boldsymbol{\tau}_n'\right)} \\
&= log\frac{\dfrac{\Gamma\left(\hat{c}_n + x_{n+1}\right)}{\left(\hat{d}_n/k + s_{n+1}\right)^{\hat{c}_n + x_{n+1}}}\cdot\dfrac{\Gamma\left(\hat{c}_n\right)}{\hat{d}_n^{\hat{c}_n}}}{\dfrac{\Gamma\left(\hat{c}_n + x_{n+1}\right)}{\left(\hat{d}_n + s_{n+1}\right)^{\hat{c}_n + x_{n+1}}}\cdot\dfrac{\Gamma\left(\hat{c}_n\right)}{\left(\hat{d}_n/k\right)^{\hat{c}_n}}} \\
&= \left(\hat{c}_n + x_{n+1}\right) log\frac{\hat{d}_n + s_{n+1}}{\hat{d}_n/k + s_{n+1}} - \hat{c}_n \cdot logk
\end{aligned}
$$

A2: PRC for the probability of success of a Binomial likelihood.

Let $X_i|\theta \sim Bin(N_i, \theta)$, where N_i is the known number of Bernoulli trials of the i^{th} observation and for the unknown success probability we assume $\theta \sim Beta\left(a, b\right)$. The IC posterior is $\theta|\boldsymbol{\tau}_n \sim Beta\left(\hat{a}_n, \hat{b}_n\right)$, while the predictive is $f\left(X_{n+1}|\boldsymbol{X}_n\right) = BetaBin\left(\hat{a}_n,\ \hat{b}_n,\ N_{n+1}\right)$, where $\hat{a}_n = a + \sum_{j=1}^{N_D} w_j d_j$ and $\hat{b}_n = b + \sum_{j=1}^{N_D} w_j N_j - \sum_{j=1}^{N_D} w_j d_j$. Thus, the vector of IC posterior parameters, the predictive's sufficient statistic and $K\left(\boldsymbol{\tau_n}\right)$, needed in PRC are

$$
\boldsymbol{\tau}_n = \left(\frac{\hat{a}_n + \hat{b}_n - 2}{\sum_{i=1}^{n} N_i}, \hat{a}_n - 1\right), \quad \boldsymbol{t}_f(x_{n+1}) = \left(N_{n+1}, x_{n+1}\right) \quad \text{and}
$$

$$
K\left(\boldsymbol{\tau_n}\right) = \frac{\Gamma(\tau_{n,1} + 1)\Gamma(N_i \cdot \tau_{n,0} - \tau_{n,1} + 1)}{\Gamma\left(\sum_{i=1}^{n} N_i \cdot \tau_{n,0} + 2\right)}
$$

For the OOC scenario we multiply the expected odds of θ by k, i.e. the OOC shift is $k \cdot E\left(\frac{\theta}{1-\theta}\right)$. This shift corresponds to a $(k-1)\cdot 100\%$ expected odds increase if $k > 1$, or to a $(1-k)\cdot 100\%$ decrease when $k < 1$. The OOC posterior will be $\theta|\boldsymbol{\tau}_n' \sim Beta\left(k \cdot \hat{a}_n, \hat{b}_n\right)$ and the corresponding predictive $f'\left(X_{n+1}|\boldsymbol{X}_n\right) = BetaBin\left(k \cdot \hat{a}_n, \hat{b}_n, N_{n+1}\right)$. Therefore, the vector of the intervened posterior parameters will be $\boldsymbol{\tau}'_n = \left(\frac{k \cdot \hat{a}_n + \hat{b}_n - 2}{\sum_{i=1}^{n} N_i}, k \cdot \hat{a}_n - 1 \right)$.

The score function $log\left(L_{n+1}\right)$ will be

$$\begin{aligned} log\left(L_{n+1}\right) &= log \frac{\dfrac{\Gamma\left(k \cdot \hat{a}_n + x_{n+1}\right) \Gamma\left(\hat{b}_n + N_{n+1} - x_{n+1}\right)}{\Gamma\left(k \cdot \hat{a}_n + \hat{b}_n + N_{n+1}\right)} \cdot \dfrac{\Gamma\left(\hat{a}_n\right) \Gamma\left(\hat{b}_n\right)}{\Gamma\left(\hat{a}_n + \hat{b}_n\right)}}{\dfrac{\Gamma\left(\hat{a}_n + x_{n+1}\right) \Gamma\left(\hat{b}_n + N_{n+1} - x_{n+1}\right)}{\Gamma\left(\hat{a}_n + \hat{b}_n + N_{n+1}\right)} \cdot \dfrac{\Gamma\left(k \cdot \hat{a}_n\right) \Gamma\left(\hat{b}_n\right)}{\Gamma\left(k \cdot \hat{a}_n + \hat{b}_n\right)}} \\ &= log \frac{B\left(k \cdot \hat{a}_n + \hat{b}_n, N_{n+1}\right) \cdot B\left(\hat{a}_n, x_{n+1}\right)}{B\left(\hat{a}_n + \hat{b}_n, N_{n+1}\right) \cdot B\left(k \cdot \hat{a}_n, x_{n+1}\right)} \end{aligned}$$

A3: PRC for the probability of success of a Negative Binomial likelihood.

Let $X_i|\theta \sim NBin(r, \theta)$, where r represents the known number of failures until the experiment stops and for the unknown probability of success we assume $\theta \sim Beta\left(a, b\right)$. The IC posterior and predictive will be $\theta|\boldsymbol{\tau}_n \sim Beta\left(\hat{a}_n, \hat{b}_n\right)$ and $f\left(X_{n+1}|\boldsymbol{X}_n\right) = NBetaBin\left(\hat{a}_n, \hat{b}_n, r\right)$ respectively, where $\hat{a}_n = a + r \sum_{j=1}^{N_D} w_j$ and $\hat{b}_n = b + \sum_{j=1}^{N_D} w_j d_j$. Thus, the vector of IC posterior parameters, the predictive's sufficient statistic and $K\left(\boldsymbol{\tau_n}\right)$, needed in PRC are

$$\boldsymbol{\tau}_n = \left(\frac{\hat{a}_n - 1}{r}, \hat{b}_n - 1 \right), \;\; \boldsymbol{t}_f(x_{n+1}) = (1, x_{n+1}) \quad \text{and} \quad K\left(\boldsymbol{\tau_n}\right) = \frac{\Gamma(\tau_{n,1} + 1)\Gamma(r \cdot \tau_{n,0} + 1)}{\Gamma(r \cdot \tau_{n,0} + \tau_{n,1} + 2)}$$

As in the Binomial case, for the OOC scenario we multiply the expected odds of θ by k. This shift represents a $(k-1) \cdot 100\%$ expected odds increase if $k > 1$, or a $(1-k) \cdot 100\%$ decrease when $k < 1$. The OOC posterior is $\theta|\boldsymbol{\tau}'_n \sim Beta\left(k \cdot \hat{a}_n, \hat{b}_n\right)$ and the corresponding predictive $f'\left(X_{n+1}|\boldsymbol{X}_n\right) = NBetaBin\left(k \cdot \hat{a}_n, \hat{b}_n, r\right)$. The intervened posterior parameters are $\boldsymbol{\tau}_n' = \left(\frac{k \cdot \hat{a}_n - 1}{r}, \hat{b}_n - 1\right)$. Finally, the score function $log\,(L_{n+1})$ will be given by

$$log\,(L_{n+1}) \;=\; log\frac{B\left(k \cdot \hat{a}_n + \hat{b}_n, r + x_{n+1}\right) \cdot B\,(\hat{a}_n, r)}{B\left(\hat{a}_n + \hat{b}_n, r + x_{n+1}\right) \cdot B\,(k \cdot \hat{a}_n, r)}$$

A4: PRC for the mean of a Normal likelihood with known variance.

Let $X_i|\theta \sim N\,(\theta, \sigma^2)$, where σ^2 is the known variance, and for the unknown mean parameter we assume $\theta \sim N\,(\mu_0, \sigma_0^2)$. The IC posterior and predictive will be $\theta|\boldsymbol{\tau}_n \sim N\,(\hat{\mu}_n, \hat{\sigma}_n^2)$ and $f\,(X_{n+1}|\boldsymbol{X}_n) = N\,(\hat{\mu}_n, \hat{\sigma}_n^2 + \sigma^2)$ respectively, where
$\hat{\mu}_n = \left(\sigma^2\mu_0 + {\sigma_0}^2 \sum\limits_{j=1}^{N_D} w_j d_j\right) \Big/ \left(\sigma^2 + {\sigma_0}^2 \sum\limits_{j=1}^{N_D} w_j\right)$ and $\hat{\sigma}_n^2 = {\sigma_0}^2\sigma^2 \Big/ \left(\sigma^2 + {\sigma_0}^2 \sum\limits_{j=1}^{N_D} w_j\right)$.
The vector of IC posterior parameters, the predictive's sufficient statistic and $K\,(\boldsymbol{\tau_n})$, needed in PRC are

$$\boldsymbol{\tau}_n = \left(\frac{\sigma^2}{\hat{\sigma}_n^2}, \frac{\hat{\mu}_n}{\hat{\sigma}_n^2}\right), \;\; \boldsymbol{t}_1\,(x_{n+1}) = \left(1, \frac{x_{n+1}}{\sigma^2}\right) \quad \text{and} \quad K\,(\boldsymbol{\tau_n}) = \sqrt{\frac{2\pi\sigma^2}{\tau_{n,0}}} exp\left\{\frac{\sigma^2\tau_{n,1}^2}{2\tau_{n,0}}\right\}$$

For the OOC shift, we introduce a step change of size $k \cdot \sigma$ on the mean, i.e. the OOC mean is $\theta + k \cdot \sigma$ and the mean shift is upward or downward, depending on whether $k > 0$ or $k < 0$ respectively. Since Normal is a location family, the OOC posterior is $\theta|\boldsymbol{\tau}_n' \sim N\,(\hat{\mu}_n + k \cdot \sigma, \hat{\sigma}_n^2)$ and the corresponding OOC predictive will be $f'\,(X_{n+1}|\boldsymbol{X}_n) = N\,(\hat{\mu}_n + k \cdot \sigma, \hat{\sigma}_n^2 + \sigma^2)$. The vector of the intervened posterior parameters is $\boldsymbol{\tau}_n' = \left(\frac{\sigma^2}{\hat{\sigma}_n^2}, \frac{\hat{\mu}_n + k \cdot \sigma}{\hat{\sigma}_n^2}\right)$. If we will standardize the future observable, setting $Z_{n+1} = (X_{n+1} - \hat{\mu}_n) / \sqrt{\hat{\sigma}_n^2 + \sigma^2}$, then the standardized predictives will be $f\,(Z_{n+1}|\boldsymbol{X}_n) = N\,(0, 1)$ and $f'\,(Z_{n+1}|\boldsymbol{X}_n) = N\left(k \cdot \sigma / \sqrt{\hat{\sigma}_n^2 + \sigma^2}, 1\right)$. The score function $log\,(L_{n+1})$ will be given by

$$\begin{aligned} log\,(L_{n+1}) \;&=\; log\frac{f'\,(X_{n+1}|\boldsymbol{X}_n)}{f\,(X_{n+1}|\boldsymbol{X}_n)} = log\frac{f'\,(Z_{n+1}|\boldsymbol{X}_n)}{f\,(Z_{n+1}|\boldsymbol{X}_n)} \\ &=\; \left(z_{n+1} - \frac{k}{2} \cdot \frac{\sigma}{\sqrt{\hat{\sigma}_n^2 + \sigma^2}}\right) \cdot \frac{k \cdot \sigma}{\sqrt{\hat{\sigma}_n^2 + \sigma^2}} \end{aligned}$$

A5: PRC for the variance of a Normal likelihood with known mean.

Let $X_i|\theta^2 \sim N(\mu, \theta^2)$, where μ is the known mean, and for the unknown variance parameter we assume $\theta^2 \sim IG(a, b)$. The IC posterior and predictive distributions will be $\theta^2|\boldsymbol{\tau}_n \sim IG\left(\hat{a}_n, \hat{b}_n\right)$ and $f(X_{n+1}|\boldsymbol{X}_n) = t_{2\hat{a}_n}\left(\mu, \hat{b}_n/\hat{a}_n\right)$ respectively, where $\hat{a}_n = a + \sum_{j=1}^{N_D} w_j/2$ and $\hat{b}_n = b + \sum_{j=1}^{N_D} w_j (d_j - \mu)^2/2$. The vector of IC posterior parameters, the predictive's sufficient statistic and $K(\boldsymbol{\tau_n})$, needed in PRC are

$$\boldsymbol{\tau}_n = \left(2(\hat{a}_n + 1), 2\hat{b}_n\right), \;\; \boldsymbol{t}_1(x_{n+1}) = \left(1, (x_{n+1} - \mu)^2\right) \quad \text{and} \quad K(\boldsymbol{\tau_n}) = \frac{\Gamma\left(\frac{\tau_{n,0}}{2} - 1\right)}{\left(\frac{\tau_{n,1}}{2}\right)^{\frac{\tau_{n,0}}{2} - 1}}$$

For the OOC shift, we multiply the variance by k, i.e. the OOC parameter is $k \cdot \theta^2$ and this shift corresponds to a $(k-1) \cdot 100\%$ variance increase if $k > 1$ or a $(1-k) \cdot 100\%$ decrease if $k < 1$. Since the Inverse Gamma is a scale family, the OOC posterior will be $\theta|\boldsymbol{\tau}'_n \sim IG\left(\hat{a}_n, k \cdot \hat{b}_n\right)$ with the corresponding predictive being $f'(X_{n+1}|\boldsymbol{X}_n) = t_{2\hat{a}_n}\left(\mu, k \cdot \hat{b}_n/\hat{a}_n\right)$. Thus, the intervened parameters are given by $\boldsymbol{\tau}'_n = \left(2(\hat{a}_n + 1), k \cdot 2\hat{b}_n\right)$. Standardizing the future observable we have $Z_{n+1} = (X_{n+1} - \hat{\mu}_n)/\sqrt{\hat{b}_n/\hat{a}_n}$, resulting the IC and OOC predictive distributions to be $f(Z_{n+1}|\boldsymbol{X}_n) = t_{2\hat{a}_n}(0, 1)$ and $f'(Z_{n+1}|\boldsymbol{X}_n) = t_{2\hat{a}_n}(0, k)$ respectively. Finally, the score function will be

$$log(L_{n+1}) = (\hat{a}_n + 1/2) \cdot log\frac{2\hat{a}_n + z_{n+1}^2}{2\hat{a}_n + z_{n+1}^2/k} - log\sqrt{k}$$

A6: PRC for the mean of a Normal likelihood with both parameters unknown.

Let $X_i|(\theta_1, \theta_2^2) \sim N(\theta_1, \theta_2^2)$ with both parameters being unknown and assumed $(\theta_1, \theta_2^2) \sim NIG(\mu_0, \lambda, a, b)$. The IC posterior and predictive distributions will be given by $(\theta_1, \theta_2^2)|\boldsymbol{\tau}_n \sim NIG\left(\hat{\mu}_n, \hat{\lambda}_n, \hat{a}_n, \hat{b}_n\right)$ and $f(X_{n+1}|\boldsymbol{X}_n) = t_{2\hat{a}_n}\left(\hat{\mu}_n, (\lambda_n + 1) \cdot \hat{b}_n/(\lambda_n \cdot \hat{a}_n)\right)$ respectively, where $\hat{\mu}_n = \left(\lambda\mu_0 + \sum_{j=1}^{N_D} w_j d_j\right) \Big/ \left(\lambda + \sum_{j=1}^{N_D} w_j\right)$, $\hat{\lambda}_n = \lambda + \sum_{j=1}^{N_D} w_j$, $\hat{a}_n = a + \sum_{j=1}^{N_D} w_j/2$

and $\hat{b}_n = b + \left(\lambda\mu_0^2 + \sum_{j=1}^{N_D} w_j d_j^2\right) \Big/ 2 - \left(\lambda\mu_0 + \sum_{j=1}^{N_D} w_j d_j\right)^2 \Big/ \left(2\left(\lambda + \sum_{j=1}^{N_D} w_j\right)\right)$. The vector of IC posterior parameters, the predictive's sufficient statistic and $K(\boldsymbol{\tau_n})$, needed in PRC are

$$\boldsymbol{\tau}_n = \left(2(\hat{a}_n + 1), 2\hat{b}_n + \hat{\lambda}_n\hat{\mu}_n^2, \hat{\lambda}_n\hat{\mu}_n, \hat{\lambda}_n\right), \quad \boldsymbol{t}_1(x_{n+1}) = \left(1, x_{n+1}^2, x_{n+1}, 1\right) \quad \text{and}$$

$$K(\boldsymbol{\tau_n}) = \frac{\sqrt{2\pi}}{\tau_{n,3}} \cdot \frac{\Gamma\left(\frac{\tau_{n,0} - 3}{2}\right)}{\left(\frac{\tau_{n,1}}{2} - \frac{\tau_{n,2}^2}{2\tau_{n,3}}\right)^{\frac{\tau_{n,0} - 3}{2}}}$$

For the OOC shift, we introduce a step change of size $k \cdot \hat{\theta}_2$ to the mean (i.e. the OOC parameter will be $\theta_1 + k\hat{\theta}_2$), where $\hat{\theta}_2 = \sqrt{\hat{b}_n/\hat{a}_n}$ (the shift will be upward or downward, depending on whether $k > 0$ or $k < 0$ respectively). The parameter $\hat{\theta}_2$ is the mean of the posterior marginal for the standard deviation θ_2. This choice preserves the conjugacy and expresses the shift in terms of the estimated standard deviation. Furthermore, it is always well defined when the predictive is available and it allows the pivotal statistic to depend only on $\hat{\lambda}_n$. Given that the posterior marginal Student t is a location family, the OOC posterior is $(\theta_1, \theta_2^2)\,|\boldsymbol{\tau}_n' \sim NIG\left(\hat{\mu}_n + k \cdot \hat{\theta}_2, \hat{\lambda}_n, \hat{a}_n, \hat{b}_n\right)$, while the corresponding predictive and the intervened posterior parameters are
$f'(X_{n+1}|\boldsymbol{X}_n) = t_{2\hat{a}_n}\left(\hat{\mu}_n + k \cdot \hat{\theta}_2, \left(\hat{\lambda}_n + 1\right) \cdot \hat{b}_n / \left(\hat{\lambda}_n \cdot \hat{a}_n\right)\right)$ and
$\boldsymbol{\tau}_n' = \left(2(\hat{a}_n + 1), 2\hat{b}_n + \hat{\lambda}_n\hat{\mu}_n^2, \hat{\lambda}_n(\hat{\mu}_n + k \cdot \hat{\theta}_2), \hat{\lambda}_n\right)$ respectively. Standardizing the future observable (using the IC parameters) we get $Z_{n+1} = (X_{n+1} - \hat{\mu}_n) / \sqrt{\left(\hat{\lambda}_n + 1\right) \cdot \hat{b}_n / \left(\hat{\lambda}_n \cdot \hat{a}_n\right)}$. Then the IC and OOC predictive will be $f(Z_{n+1}|\boldsymbol{X}_n) = t_{2\hat{a}_n}(0, 1)$ and $f'(Z_{n+1}|\boldsymbol{X}_n) = t_{2\hat{a}_n}\left(k \cdot \sqrt{\hat{\lambda}_n / \left(\hat{\lambda}_n + 1\right)}, 1\right)$ respectively. The score function $log(L_{n+1})$ will be given by

$$log(L_{n+1}) \quad = \quad (\hat{a}_n + 1/2) \cdot log\frac{2\hat{a}_n + z_{n+1}^2}{2\hat{a}_n + \left(z_{n+1} - k \cdot \hat{\lambda}_n/(\hat{\lambda}_n + 1)\right)^2}$$

A7: PRC for the variance of a Normal likelihood with both parameters unknown.

In this scenario, the likelihood, prior and the IC posterior distributions are identical to the ones of scenario **A6**. However, here we consider the PRC for the variance term and so for the OOC shift, we multiply the variance by k, i.e. $k \cdot \theta_2^2$. The shift corresponds to a $(k-1) \cdot 100\%$ variance increase if $k > 1$ or a $(1-k) \cdot 100\%$ decrease when $k < 1$. Furthermore, as the posterior marginal of θ_2^2 is Inverse Gamma, i.e. a scale family, the OOC posterior will be given by $(\theta_1, \theta_2^2) \, |\boldsymbol{\tau}_n' \sim NIG\left(\hat{\mu}_n, \hat{\lambda}_n, \hat{a}_n, k \cdot \hat{b}_n\right)$, while the corresponding predictive will be $f'\left(X_{n+1}|\boldsymbol{X}_n\right) = t_{2\hat{a}_n}\left(\hat{\mu}_n, k \cdot \left(\hat{\lambda}_n + 1\right) \cdot \hat{b}_n / \left(\hat{\lambda}_n \cdot \hat{a}_n\right)\right)$. Thus the vector of the intervened posterior parameters will be $\boldsymbol{\tau}_n' = \left(2(\hat{a}_n+1), 2k \cdot \hat{b}_n + \hat{\lambda}_n \hat{\mu}_n^2, \hat{\lambda}_n \hat{\mu}_n, \hat{\lambda}_n\right)$. Standardizing the future observable (just as in **A6**) we get the standardized IC and OOC predictive distributions to be $f\left(Z_{n+1}|\boldsymbol{X}_n\right) = t_{2\hat{a}_n}(0,1)$ and $f'\left(Z_{n+1}|\boldsymbol{X}_n\right) = t_{2\hat{a}_n}(0,k)$ respectively. Finally, the score function $log\left(L_{n+1}\right)$ will be

$$log\left(L_{n+1}\right) = (\hat{a}_n + 1/2) \cdot log\frac{2\hat{a}_n + z_{n+1}^2}{2\hat{a}_n + z_{n+1}^2/k} - log\sqrt{k}$$

A8: PRC for the rate of a Gamma likelihood.

Let $X_i|\theta \sim G(\alpha, \theta)$, where α is the known shape parameter, and for the unknown rate parameter we assume that $\theta \sim G(c,d)$. Then, the resulting IC posterior and predictive will be $\theta|\boldsymbol{\tau}_n \sim G\left(\hat{c}_n, \hat{d}_n\right)$ and $f\left(X_{n+1}|\boldsymbol{X}_n\right) = CompG\left(\alpha, \hat{c}_n, \hat{d}_n\right)$ (i.e. Compound Gamma) respectively, where $\hat{c}_n = c + \alpha \sum_{j=1}^{N_D} w_j$ and $\hat{d}_n = d + \sum_{j=1}^{N_D} w_j d_j$. Therefore, the vector of IC posterior parameters, the predictive's sufficient statistic and $K\left(\boldsymbol{\tau_n}\right)$, needed in PRC are

$$\boldsymbol{\tau}_n = \left(\frac{\hat{c}_n - 1}{\alpha}, \hat{d}_n\right), \;\; \boldsymbol{t}_f(x_{n+1}) = (1, x_{n+1}) \quad \text{and} \quad K\left(\boldsymbol{\tau_n}\right) = \frac{\Gamma(\alpha\tau_{n,0}+1)}{\tau_{n,1}^{\alpha\tau_{n,0}+1}}$$

Just as in the Poisson case, the OOC scenario is introduced as a shift to the rate θ parameter, by multiplying it by k, representing a $(k-1) \cdot 100\%$ rate increase if $k > 1$ or a $(1-k) \cdot 100\%$ decrease when $k < 1$. As Gamma is a scale family it follows that the OOC posterior will be $\theta|\boldsymbol{\tau}_n' \sim G\left(\hat{c}_n, \hat{d}_n/k\right)$, and the corresponding predictive will be $f'\left(X_{n+1}|\boldsymbol{X}_n\right) = CompG\left(\alpha, \hat{c}_n, \hat{d}_n/k\right)$. Therefore, the vector of intervened posterior parameters will be $\boldsymbol{\tau}_n' = \left(\frac{\hat{c}_n - 1}{\alpha}, \frac{\hat{d}_n}{k}\right)$. Finally, the score function $log\left(L_{n+1}\right)$ will be given by

$$log\left(L_{n+1}\right) \;=\; \left(\hat{c}_n + \alpha\right) \cdot log \frac{\hat{d}_n + x_{n+1}}{\hat{d}_n/k + x_{n+1}} - \hat{c}_n \cdot logk$$

A9: PRC for the scale of a Weibull likelihood.

If $X_i|\theta \sim W(\theta, \kappa)$, where κ is the known shape parameter, and for the unknown scale parameter we assume $\theta \sim IG\left(a, b\right)$. The IC posterior and predictive distributions will be $\theta|\boldsymbol{\tau}_n \sim IG\left(\hat{a}_n, \hat{b}_n\right)$ and $f\left(X_{n+1}|\boldsymbol{X}_n\right) = Burr\left(\kappa, \hat{a}_n, \hat{b}_n^{1/\kappa}\right)$ respectively, where $\hat{a}_n = a + \sum_{j=1}^{N_D} w_j$ and $\hat{b}_n = b + \sum_{j=1}^{N_D} w_j d_j^{\kappa}$. Thus, the vector of IC posterior parameters, the predictive's sufficient statistic and $K\left(\boldsymbol{\tau_n}\right)$, needed in PRC are

$$\boldsymbol{\tau}_n = \left(\hat{a}_n + 1, \hat{b}_n\right), \;\; \boldsymbol{t}_f(x_{n+1}) = \left(1, x_{n+1}^{\kappa}\right) \quad \text{and} \quad K\left(\boldsymbol{\tau_n}\right) = \frac{\Gamma(\tau_{n,0} - 1)}{\tau_{n,1}^{\tau_{n,0}-1}}$$

Similarly to scenario **A5**, we introduce the OOC shift by multiplying the scale parameter θ^{κ} by k. The shift corresponds to a $(k-1)\cdot 100\%$ scale increase if $k > 1$ or a $(1-k)\cdot 100\%$ decrease when $k < 1$. The Inverse Gamma is a scale family, thus the OOC posterior will be $\theta^{\kappa}|\boldsymbol{\tau}_n' \sim IG\left(\hat{a}_n, k \cdot \hat{b}_n\right)$ and the corresponding OOC predictive will be given by $f'\left(X_{n+1}|\boldsymbol{X}_n\right) = Burr\left(\kappa, \hat{a}_n, \left(k \cdot \hat{b}_n\right)^{1/\kappa}\right)$. Finally, the vector of the intervened posterior parameters is $\boldsymbol{\tau}_n' = \left(\hat{a}_n + 1, k \cdot \hat{b}_n\right)$, while the score function becomes

$$log\left(L_{n+1}\right) \;=\; \left(\hat{a}_n + 1\right) \cdot log \frac{\hat{b}_n + x_{n+1}^{\kappa}}{\hat{b}_n + x_{n+1}^{\kappa}/k} - logk$$

A10: PRC for the scale of an Inverse Gamma likelihood.

Let $X_i|\theta \sim IG(\alpha,\theta)$, where α is the known shape parameter while for the unknown scale parameter we assume $\theta \sim G(c,d)$. The IC posterior is $\theta|\boldsymbol{\tau}_n \sim G\left(\hat{c}_n, \hat{d}_n\right)$, while the resulting predictive is $f(X_{n+1}|\boldsymbol{X}_n) = GB2\left(-1, 1/\hat{d}_n, \alpha, \hat{c}_n\right)$ (i.e. Generalized Beta of the second kind), where $\hat{c}_n = c+\alpha\sum_{j=1}^{N_D} w_j$ and $\hat{d}_n = d+\sum_{j=1}^{N_D} w_j/d_j$. The vector of IC posterior parameters, the predictive's sufficient statistic and $K(\boldsymbol{\tau_n})$, needed in PRC are

$$\boldsymbol{\tau}_n = \left(\frac{\hat{c}_n - 1}{\alpha}, \hat{d}_n\right), \ \ \boldsymbol{t}_f(x_{n+1}) = \left(1, \frac{1}{x_{n+1}}\right) \quad \text{and} \quad K(\boldsymbol{\tau_n}) = \frac{\Gamma(\alpha\tau_{n,0}+1)}{(\tau_{n,1})^{\alpha\tau_{n,0}+1}}$$

Similarly to earlier scenarios, where Gamma was the prior, we introduce the shift to the shape θ by multiplying it by k, which represents a $(k-1)\cdot 100\%$ scale increase if $k>1$ or a $(1-k)\cdot 100\%$ decrease if $k<1$. Gamma is a scale family, thus the OOC posterior will be $\theta|\boldsymbol{\tau}'_n \sim G\left(\hat{c}_n, \hat{d}_n/k\right)$, and the corresponding predictive will be $f'(X_{n+1}|\boldsymbol{X}_n) = GB2\left(-1, k/\hat{d}_n, \alpha, \hat{c}_n\right)$. The intervened posterior parameters will be $\boldsymbol{\tau}'_n = \left(\frac{\hat{c}_n - 1}{\alpha}, \hat{d}_n/k\right)$ and the score function $log(L_{n+1})$ will be given by

$$log(L_{n+1}) \;=\; (\hat{c}_n+\alpha)\cdot log\frac{\hat{d}_n\cdot x_{n+1}+1}{\hat{d}_n\cdot x_{n+1}/k+1} - \hat{c}_n\cdot logk$$

A11: PRC for the shape of Pareto likelihood.

Let $X_i|\theta \sim Pa(\theta,m)$, where m is the known minimum parameter, and for the shape parameter we assume $\theta \sim G(c,d)$. The IC posterior and predictive distribution are $\theta|\boldsymbol{\tau}_n \sim G\left(\hat{c}_n, \hat{d}_n\right)$ and $f(X_{n+1}|\boldsymbol{X}_n) = expGPD\left(\hat{d}_n/(m\cdot\hat{c}_n), \hat{c}_n^{-1}\right)$ (i.e. exponentiated Generalized Pareto Distribution) respectively, where $\hat{c}_n = c+\sum_{j=1}^{N_D} w_j$ and $\hat{d}_n = d+\sum_{j=1}^{N_D} w_j log(d_j/m)$. The vector of IC posterior parameters, the predictive's sufficient statistic and $K(\boldsymbol{\tau_n})$, needed in

PRC are

$$\boldsymbol{\tau}_n = \left(\hat{c}_n - 1, \hat{d}_n\right), \;\; \boldsymbol{t}_f(x_{n+1}) = (1, log(x_{n+1}/m)) \quad \text{and} \quad K\left(\boldsymbol{\tau_n}\right) = \frac{\Gamma(\tau_{n,0}+1)}{\tau_{n,1}^{\tau_{n,0}+1}}$$

Just as it was done in the earlier cases where Gamma was involved as prior, we multiply the shape θ by k, which represents a $(k-1)\cdot 100\%$ shape increase if $k > 1$ or to a $(1-k)\cdot 100\%$ decrease when $k < 1$. As Gamma is a scale family, the OOC posterior $\theta|\boldsymbol{\tau}'_n \sim G\left(\hat{c}_n, \hat{d}_n/k\right)$, and the OOC predictive: $f'\left(X_{n+1}|\boldsymbol{X}_n\right) = expGPD\left(\hat{d}_n/\left(k\cdot m\cdot \hat{c}_n\right), \hat{c}_n^{-1}\right)$. The intervened posterior parameters will be $\boldsymbol{\tau}'_n = \left(\dfrac{\hat{c}_n - 1}{\alpha}, \hat{d}_n/k\right)$ and the score function $log\left(L_{n+1}\right)$ will be given by

$$log\left(L_{n+1}\right) \;\; = \;\; (\hat{c}_n + 1)\cdot log\frac{\hat{d}_n + log(x_{n+1}/m)}{\hat{d}_n/k + log(x_{n+1}/m)} - \hat{c}_n \cdot logk$$

A12: PRC for the scale of Lognormal likelihood with known shape parameter. Let $X_i|\theta \sim LogN\left(\theta, \sigma^2\right)$, where σ^2 is the known shape parameter, and for the scale parameter we assume $\theta \sim N\left(\mu_0, \sigma_0^2\right)$. Similarly to the corresponding Normal case (scenario **A4**) we have that the IC posterior and predictive distributions to be $\theta|\boldsymbol{\tau_n} \sim N\left(\hat{\mu}_n,\ \hat{\sigma}_n^2\right)$ and $f\left(X_{n+1}|\boldsymbol{X}_n\right) = LogN\left(\hat{\mu}_n,\ \hat{\sigma}_n^2 + \sigma^2\right)$ respectively, where $\hat{\mu}_n = \left(\sigma^2\mu_0 + {\sigma_0}^2 \sum\limits_{j=1}^{N_D} w_j log(d_j)\right) \Big/ \left(\sigma^2 + {\sigma_0}^2 \sum\limits_{j=1}^{N_D} w_j\right)$ and $\hat{\sigma}_n^2 = {\sigma_0}^2\sigma^2 \Big/ \left(\sigma^2 + {\sigma_0}^2 \sum\limits_{j=1}^{N_D} w_j\right)$. The vector of IC posterior parameters, the predictive's sufficient statistic and $K\left(\boldsymbol{\tau_n}\right)$, needed in PRC are

$$\boldsymbol{\tau}_n = \left(\frac{\sigma^2}{\hat{\sigma}_n^2}, \frac{\hat{\mu}_n}{\hat{\sigma}_n^2}\right), \;\; \boldsymbol{t}_1\left(x_{n+1}\right) = \left(1, \frac{log(x_{n+1})}{\sigma^2}\right) \quad \text{and} \quad K\left(\boldsymbol{\tau_n}\right) = \sqrt{\frac{2\pi\sigma^2}{\tau_{n,0}}} exp\left\{\frac{\sigma^2\tau_{n,1}^2}{2\tau_{n,0}}\right\}$$

For the OOC shift, we introduce a step change of size of $k\cdot\sigma$ for θ, i.e. the OOC parameter is $\theta + k\cdot\sigma$ with the shift being upwards or downwards depending if $k > 0$ or $k < 0$ respectively. Since the Normal is a location family, the OOC posterior will be $\theta|\boldsymbol{\tau}'_n \sim N\left(\hat{\mu}_n + k\cdot\sigma,\ \hat{\sigma}_n^2\right)$

with the corresponding predictive $f'(X_{n+1}|\boldsymbol{X}_n) = LogN(\hat{\mu}_n + k \cdot \sigma,\ \hat{\sigma}_n^2 + \sigma^2)$. The vector of the intervened posterior parameters will be $\boldsymbol{\tau}_n' = \left(\frac{\sigma^2}{\hat{\sigma}_n^2}, \frac{\hat{\mu}_n + k \cdot \sigma}{\hat{\sigma}_n^2}\right)$. If we will standardize the log-transformed future observable, setting $Z_{n+1} = (log(X_{n+1}) - \hat{\mu}_n) \big/ \sqrt{\hat{\sigma}_n^2 + \sigma^2}$, then the standardized predictives will be $f(Z_{n+1}|\boldsymbol{X}_n) = N(0,1)$ and $f'(Z_{n+1}|\boldsymbol{X}_n) = N\left(k \cdot \sigma / \sqrt{\hat{\sigma}_n^2 + \sigma^2}, 1\right)$. The score function $log(L_{n+1})$ will be given by:

$$log(L_{n+1}) = \left(z_{n+1} - \frac{k}{2} \cdot \frac{\sigma}{\sqrt{\hat{\sigma}_n^2 + \sigma^2}}\right) \cdot \frac{k \cdot \sigma}{\sqrt{\hat{\sigma}_n^2 + \sigma^2}}$$

A13: PRC for the shape of Lognormal likelihood with known scale parameter. Let $X_i|\theta^2 \sim LogN(\mu, \theta^2)$, where μ is the known scale, and for the shape parameter we assume $\theta^2 \sim IG(a,b)$. Similarly to the corresponding Normal case (scenario **A5**) we have that the IC posterior and predictive distributions to be $\theta^2|\boldsymbol{\tau}_n \sim IG\left(\hat{a}_n,\ \hat{b}_n\right)$, and $f(X_{n+1}|\boldsymbol{X}_n) = Logt_{2\hat{a}_n}\left(\mu,\ \hat{b}_n/\hat{a}_n\right)$ respectively, where $\hat{a}_n = a + \sum_{j=1}^{N_D} w_j/2$ and $\hat{b}_n = b + \sum_{j=1}^{N_D} w_j \left(log(d_j) - \mu\right)^2/2$. The vector of IC posterior parameters, the predictive's sufficient statistic and $K(\boldsymbol{\tau_n})$, needed in PRC are

$$\boldsymbol{\tau}_n = \left(2(\hat{a}_n + 1), 2\hat{b}_n\right), \quad \boldsymbol{t}_1(x_{n+1}) = \left(1, (log(x_{n+1}) - \mu)^2\right) \quad \text{and} \quad K(\boldsymbol{\tau_n}) = \frac{\Gamma\left(\frac{\tau_{n,0}}{2} - 1\right)}{\left(\frac{\tau_{n,1}}{2}\right)^{\frac{\tau_{n,0}}{2} - 1}}$$

For the OOC shift, we multiply the shape parameter by k, i.e. $k \cdot \theta^2$. The shift corresponds to a $(k-1) \cdot 100\%$ increase if $k > 1$ or to a $(1-k) \cdot 100\%$ decrease when $k < 1$. Since, Inverse Gamma is a scale family, the OOC posterior and predictive will be $\theta|\boldsymbol{\tau}_n' \sim IG\left(\hat{a}_n,\ k \cdot \hat{b}_n\right)$ and $f'(X_{n+1}|\boldsymbol{X}_n) = Logt_{2\hat{a}_n}\left(\mu,\ k \cdot \hat{b}_n/\hat{a}_n\right)$. The vector of the intervened posterior parameters will be $\boldsymbol{\tau}_n' = \left(2(\hat{a}_n + 1),\ k \cdot 2\hat{b}_n\right)$. Standardizing the log-transformed future observable we have $Z_{n+1} = (log(X_{n+1}) - \mu) \big/ \sqrt{\hat{b}_n/\hat{a}_n}$, resulting the IC and OOC predictive distributions to be $f(Z_{n+1}|\boldsymbol{X}_n) = t_{2\hat{a}_n}(0,1)$ and $f'(Z_{n+1}|\boldsymbol{X}_n) = t_{2\hat{a}_n}(0,k)$ respectively. Finally, the score function will be

$$log\left(L_{n+1}\right) = \left(\hat{a}_n+1/2\right)\cdot log\frac{2\hat{a}_n+z_{n+1}^2}{2\hat{a}_n+z_{n+1}^2/k} - log\sqrt{k}$$

A14: PRC for the scale of Lognormal likelihood with both parameters unknown. Let $X_i|\left(\theta_1,\theta_2^2\right)\sim LogN\left(\theta_1,\theta_2^2\right)$, where both parameters are being unknown and we assume $\left(\theta_1,\theta_2^2\right)\sim NIG\left(\mu_0,\lambda,a,b\right)$. Similarly to the corresponding Normal case (scenario **A6**) we have that the IC posterior and predictive distributions will be $\left(\theta_1,\theta_2^2\right)|\boldsymbol{\tau}_n\sim NIG\left(\hat{\mu}_n,\hat{\lambda}_n,\hat{a}_n,\hat{b}_n\right)$ and $f\left(X_{n+1}|\boldsymbol{X}_n\right)=Logt_{2\hat{a}_n}\left(\hat{\mu}_n,\ (\lambda_n+1)\cdot\hat{b}_n/\left(\lambda_n\cdot\hat{a}_n\right)\right)$ respectively, where $\hat{\mu}_n = \left(\lambda\mu_0+\sum_{j=1}^{N_D}w_j log(d_j)\right)\Big/\left(\lambda+\sum_{j=1}^{N_D}w_j\right)$, $\hat{\lambda}_n = \lambda+\sum_{j=1}^{N_D}w_j$, $\hat{a}_n = a+\sum_{j=1}^{N_D}w_j/2$ and $\hat{b}_n = b+\left(\lambda\mu_0^2+\sum_{j=1}^{N_D}w_j(log(d_j))^2\right)\Big/2-\left(\lambda\mu_0+\sum_{j=1}^{N_D}w_j log(d_j)\right)^2\Big/\left(2\left(\lambda+\sum_{j=1}^{N_D}w_j\right)\right)$.
The vector of IC posterior parameters, the predictive's sufficient statistic and $K\left(\boldsymbol{\tau_n}\right)$, needed in PRC are

$$\boldsymbol{\tau}_n=\left(2(\hat{a}_n+1),2\hat{b}_n+\hat{\lambda}_n\hat{\mu}_n^2,\hat{\lambda}_n\hat{\mu}_n,\hat{\lambda}_n\right),\ \ \boldsymbol{t}_1\left(x_{n+1}\right)=\left(1,(log(x_{n+1}))^2,log(x_{n+1}),1\right)\ \text{and}$$

$$K\left(\boldsymbol{\tau_n}\right)=\frac{\sqrt{2\pi}}{\tau_{n,3}}\cdot\frac{\Gamma\left(\frac{\tau_{n,0}-3}{2}\right)}{\left(\frac{\tau_{n,1}}{2}-\frac{\tau_{n,2}^2}{2\tau_{n,3}}\right)^{\frac{\tau_{n,0}-3}{2}}}$$

For the OOC shift, we introduce a step change of size of $k\cdot\hat{\theta}_2$ to the mean, where $\hat{\theta}_2=\sqrt{\hat{b}_n/\hat{a}_n}$, (i.e the expected value of the posterior marginal for the θ_2) and so the OOC parameter will be $\theta_1+k\hat{\theta}_2$. The shift is upward or downward depending on whether $k>0$ or $k<0$ respectively. As the posterior marginal Student t is a location family, the OOC posterior is $\left(\theta_1,\theta_2^2\right)|\boldsymbol{\tau}_n'\sim NIG\left(\hat{\mu}_n+k\cdot\hat{\theta}_2,\hat{\lambda}_n,\hat{a}_n,\hat{b}_n\right)$, while the corresponding predictive is $f'\left(X_{n+1}|\boldsymbol{X}_n\right)=Logt_{2\hat{a}_n}\left(\hat{\mu}_n+k\cdot\hat{\theta}_2,\ (\lambda_n+1)\cdot\hat{b}_n/\left(\lambda_n\cdot\hat{a}_n\right)\right)$. The vector of intervened posterior parameters is $\boldsymbol{\tau}_n' = \left(2(\hat{a}_n+1), 2\hat{b}_n + \hat{\lambda}_n\hat{\mu}_n^2, \hat{\lambda}_n(\hat{\mu}_n + k\cdot\hat{\theta}_2), \hat{\lambda}_n\right)$. Standardizing the the log-transformed future observable (using the IC parameters) we get $Z_{n+1} = (log(X_{n+1}) - \hat{\mu}_n) \Big/ \sqrt{\left(\hat{\lambda}_n+1\right)\cdot\hat{b}_n / \left(\hat{\lambda}_n\cdot\hat{a}_n\right)}$. Then the IC and OOC predictive will be $f\left(Z_{n+1}|\boldsymbol{X}_n\right) = t_{2\hat{a}_n}(0,1)$ and $f'\left(Z_{n+1}|\boldsymbol{X}_n\right) = t_{2\hat{a}_n}\left(k\cdot\sqrt{\hat{\lambda}_n/\left(\hat{\lambda}_n+1\right)}, 1\right)$ respectively. The score function $log\left(L_{n+1}\right)$ will be given by

$$log\left(L_{n+1}\right) = (\hat{a}_n + 1/2)\cdot log\frac{2\hat{a}_n + z_{n+1}^2}{2\hat{a}_n + \left(z_{n+1} - k\cdot\hat{\lambda}_n/(\hat{\lambda}_n+1)\right)^2}$$

A15: PRC for the shape of Lognormal likelihood with both parameters unknown.

The likelihood and the IC distributions and parameters are identical with the ones presented in scenario **A14**, but for the OOC shift, we multiply the shape parameter θ^2 by k, referring to a $(k-1)\cdot 100\%$ increase if $k>1$ or $(1-k)\cdot 100\%$ decrease when $k<1$. Furthermore, as the posterior marginal (Inverse Gamma) is a scale family, the OOC posterior and predictive will be $(\theta_1, \theta_2^2)\,|\boldsymbol{\tau}_n' \sim NIG\left(\hat{\mu}_n, \hat{\lambda}_n, \hat{a}_n, k\cdot\hat{b}_n\right)$ and $f'\left(X_{n+1}|\boldsymbol{X}_n\right) = Logt_{2\hat{a}_n}\left(\hat{\mu}_n,\ k\cdot(\lambda_n+1)\cdot\hat{b}_n/\left(\lambda_n\cdot\hat{a}_n\right)\right)$ respectively. The intervened posterior parameters will be $\boldsymbol{\tau}_n' = \left(2(\hat{a}_n+1), 2k\cdot\hat{b}_n + \hat{\lambda}_n\hat{\mu}_n^2, \hat{\lambda}_n\hat{\mu}_n, \hat{\lambda}_n\right)$. Standardizing the the log-transformed future observable (just as in **A14**) we get the standardized IC and OOC predictive distributions to be $f\left(Z_{n+1}|\boldsymbol{X}_n\right) = t_{2\hat{a}_n}(0,1)$ and $f'\left(Z_{n+1}|\boldsymbol{X}_n\right) = t_{2\hat{a}_n}(0,k)$ respectively. Finally, the score function $log\left(L_{n+1}\right)$ will be

$$log\left(L_{n+1}\right) = (\hat{a}_n + 1/2)\cdot log\frac{2\hat{a}_n + z_{n+1}^2}{2\hat{a}_n + z_{n+1}^2/k} - log\sqrt{k}$$

Appendix B: PRC Algorithm

Algorithm 1 PRC algorithm

Determine the PRC model from the Table 1 and define the size of the shift k ▷ *Model*
Is prior information available? ▷ *initial prior* $\pi_0(\cdot)$
YES
    Determine the hyperparameters of the initial prior $\boldsymbol{\tau}$
NO
    Set the initial reference prior
Are prior data available? ▷ *power prior*
YES
    Provide the historical data $\boldsymbol{Y}$ and determine α_0
NO
    Set $\alpha_0 = 0$
Choose the appropriate threshold h from Bourazas et al. (2023) ▷ *Decision Threshold*
Once the data point x_n ($n \geq 1^{\bigstar}$) arrives, derive the predictive distribution of next observable $X_{n+1} | \, (\boldsymbol{X}_n, \boldsymbol{Y}, \alpha_0, \boldsymbol{\tau})$
Obtain x_{n+1} and calculate $log(L_{n+1})$ in (7) and S_{n+1} in (8) ▷ S_{n+1}
if $S_{n+1} \leq h$ (or $S_{n+1} \geq h$ for downward shifts) **then** ▷ *test*
    $n \leftarrow n + 1$
    goto 13
else ▷ *Stopping time alarm*
    Raise an Alarm
    if you do not take a corrective action **then**
        goto 16
    else
        end PRC scheme
    endif
endif

$^{\bigstar}$For the likelihoods with two unknown parameters and total prior ignorance (i.e. initial reference prior and $\alpha_0 = 0$ in the power prior) we need $n = 3$ to initiate PRC, while for all other cases PRC starts right after x_1 becomes available.

Appendix C: Tabulated Simulation Results

Table S1 provides the simulation results depicted in Figure 2 of section 4.

Data	Jump	ω	SSC $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	CBF_r $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	CBF_{mi} $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	PRC_r $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	PRC_{mi} $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$
Normal (mean)	$+1\sigma$	IC	4.973%	4.824%	4.871%	4.879%	4.917%
		11	31.724% 13.908 (8.738)	4.080% 20.838 (10.522)	5.913% 22.275 (9.441)	35.053% 14.355 (8.562)	58.009% 14.013 (8.234)
		26	61.487% 11.509 (5.720)	16.479% 11.722 (6.594)	24.513% 12.357 (6.436)	64.343% 11.534 (5.657)	74.954% 11.004 (5.552)
		41	37.571% 6.885 (2.198)	12.805% 5.985 (2.701)	16.832% 6.034 (2.680)	38.804% 6.926 (2.171)	45.088% 6.778 (2.200)
	$+1.5\sigma$	IC	4.968%	4.824%	4.871%	4.889%	4.926%
		11	38.256% 8.278 (6.641)	4.820% 12.691 (8.906)	12.768% 13.750 (8.034)	45.335% 8.958 (6.411)	74.709% 8.476 (6.004)
		26	81.899% 7.798 (4.818)	39.069% 9.684 (5.890)	57.505% 9.867 (5.825)	84.536% 7.765 (4.690)	91.916% 7.158 (4.399)
		41	74.124% 5.525 (2.250)	38.075% 5.897 (2.584)	47.120% 5.810 (2.581)	75.624% 5.557 (2.210)	80.342% 5.372 (2.203)

Data	Jump	ω	SSC $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	CBF_r $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	CBF_{mi} $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	PRC_r $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	PRC_{mi} $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$
Normal (sd)	+50%	IC	4.885%	4.894%	4.804%	4.894%	4.804%
		11	23.093% 16.891 (10.708)	24.591% 17.647 (10.363)	43.089% 17.304 (9.981)	24.591% 17.647 (10.363)	43.089% 17.304 (9.981)
		26	40.032% 11.756 (6.655)	42.066% 11.859 (6.607)	53.937% 11.622 (6.551)	42.066% 11.859 (6.607)	53.937% 11.622 (6.551)
		41	23.678% 5.912 (2.692)	24.372% 5.946 (2.680)	29.146% 5.873 (2.691)	24.372% 5.946 (2.680)	29.146% 5.873 (2.691)
	+100%	IC	5.001%	4.824%	4.871%	4.824%	4.871%
		11	42.764% 11.443 (9.579)	46.051% 11.381 (9.229)	75.060% 10.204 (8.411)	46.051% 11.381 (9.229)	75.060% 10.204 (8.411)
		26	76.465% 8.355 (6.015)	77.453% 8.392 (5.971)	86.604% 7.840 (5.738)	77.453% 8.392 (5.971)	86.604% 7.840 (5.738)
		41	61.593% 4.957 (2.678)	61.548% 5.004 (2.671)	66.863% 4.897 (2.667)	61.548% 5.004 (2.671)	66.863% 4.897 (2.667)

Data	Jump	ω	SSC $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	CBF_r $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	CBF_{mi} $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	PRC_r $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	PRC_{mi} $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$
Poisson (rate)	+50%	IC	5.175%	49.67%	4.951%	4.975%	5.021%
		11	8.681% 31.006 (7.190)	8.126% 20.369 (11.433)	9.271% 20.178 (11.260)	28.482% 20.335 (9.778)	37.723% 20.262 (9.714)
		26	14.299% 17.710 (5.788)	8.762% 12.318 (7.130)	9.531% 12.493 (7.101)	28.482% 13.676 (6.272)	37.782% 13.819 (6.215)
		41	7.803% 6.491 (2.698)	4.441% 5.495 (2.841)	4.677% 5.508 (2.839)	15.437% 6.636 (2.538)	16.528% 6.653 (2.528)
	+100%	IC	5.175%	4.967%	4.951%	5.004%	5.036%
		11	16.492% 23.813 (9.948)	12.015% 15.527 (11.393)	15.377% 15.301 (11.068)	49.866% 13.378 (9.326)	65.412% 13.107 (8.979)
		26	31.298% 15.645 (6.379)	17.045% 10.489 (6.839)	19.643% 10.690 (6.856)	70.240% 10.174 (5.896)	76.212% 10.114 (5.831)
		41	12.363% 6.502 (2.666)	11.307% 5.382 (2.818)	12.345% 5.418 (2.819)	46.594% 6.134 (2.444)	49.006% 6.133 (2.442)

Data	Jump	ω	SSC $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	CBF_r $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	CBF_{mi} $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	PRC_r $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$	PRC_{mi} $PSD(\omega)\%$ $tCED(\omega)$ $(sd(tCED(\omega)))$
Binomial (odds)	+50%	IC	5.191%	5.069%	49.76%	5.030%	5.008%
		11	8.865% 31.234 (7.046)	8.025% 20.606 (11.405)	9.180% 20.467 (11.283)	26.472% 21.411 (9.425)	36.262% 20.856 (9.566)
		26	14.343% 17.836 (5.725)	8.399% 12.468 (7.140)	9.084% 12.629 (7.089)	36.033% 14.008 (6.201)	40.734% 14.056 (6.174)
		41	7.817% 6.509 (2.682)	4.311% 5.470 (2.831)	4.562% 5.510 (2.825)	14.279% 6.704 (2.539)	15.535% 6.704 (2.534)
	+100%	IC	5.200%	5.069%	4.976%	5.069%	5.046%
		11	16.731% 24.719 (9.692)	11.703% 15.867 (11.385)	15.107% 15.852 (11.172)	48.631% 14.042 (9.247)	64.529% 13.552 (8.969)
		26	30.966% 15.834 (6.326)	16.556% 10.691 (6.880)	19.111% 10.885 (6.859)	68.966% 10.502 (5.890)	74.943% 10.412 (5.836)
		41	12.431% 6.511 (2.671)	10.729% 5.429 (2.811)	11.658% 5.466 (2.803)	44.227% 6.279 (2.414)	46.715% 6.254 (2.416)

Table S1: The $FWER$ at $N = 50$, the probability of successful detection, $PSD(\omega)$, the truncated conditional expected delay, $tCED(\omega)$ and its corresponding standard deviation (in parenthesis) for shifts at locations $\omega = \{11, 26 \text{ or } 41\}$, of SSC, CBF and PRC, under a reference (CBF_r, PRC_r) or a moderately informative (CBF_{mi}, PRC_{mi}) prior. The results refer to Normal data with step changes for the mean of size $\{1\theta_2, 1.5\theta_2\}$, Normal data with inflated standard deviation of size $\{50\%, 100\%\}$, Poisson data with rate increase of size $\{50\%, 100\%\}$ and Binomial data with increases for the odds of size $\{50\%, 100\%\}$.