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Nonlinear energy sink and Eurocode 8: an optimal design approach based on elastic response spectra

Oliva M.^a, Barone G.^b, Lo Iacono F.^a, Navarra G.^a

^a*Faculty of Engineering and Architecture, University of Enna Kore, Cittadella Universitaria, Enna, Italy*

^b*School of Architecture, Building and Civil Engineering, Loughborough University, Loughborough, Leicestershire, UK*

Abstract

In this paper, the authors propose novel empirical formulae for the optimal design of Nonlinear Energy Sinks (NESs) to control seismic induced vibrations in structures. While large part of the existing literature on NES for seismic applications is based on deterministic analyses of the structural response to recorded ground motions, herein the effectiveness of the device is studied in a stochastic framework. In fact, the seismic excitation has been modeled using a response spectra consistent power spectral density function, taking into account design spectra prescribed into Eurocode 8. Statistics of the structural response have been determined by Monte Carlo Simulations, performing nonlinear time-domain analyses to investigate the dependence of the optimal NES parameters on the main characteristics of the primary system and on the seismic load amplitude. The outcomes of these analyses have been translated into empirical expressions allowing for computing the optimal NES stiffness and damping. Since these are closed-form relationships depending on the main structure and the response spectra parameters, their use allows for an immediate estimate of the NES optimal design configuration, avoiding demanding numerical analyses and making their implementation suitable for practical engineering purposes. The accuracy of the design formulae has been tested by comparison with numerical optimisations for a set of possible configurations of the main structure and for varying intensity of the external excitation. Although the proposed relationships have been extrapolated from single degree of freedom system responses, applications to multi degree of freedom structures is quite straightforward, as shown through two case studies involving the analysis of a 2D multi-storey structure and a 3D multi-bay frame controlled by two orthogonal NESs.

Keywords: Nonlinear Energy Sink, Optimisation, Spectrum-compatible PSD, Structural dynamics

1. Introduction

In the last decades, a multitude of structural control strategies have been developed to protect civil structures against environmental agents [1]. Passive control systems are the most extensively researched and widely used devices across the world due to their capacity to enhance energy dissipation in the controlled structures without needing an external energy supply to trigger the damping process. Further, due to the low costs of installation and maintenance compared to alternative control strategies such as active devices, passive dampers often result as the most convenient choice also from an economic perspective.

The Nonlinear Energy Sink (NES) is an extremely interesting passive dynamic vibration absorber due to its peculiar features. Similarly to the most commonly known Tuned Mass Damper (TMD), the NES is a small secondary mass connected to the primary structure through a damping element and a spring, but with the substantial difference that the latter is essentially nonlinear (type-I NES in Figure 1a) [2]. This enables the NES to be effective on broad frequency bands and to resonate with multiple modes of the main structure,

Figure 1: Essentially nonlinear energy sinks: (a) type-I NES; (b) type-II NES; (c) type-III NES.

thus resulting in a robust design choice with reduced sensitivity to changes in the structural frequencies (e.g. due to structural damage) or in the environmental loads.

One of the specific features of this device is to absorb and dissipate energy through Targeted Energy Transfer (TET), irreversibly absorbing energy from the primary structure [3]. Several theoretical and experimental studies have been conducted to achieve a better understanding of this phenomenon. The basic mechanism for the TET occurs through single mode or cascade of mode resonance captures [4–11].

In addition to the simplest and best known type-I NES, other configurations have been proposed in literature. The type-II NES consists of a mass coupled to a linear system with nonlinear stiffness and nonlinear damping [12], the type-III consists of two NES arranged in series with strong stiffness asymmetry, connected to a linear structure by essentially nonlinear cubic stiffness in parallel with a linear viscous damper [13]. Schematic representations of type-II and type-III NESs are respectively depicted in Figures 1b and 1c. Non-smooth nonlinearities in the form of clearances or vibro-impacts have been investigated in Refs. [14–16], whereas non-polynomial nonlinearity has been employed in Ref. [17]. A rotating NES in a horizontal plane has been proposed in Refs. [18, 19], whereas a vertical curve track NES has been introduced in Ref. [20].

The optimal design of a NES is not an easy task because the optimal nonlinear stiffness is strongly dependent on the amplitude of the load [21, 22]. Closed-form solutions and design criteria to determine the optimal nonlinear stiffness have been proposed for transient, harmonic and impulsive excitations [23–26]. Empirical formulae for determining the optimal parameters of a type-I NES installed on a Single Degree Of Freedom (SDOF) structure under white noise excitations have been derived for pre-design purposes in Ref. [22]. The application of these empirical formulae has been extended to MDOF structures by considering dummy oscillators whose masses and natural frequencies depend on the effective modal masses and modal frequencies of the main structure [27].

Differently from previous works, this paper focuses on practical applications in seismic engineering, by investigating multi-storey structures excited by ground excitations compatible with Eurocode 8 prescriptions. The design of earthquake-resistant structures with nonlinear components usually requires nonlinear

time domain analysis, computing the structural response due either to selected earthquake data records, or to artificially simulated ground motion time histories. Although accelerograms of historically renowned earthquakes, such as El Centro (May, 1940), Northridge (January, 1994), Kobe (January, 1995) among others, are often used as input excitations to study the structural response [2, 9, 11], the use of recorded seismic time histories could be debatable for the analysis of structures built on very different types of ground [28]. Depending on the study at hand, a number of earthquake records with similar characteristics may be needed for time-domain analyses. This approach can easily become unfeasible due to the limited number of historical data available. Accordingly, many international seismic codes and guidelines allow the use of artificial accelerograms compatible with design response spectra. Therefore, several methods to obtain response-spectrum-compatible Power Spectral Density (PSD) functions of ground acceleration processes have been proposed in literature, such as filter equations, iterative, non-iterative recursive and analytic methods [29–37].

In this work, taking advantage of the analytic model of Eurocode 8-compatible ground motion PSD function, introduced in Refs. [36, 37], the authors propose empirical formulae for the optimal design of a type-I NES to mitigate seismic induced vibrations on a primary structure. For each ground type defined in the Eurocode 8 (A to E), the dependence of the NES performance on mass, natural frequency and damping of the primary system has been investigated by varying the amplitude of the external excitation.

The results of this parametric analysis have been implemented into empirical formulae allowing for immediate design of the NES, substituting computational demanding optimisation processes based on non-linear time history analysis. The formulae can also, conveniently, be translated into design charts depending on few selected parameters.

Numerical results are initially presented based on the study of a SDOF shear-type frame. Then, two further cases studies are reported to emphasize the applicability of the proposed methodology to MDOF structures.

2. Definition of the mathematical problem

2.1. Governing equations

Figure 2 shows the type-I NES configuration. It consists of a SDOF linear primary structure, modelled herein as a shear type frame, connected to an ungrounded and lightweight NES through a pure cubic nonlinear stiffness and a linear viscous damper.

The system subjected to a sample of the seismic ground motion acceleration process $\ddot{U}_g(t)$ is described by the following equations of motion:

$$\begin{cases} m_1 \ddot{X}_1(t) + c_1 \dot{X}_1(t) + k_1 X_1(t) + c_N (\dot{X}_1(t) - \dot{X}_N(t)) + k_N (X_1(t) - X_N(t))^3 = -m_1 \ddot{U}_g(t) \\ m_N \ddot{X}_N(t) + c_N (\dot{X}_N(t) - \dot{X}_1(t)) + k_N (X_N(t) - X_1(t))^3 = -m_N \ddot{U}_g(t) \end{cases} \quad (1)$$

in which m_1 , c_1 , k_1 are mass, viscous damping and stiffness coefficients of the primary structure and m_N , c_N , k_N are mass, viscous damping and nonlinear stiffness coefficients of the NES, respectively; X_1 and X_N are the displacements of the primary structure and the NES, respectively. As customary, overdots indicate derivative with respect to time and capital letters stochastic processes.

Dividing Eq. (1) by m_1 , the set of nonlinear equations becomes:

$$\begin{cases} \ddot{X}_1(t) + \lambda_1 \dot{X}_1(t) + \lambda_N (\dot{X}_1(t) - \dot{X}_N(t)) + \omega_1^2 X_1(t) + \kappa (X_1(t) - X_N(t))^3 = -\ddot{U}_g(t) \\ \varepsilon \ddot{X}_N(t) + \lambda_N (\dot{X}_N(t) - \dot{X}_1(t)) + \kappa (X_N(t) - X_1(t))^3 = -\varepsilon \ddot{U}_g(t) \end{cases} \quad (2)$$

Figure 2: Schematic representation of the type-I NES connected to a SDOF shear type frame.

in which the new parameters introduced are:

$$\lambda_1 = \frac{c_1}{m_1} = 2\zeta_1\omega_1; \quad \lambda_N = \frac{c_N}{m_1}; \quad \omega_1^2 = \frac{k_1}{m_1}; \quad \kappa = \frac{k_N}{m_1}; \quad \varepsilon = \frac{m_N}{m_1}. \quad (3)$$

where ζ_1 and ω_1 are the damping ratio and the circular frequency of the primary structure, respectively, and ε is the mass ratio between the NES and the main structure, in the following assumed as $\varepsilon \ll 1$.

In the most general case, a n -DOF linear structure can be connected to p -NESs through pure cubic stiffnesses and linear viscous dampers. In this case, the governing equations of motion in the uncontrolled configuration (i.e. without NES) can be written as:

$$\mathbf{M}_u \ddot{\mathbf{X}}_u(t) + \mathbf{C}_u \dot{\mathbf{X}}_u(t) + \mathbf{K}_u \mathbf{X}_u(t) = -\mathbf{M}_u \boldsymbol{\tau}_u \ddot{U}_g(t) \quad (4)$$

where $\mathbf{X}_u(t)$, $\dot{\mathbf{X}}_u(t)$, $\ddot{\mathbf{X}}_u(t)$ are the n -dimensional generalised displacement vector and its derivatives, respectively; $\boldsymbol{\tau}_u$ is the influence vector; $\mathbf{M}_u$, $\mathbf{C}_u$ and $\mathbf{K}_u$ are the mass, damping and linear stiffness matrices of the primary system, respectively. When the main structure is controlled by p -NESs, the equations of motion become:

$$\mathbf{M} \ddot{\mathbf{X}}(t) + \mathbf{C} \dot{\mathbf{X}}(t) + \mathbf{K} \mathbf{X}(t) + \mathbf{g}(\mathbf{X}(t)) = -\mathbf{M} \boldsymbol{\tau} \ddot{U}_g(t) \quad (5)$$

in which $\mathbf{X}(t)$, $\dot{\mathbf{X}}(t)$, $\ddot{\mathbf{X}}(t)$ are the r -dimensional generalised displacement vector and its derivatives, respectively; $r = n + p$ is the number of degrees of freedom of the combined system; $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and linear stiffness matrices of the combined system, respectively, i.e.:

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{X}_u(t) \\ \mathbf{X}_{NES}(t) \end{bmatrix} \quad (6)$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_u & \mathbf{0}_{(n \times p)} \\ \mathbf{0}_{(p \times n)} & \mathbf{M}_{NES} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_u & \mathbf{0}_{(n \times p)} \\ \mathbf{0}_{(p \times n)} & \mathbf{0}_{(p \times p)} \end{bmatrix} + \mathbf{L} \mathbf{C}_{NES} \mathbf{L}^T \quad \mathbf{K} = \begin{bmatrix} \mathbf{K}_u & \mathbf{0}_{(n \times p)} \\ \mathbf{0}_{(p \times n)} & \mathbf{0}_{(p \times p)} \end{bmatrix} \quad (7)$$

In Eq. (7), $\mathbf{L}$ is the $r \times p$ linear transformation matrix returning the vector of relative displacements $\mathbf{Y}(t)$ at the ends of each NES device as a function of the selected degrees of freedom representing the motion of the points to which the devices are attached:

$$\mathbf{Y}(t) = \mathbf{L} \mathbf{X}(t) \quad (8)$$

with the i -th column of $\mathbf{L}$ corresponding to the i -th NES. $\mathbf{M}_{NES}$ and $\mathbf{C}_{NES}$ are the diagonal matrices of mass and linear damping of the NESs, respectively, whereas $\mathbf{g}(\mathbf{X})(t)$ is the vector collecting all the nonlinear forces. The latter can be expressed, in general, as a function of the relative displacements $\mathbf{Y}(t)$ as:

$$\mathbf{g}(\mathbf{X}(t)) = \mathbf{L}^T \mathbf{g}_L(\mathbf{Y}(t)) \quad (9)$$

where the i -th term of the vector $\mathbf{g}_L(\mathbf{Y}(t))$ is defined as:

$$\{\mathbf{g}_L(\mathbf{Y}(t))\}_i = k_{N,i} Y_i(t)^3 \quad (10)$$

being $k_{N,i}$ the nonlinear stiffness of the i -th NES.

2.2. Spectra-compatible ground motions

International building codes generally define the seismic excitation by means of pseudo-acceleration response spectra. Figure 3a illustrates a qualitative representation of the Eurocode 8 (EC8) elastic response spectrum $S_a(T)$. The latter can be expressed as [38]:

$$S_a(T) = \begin{cases} a_g S \left[1 + \frac{T}{T_B} (\eta 2.5 - 1) \right] & 0 \leq T \leq T_B \\ a_g S \eta 2.5 & T_B < T \leq T_C \\ a_g S \eta 2.5 \left[\frac{T_C}{T} \right] & T_C < T \leq T_D \\ a_g S \eta 2.5 \left[\frac{T_C T_D}{T^2} \right] & T > T_D \end{cases} \quad (11)$$

in which T is the natural period of a linear SDOF system, a_g is the design peak ground acceleration on soil type A, η is the damping correction factor, S is the soil factor, and T_B , T_C and T_D are the periods delimitating the various branches. The values of the periods and of the soil factor depend on the ground type as reported in the EC8 [38].

Figure 3: (a) EC8 response spectra and (b) PSD compatible with EC8 response spectra for ground types A to E.

One of the intrinsic limits of the response spectrum analysis is that response spectra provided by building codes are determined for linear SDOF systems with assigned damping, therefore being non-representative of the effect of the ground motion on nonlinear MDOF structures. However, several codes allow to model the seismic ground motion by means of artificial accelerograms that can be generated as samples of a zero-mean Gaussian stationary process fully characterized by its one-sided PSD function $G_{\ddot{U}_g}(\omega)$. The latter is related only to the ground acceleration and it is independent of the damping ratio as well as the inherent non-linearities of the superimposed structure, thus allowing to overcome the aforementioned limitations. Nevertheless, it is right to emphasize that real ground motions exhibit non-stationary characteristics both in time and in frequency domain making the assumption of stationarity a limitation of this approach.

The analytical model of PSD function compatible with building code response spectra proposed in Refs. [36, 37] is herein reported. Figure 3b reports the PSD function, $G_{\ddot{U}_g}(\omega)$, compatible with elastic response spectra provided by EC8. These have been obtained by using the following four-branches piecewise function [36]:

$$G_{\ddot{U}_g}(\omega) = \begin{cases} G_0 \left(\frac{\omega_D}{\omega_C} \right)^{e_2} \left(\frac{\omega}{\omega_D} \right)^{e_1} & 0 \leq \omega \leq \omega_D \\ G_0 \left(\frac{\omega}{\omega_C} \right)^{e_2} & \omega_D < \omega \leq \omega_C \\ G_0 \left(\frac{\omega}{\omega_C} \right)^{e_3} & \omega_C < \omega \leq \omega_B \\ G_0 \left(\frac{\omega_B}{\omega_C} \right)^{e_3} \left(\frac{\omega}{\omega_B} \right)^{e_4} & \omega > \omega_B \end{cases} \quad (12)$$

where G_0 is the peak value of the PSD function at the frequency $\omega = \omega_C$. This model is dependent on five parameters, i.e. G_0 and the exponents e_i ($i = 1, \dots, 4$), that have been evaluated as closed-form functions of the seismic response spectra parameters using the formulae proposed in Refs. [36, 37]. While G_0 is dependent on the peak ground acceleration a_g , the four exponents e_i are not dependent on the amplitude of the excitation and therefore need to be determined only once for each type of ground. Table 1 shows the values of the 5 parameters defining the PSD function compatible with EC8 spectra for each ground type.

Table 1: Values of the parameters describing the PSD consistent to Type 1 elastic response spectra provided by EC8.

Soil Type	G_0/a_g^2	e_1	e_2	e_3	e_4
A	$3.0194 \cdot 10^{-3}$	2.5688	0.75259	-1.3247	-2.6468
B	$5.7546 \cdot 10^{-3}$	2.5688	0.73716	-1.3406	-2.6848
C	$6.6646 \cdot 10^{-3}$	2.5688	0.72302	-1.3552	-2.6829
D	$13.325 \cdot 10^{-3}$	2.5688	0.69726	-1.3815	-2.7376
E	$7.8327 \cdot 10^{-3}$	2.5688	0.73716	-1.3406	-2.6848

2.3. Optimisation problem

In the following section, the optimal NES parameters, i.e. damping and nonlinear stiffness, have been determined by means of the NelderMead optimisation method [39], a heuristic search method used in non-linear problems to find the minimum of a selected objective function (herein, the implementation available

in MATLAB toolboxes has been used). Several functions have been proposed in literature for the optimisation of structural control devices in order to maximise their performance [15, 22, 40, 41]. These are generally defined as the ratio between a controlled and uncontrolled characteristic relative to the dynamic response of the structure to be protected. Hence, their minima correspond to the maximum reduction of the response of the primary structure (i.e. maximum efficiency of the device).

Since the stresses on the structural members depend on the relative displacements of the storey with respect to the ground, herein the objective function of the optimisation problem has been defined as the mean value of the dimensionless ratio between the RMS of displacements of the primary structure in its controlled (i.e. with attached NES) and uncontrolled (i.e. without NES) configurations over n_c samples of a Monte Carlo Simulation (MCS). The latter has been used to determine the statistics of the structural response by integrating the equations of motion in the time domain by 4th order Runge Kutta method. Therefore, the objective function J can be expressed as:

$$J = E \left[\frac{RMS \{ \mathbf{X}(t) \}^i}{RMS \{ \mathbf{X}_u(t) \}^i} \right] \quad (13)$$

in which $\{ \mathbf{X}(t) \}^i$ and $\{ \mathbf{X}_u(t) \}^i$ are the i -th samples of displacements processes of the primary structure in the controlled and the uncontrolled configurations, respectively. Hence, the optimisation problem may be summarised as follows:

Given: $\mathbf{M}_u, \mathbf{C}_u, \mathbf{K}_u, \varepsilon, G \ddot{U}_g$;

Find: c_N, k_N or λ_N, κ ;

To minimise: $J(\mathbf{X}(t), \mathbf{X}_u(t))$.

To ensure the results of the optimisation algorithm to return design variables with actual physical meaning, the following constraints must be set: $c_N > 0$ (alternatively $\lambda_N > 0$) and $k_N > 0$ (alternatively $\kappa > 0$).

3. Optimal NES Parameters for EC8 Spectra-Compatible Ground Motion

Aiming at understanding the relationship among the parameters defining the optimal NES configuration, the primary structure and the seismic excitation (modelled following Eurocode 8 prescriptions and considering different ground types), an actual small-scale experimental model has been selected as reference structure [42]. For this same structure, optimal NES parameters have been investigated for the case of white noise excitations in Ref. [22]. The frame is made of steel columns and nylon rigid plates and, in the following, it is mathematically described as a SDOF shear type frame having mass, stiffness and damping ratio respectively $m_{1,R} = 2.135$ kg, $k_{1,R} = 890$ N/m and $\zeta_{1,R} = 1.8\%$ (i.e., $\omega_{1,R} = 20.42$ rad/s and $\lambda_{1,R} = 0.735$ s⁻¹). The subscript R indicates the reference structure.

The optimisation problem has been solved for a number of possible cases. For each of them, the normalised optimal parameters κ_{opt} and $\lambda_{N,opt}$ have been determined by minimizing Eq. (13) particularized for the case of SDOF structures, i.e.:

$$J_1 = E \left[\frac{RMS \{ X_1(t) \}^i}{RMS \{ X_{1,u}(t) \}^i} \right] \quad (14)$$

in which $X_1(t)$ and $X_{1,u}(t)$ are the displacements processes of the primary structure in the controlled and uncontrolled configurations. J_1 has been calculated by means of MCSs considering 10000 samples of EC8 spectra-compatible ground motion accelerations, 20 s long and with a sampling rate of 0.01 s.

Figure 4: J_1 function with respect to κa_g^2 and λ_N .

Although all ground types proposed in the Eurocode 8 have been analysed, in the following section only the numerical results obtained for ground type A are discussed in detail, since the results for all ground types have shown the same qualitative behaviour for the dynamic coupled system frame-NES.

3.1. Sensitivity analysis: ground type A

In Ref. [22], the authors have identified a polynomial relationship between the logarithmic values of the optimal NES nonlinear stiffness and the intensity of the ground motion for the specific case of white noise excitation. However, the latter does not affect the shape of the function J_1 in the κ - λ_N domain and its minimum value (optimal performance of the NES).

In order to investigate if a similar relationship can be found for the case of response spectra compatible ground motion processes, a similar approach has been followed herein. An initial mass ratio $\varepsilon = 0.05$ between the NES and the reference primary structure has been arbitrarily set. Then, the objective function J_1 has been determined by varying damping and stiffness of the NES for $a_g \in [0.2 \text{ g}, 1.0 \text{ g}]$.

In fact, also in this case, the shape of J_1 results to be independent from a_g and the position of its minimum only dependent on κ . Taking that into account, results of these numerical analyses are summarised in Figure 4, where values of J_1 are reported, in the logarithmic domain, as a function of the product $\kappa \cdot a_g^2$ and the optimal damping parameter λ_N .

Figures 5 and 6 show the optimal design variables obtained varying the primary structure natural frequency first, and its damping ratio later. The natural period of the structure has been considered in the range $[0.1 \text{ s}, 1.5 \text{ s}]$, since outside this range (very rigid or very flexible structures, comparable to base isolated systems) the use of the NES would normally be unnecessary. Figure 5b confirms $\lambda_{N,opt}$ to be independent on a_g . Both κ_{opt} and $\lambda_{N,opt}$ values increase for increasing natural frequency ω_1 . Conversely, κ_{opt} increases with increasing ζ_1 , while $\lambda_{N,opt}$ decreases. Table 2 reports the numerical results for the case of $\zeta_1 = \zeta_{1,R}$ and $\varepsilon = 0.05$.

Finally, Figures 7a-b show the effects of the variation of the main frequency ω_1 on the optimal parameters κ_{opt} and $\lambda_{N,opt}$ for different values of the mass ratio ε assuming $\zeta_1 = \zeta_{1,R}$. The values of the mass ratio considered in the analysis have been set at 0.025, 0.05 and 0.1, to ensure the assumption of lightweight NES. Both the optimal nonlinear parameters increase increasing the mass ratio.

Figure 5: Optimal NES parameters obtained by numerical optimisations (symbols) and fitted curves (solid line) versus ω_1 varying the peak ground acceleration a_g ($\zeta_1 = \zeta_{1,R}$, $\varepsilon = 0.05$): (a) normalised optimal NES stiffness κ_{opt} ; (b) normalised optimal NES damping $\lambda_{N,opt}$.

Figure 6: Optimal parameters of the NES obtained by numerical optimisations (symbols) and fitted curves (solid line) versus ω_1 varying the damping ratio ζ_1 for ground type A and $\varepsilon = 0.05$: (a) normalised optimal NES stiffness κ_{opt} ; (b) normalised optimal NES damping $\lambda_{N,opt}$.

Figure 7: Optimal parameters of the NES obtained by numerical optimisations (symbols) and fitted curves (solid line) versus ω_1 varying the mass ratio ε for ground type A and $\zeta_1 = \zeta_{1,R}$: (a) normalised optimal NES stiffness κ_{opt} ; (b) normalised optimal NES damping $\lambda_{N,opt}$.

3.2. Empirical formulae for optimal NES design

Analysis of the numerical results obtained, in analogous way, varying opportunely ω_1 , ζ_1 , a_g , ε and ground type (A to E), leads to the following findings:

- A linear piecewise function can be defined between the logarithms of the quantities $\kappa_{opt} \cdot a_g^2$ and ω_1 accurately fitting all the computed numerical results in the three sub-domains, i.e. $T_{min} \leq T \leq T_B$, $T_B < T \leq T_C$ and $T_C < T \leq T_{max}$;
- A linear function between $\lambda_{N,opt}$ and ω_1 can be defined throughout the whole domain $[T_{min}, T_{max}]$.

Hence, the empirical formulae for the nonlinear stiffness and the damping of the optimal NES can be written in the following form:

$$\log_{10} [\kappa_{opt} a_g^2] = (a_1 \log_{10} \varepsilon + a_2 \log_{10} \zeta_1 + a_3) \log_{10} \omega_1 + b_1 \log_{10} \varepsilon + b_2 \log_{10} \zeta_1 + b_3 \quad (15)$$

$$\lambda_{N,opt} = (d_1 \varepsilon + d_2 \zeta_1 + d_3) \omega_1 + g_1 \varepsilon + g_2 \zeta_1 + g_3 \quad (16)$$

The coefficients in Eq. (15) depend on the ground type and on the interval to which the natural period of the structure under investigation belongs, and are listed in Table 3.

Although the expression in Eq. (16) is the most correct, it has been observed that negligible errors in terms of efficiency of the NES are achieved when the optimal damping $\lambda_{N,opt}$ is estimated through the significantly simpler formula obtained for white noise excitations [22]:

$$\lambda_{N,opt} = (0.204 \varepsilon - 0.001) \zeta_1^{-0.1} \omega_1 \quad (17)$$

This is because the J_1 function is not very sensitive to small variations of the damping of the NES in the neighbourhood of its minimum value, as shown in Figures 4 and 5b. Hence, Eq. (17) can be used for the design of the NES.

In order to show the degree of accuracy of these formulae, Figures 8a-b report J_1 values obtained by numerical optimisations (marked with dots), by Eqs. (15) and (16) (marked with asterisks), and Eqs.

Table 2: Optimal NES parameters obtained by numerical optimisations for $\zeta_1 = \zeta_{1,R}$ and $\varepsilon = 0.05$.

	T	ω [rad/s]	$\log_{10}[\kappa_{opt} a_g^2]$	$\lambda_{N,opt}$	J_1
	[s]	[rad/s]	[(s m) ⁻² (m s ⁻²) ²]	[s ⁻¹]	[-]
$T_{min} \leq T \leq T_B$	0.10	60.07	8.415	0.953	0.715
	0.11	55.92	8.193	0.887	0.707
	0.12	52.05	7.971	0.826	0.697
	0.13	48.46	7.745	0.769	0.691
	0.14	45.11	7.532	0.716	0.684
	0.15	41.99	7.339	0.666	0.679
$T_B < T \leq T_C$	0.17	37.59	7.039	0.576	0.661
	0.19	33.73	6.757	0.517	0.662
	0.21	30.27	6.461	0.464	0.663
	0.23	27.16	6.181	0.416	0.664
	0.26	24.38	5.893	0.373	0.666
	0.29	21.87	5.620	0.335	0.667
	0.32	19.63	5.361	0.301	0.670
	0.36	17.62	5.092	0.270	0.674
	0.40	15.81	4.855	0.242	0.675
$T_C < T \leq T_{max}$	0.40	15.71	4.850	0.216	0.669
	0.48	13.18	4.540	0.181	0.680
	0.57	11.06	4.218	0.152	0.688
	0.68	9.28	3.893	0.128	0.697
	0.81	7.79	3.587	0.107	0.709
	0.96	6.54	3.282	0.090	0.721
	1.14	5.49	2.981	0.075	0.738
	1.36	4.60	2.686	0.063	0.752

(15) and (17) (marked with crosses) by varying the frequency ω_1 . The following parameters have been considered for the analyses: $\varepsilon = 0.05$, $\zeta_1 = \zeta_{1,R}$ and ground type A, for which the coefficients in Eq. (16) are $d_1 = 0.3323$, $d_2 = -0.0222$, $d_3 = -0.0038$, $g_1 = 0.7090$, $g_2 = -0.5556$, $g_3 = 0.0331$. The relative loss in efficiency of the NES due to using these three different design values can be determined as:

$$e_{J_1} = \frac{|J_{1,NOPT} - J_{1,EMP}|}{J_{1,NOPT}} \cdot 100 \quad [\%] \quad (18)$$

where $J_{1,NOPT}$ represents the value of J_1 obtained by numerical optimisations and MCSs, while $J_{1,EMP}$ is the corresponding value of J_1 determined by using the proposed empirical formulae.

By analysing all the combinations obtained considering three values of mass ratios ($\varepsilon = 0.025$, $\varepsilon = 0.05$ and $\varepsilon = 0.1$), three values of the primary structure damping ($\zeta_1 = 0.5 \zeta_{1,R}$, $\zeta_1 = \zeta_{1,R}$, $\zeta_1 = 2 \zeta_{1,R}$) and varying the primary structure natural period in the range $[0.1, 1.5]$ s, the maximum efficiency losses for the NES committed using (15) and (17) versus the actual numerical optimisation are: $e_{J_1max} = 1.64\%$ for ground type A, $e_{J_1max} = 1.83\%$ for type B, $e_{J_1max} = 2.36\%$ for type C, $e_{J_1max} = 2.40\%$ for type D and $e_{J_1max} = 4.25\%$ for type E. Therefore, although the empirical formulae (15) and (17) do not allow to determine the

Figure 8: (a) Comparison in terms of J_1 functions among the results obtained by Eqs. (16) and (17) (a) by varying ω_1 ; (b) for $\omega_1 = 30.27$ rad/s.

Table 3: Coefficients of the empirical formula (15) for determining the optimal nonlinear stiffness.

		Ground Type				
		A	B	C	D	E
$T_{min} \leq T \leq T_B$	Coefficients					
	a_1	-0.2345	-0.3171	-0.3475	-0.4923	-0.3540
	a_2	-0.0737	-0.1163	-0.2383	-0.1257	0.0618
	a_3	6.5018	6.0095	6.0689	6.1269	6.2413
	b_1	1.7490	1.8968	1.9953	2.1174	1.9378
	b_2	0.2402	0.3001	0.5033	0.2930	0.0953
$T_B < T \leq T_C$	b_3	-1.2616	-0.5725	-0.4937	-0.8856	-0.9612
	a_1	0.1702	0.1918	-0.0850	0.0468	0.0978
	a_2	-0.0943	-0.0958	-0.0885	-0.0804	0.1756
	a_3	5.9056	5.80526	5.4983	5.7713	6.1766
	b_1	1.1660	1.1416	1.4158	1.3236	1.2906
	b_2	0.3011	0.2948	0.2790	0.2483	-0.0658
$T_C < T \leq T_{max}$	b_3	-0.1218	-0.2342	0.0574	-0.4059	-0.8346
	a_1	0.2277	0.2882	0.5189	0.4338	0.2804
	a_2	0.0756	0.0811	0.0816	0.0807	0.0977
	a_3	4.4882	4.5137	4.7617	4.5979	4.5350
	b_1	1.1657	1.1107	1.0212	1.0328	1.1190
	b_2	0.1405	0.1325	0.1234	0.1174	0.1196
	b_3	1.7492	1.3585	1.1237	0.7765	1.2121

exact values of optimal damping ratio and nonlinear stiffness, the maximum efficiency loss is below 5%, thus confirming the convenience of use the expression (17). In particular, values obtained only for ground

type A are shown in Figure 9, showing that the NES efficiency:

- increases as the mass ratio ε increases;
- increases with decreasing damping of the main structure;
- is maximum in the range $[T_B, T_C]$ (gray area in Figure 9), i.e. where the response spectrum has maximum value.

Figure 9: Comparison in terms of J_1 obtained by numerical optimisation and the proposed empirical formulae for ground type A by varying ζ_1 and ω_1 for: (a) $\varepsilon = 0.025$; (b) $\varepsilon = 0.05$; (c) $\varepsilon = 0.1$.

For practical purposes, Eqs. (15) and (17) can be easily represented by design charts. Examples are shown in Figure 10 for ground type A and in Appendix A for the remaining types when $\varepsilon = 0.05$. In these charts, the optimal NES parameters κ_{opt} and $\lambda_{N,opt}$ are reported in the abscissa and ordinate, respectively. Continuous black and red lines correspond to constant natural frequency $f_1 = \omega_1/2\pi$ and damping ratio ζ_1 , respectively. For reasons of readability of the charts, the frequency is reported in Hertz. The use of these

diagrams is very simple: the optimal NES parameters for given mass ratio ε and peak ground acceleration a_g is immediately identified by the intersection of the f_1 and ζ_1 curves.

Figure 10: Design chart to determine the optimal parameters values for different values of f_1 and ζ_1 for soil type A and $\varepsilon = 0.05$.

3.3. Extension to MDOF systems

The proposed numerical formulae have been obtained on the basis of the analysis of SDOF structures, yet they can also be used for the design of NESs linked to MDOF structures by means of a simplified procedure.

Consider for simplicity the case of a single NES aimed to mitigate the structural response in a specific direction of motion. Then, the design procedure can be synthesized in the following steps [27]:

1. computation of natural frequencies and percentages of participants' responses of the MDOF structure;
2. selection of the main mode to mitigate, based on the predetermined direction of the NES motion;
3. definition of a dummy SDOF system, i.e. an elementary oscillator having mass equivalent to the effective modal mass at the selected mode of the main structure, $m_{1,e}$, and frequency and damping equivalent to its modal frequency $\omega_{1,e}$ and damping $\zeta_{1,e}$;
4. for an arbitrarily selected mass ratio between the NES and the dummy structure, ε_e , computation of the optimal NES parameters by multiplying for $m_{1,e}$ the normalised parameters κ_{opt} and $\lambda_{N,opt}$ obtained by Eqs. (15) and (17), i.e:

$$\begin{cases} m_N = \varepsilon_e \cdot m_{1,e} \\ c_N = \lambda_{N,opt} \cdot m_{1,e} \\ k_N = \kappa_{opt} \cdot m_{1,e} \end{cases} \quad (19)$$

For pre-design purposes, the same procedure can be simply extended to the case of more concurrent NESs acting in different directions, under the assumption of well-spaced modal frequencies. The validity of this procedure is tested through numerical applications in the following section.

4. Numerical Applications

In order to prove the effectiveness of the empirical formulae (15) and (17) to design NESs, three numerical applications are herein presented. In the first one, the NES design for the SDOF system introduced in section 3 is analysed varying the ground type (based on EC8 prescriptions). The last two applications aim to demonstrate how these formulae can be adopted for MDOF structures. In particular, the two-DOF shear type frame investigated in Refs. [20–22] and a realistic two-storeys two-bays 3D frame have been analysed [27]. For each of them, results are presented in terms of reductions of the displacements of the primary structures and compared with numerical optimisations.

4.1. SDOF Structure

Firstly, the performance of a single NES linked to the SDOF reference structure previously presented has been investigated by varying the ground type in which the primary structure arises. The primary structure features are reported in Table 4.

Table 4: Characteristics of the SDOF structure

m_1	k_1	ω_1	f_1	ζ_1
[kg]	[N/m]	[rad/s]	[Hz]	[%]
2.135	890	20.42	3.25	1.8

For all the cases, it has been assumed that the mass of the NES is equal to 5% of the mass of the primary structure. The effectiveness of the passive control system has been evaluated considering ground accelerations compatible with EC8 response spectra having $a_g = 0.25$ g.

The design of the NESs has been carried out by using by Eqs. (15) and (17). Alternatively, the same parameters can be found by using the charts illustrated in Figure 10 and in Appendix A.

Table 5: Optimal NES parameters obtained by formulae (15) and (17) (EMP) and numerical optimisation (NOPT).

Soil	$\log_{10}[\kappa_{opt} [(s\ m)^{-2}]]$		$\lambda_{N,opt} [s^{-1}]$		$J_1 [-]$	
	EMP	NOPT	EMP	NOPT	EMP	NOPT
A	4.72	4.67	0.281	0.278	0.674	0.672
B	4.51	4.47	0.281	0.264	0.673	0.672
C	4.50	4.51	0.281	0.265	0.674	0.673
D	4.32	4.35	0.281	0.226	0.681	0.677
E	4.35	4.34	0.281	0.266	0.674	0.672

The accuracy of the proposed design methodology has been verified with numerical optimisations. For each ground type, J_1 (defined in Eq.(14)) has been calculated by performing MCSs with 10000 samples ground motion accelerations, 20 s long and with a sampling rate of 0.01 s. The optimal parameters and the corresponding J_1 are listed in Table 5. The maximum error committed on the reduction of the RMS of the displacements of the primary structure is 0.6%.

Figures 11a-e show also the PSDs of the displacement response process considering the uncontrolled configuration (black line), and the controlled configuration with NES designed by empirical formulae (red

Figure 11: Comparison between the uncontrolled and controlled configurations in the frequency domain: PSD of displacements process for each ground type.

line) and by numerical optimisation (dashed blue line). It is immediately visible that the two curves corresponding to the controlled configurations (i.e. blue and red lines) are practically overlapping, thus demonstrating the accuracy of the proposed empirical formulae. In all analysed cases, the RMS of the structural response has been reduced of about 30%.

4.2. 2-DOF Structure

The second application consists of a two-DOF shear type frame coupled with a NES localised on the top storey [20–22], whose schematic representation is depicted in Figure 12. The mass and stiffness matrices are:

$$\mathbf{M}_u = \begin{bmatrix} 24.3 & 0 \\ 0 & 24.2 \end{bmatrix} \text{ kg} \quad \mathbf{K}_u = \begin{bmatrix} 15040 & -8220 \\ -8220 & 8220 \end{bmatrix} \text{ N/m} \quad (20)$$

leading to first and second natural frequencies $f_1 = 1.69$ Hz and $f_2 = 4.63$ Hz. The damping ratio is assumed as 2% for each mode and the ground acceleration has been generated from an EC8-compatible PSD function, obtained for $a_g = 0.25$ g and ground type A.

Figure 12: Two-DOF structure coupled with a NES.

The equations of motion in its uncontrolled and controlled configurations are reported in Eqs. (4) - (10), particularized for the case of $n = 2$, $p = 1$, in which:

$$\mathbf{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_N(t) \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \boldsymbol{\tau} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (21)$$

being $X_1(t)$ and $X_2(t)$ the relative displacements with respect to the ground of the first and second floor of the structure, respectively, and $X_N(t)$ the relative displacement with respect to the ground of the NES.

The design of the NES has been obtained by considering a dummy SDOF structure with mass equivalent to the effective modal mass at the first mode of the original structure $m_{1,e} = 46.65$ kg, frequency equal to the first modal frequency $f_1 = 1.69$ Hz ($\omega_1 = 10.62$ rad/s) and damping ratio 2%. Moreover, the NES mass has been defined as 5% of $m_{1,e}$. NES optimal parameters have been directly determined by substituting Eqs. (15) and (17) in Eq. (19), thus obtaining $m_N = 2.33$ kg, $k_N = 10^{5.05}$ N/m³ and $c_N = 6.74$ Ns/m.

The accuracy of the results has been evaluated against MCSs. The optimisation procedure has been performed minimising two objective functions, J_1 and J_2 , defined as the average of the ratio of the RMS of the displacements of the two storeys with respect to the base in its controlled and uncontrolled configurations over 10000 samples:

$$J_1 = E \left[\frac{RMS \{X_1(t)\}^i}{RMS \{X_{1,u}(t)\}^i} \right] \quad J_2 = E \left[\frac{RMS \{X_2(t)\}^i}{RMS \{X_{2,u}(t)\}^i} \right] \quad (22)$$

Figures 13a-b show J_1 and J_2 values for the nonlinear system having parameters optimised with the empirical approach (marked with red crosses) versus the actual minima of J_1 and J_2 obtained by numerical optimisations involving MCSs (marked with black dots). It can be noted that the relative error committed on the reduction of the RMS of the displacements of the primary structure is less than 0.5%.

Figure 13: Objective functions and their minima computed by numerical optimisations (black dot) and by using the empirical formulae (red cross): (a) J_1 ; (b) J_2 .

Figure 14: Comparison between the uncontrolled and controlled configurations in frequency domain: (a) PSD of the first storey displacements; (b) PSD of the second storey displacements.

The comparison between the controlled and uncontrolled configurations in terms of displacements of each storey has been performed in the frequency-domain in Figures 14a-b. The effectiveness of the NES

is considerable on the first mode of the structure, i.e. on the mode considered for the design of the NES parameters. Although the second modal peak in the PSD function of the response is higher in the controlled configuration, it must be noted that this is negligible corresponding to the reduction in the first modal peak, so that the displacement RMS (related to the area below the PSD curve) is strongly reduced when the NES is attached to the structure.

4.3. 3-D Structure

The proposed optimisation procedure for MDOF structures is applied herein to the case of a low-rise, multi-bay building controlled by two independent NESs located at the top storey of the structure and acting along the x and y geometrical axes. Their performance has been analysed considering seismic excitation, 20 s long, generated from an EC8-compatible PSD function, obtained for $a_g = 0.25$ g and ground type A.

Figure 15: Schematic representation of the 3-D building.

The primary structure is a two-storey steel frame having an overall height of 7.5 m and a mass of approximately 86 tons [27], whose schematic representation is shown in Figure 15. Assuming the floors diaphragm rigid in its own plane while flexible in bending, axial deformations of beams and columns negligible and masses lumped at floors height, the motion of the uncontrolled system is determined by 6 DOFs: for the i -th floor, the translations u_{xi} and u_{yi} in the x and y directions and the rotation $u_{\phi i}$ about the vertical direction with $i = 1, 2$. Hence, the generalised displacement vector of the uncontrolled structure is the following:

$$\mathbf{X}_u(t) = \begin{bmatrix} u_{x1} & u_{y1} & u_{\phi1} & u_{x2} & u_{y2} & u_{\phi2} \end{bmatrix}^T \quad (23)$$

An eccentric centre of mass has been considered, positioned at a distance equal to 5% of the overall building dimensions, both along the x and y directions. The mass and stiffness matrices, expressed in SI units, i.e. lengths are expressed in m, forces in N and masses in kg, can be written, after applying the static condensation technique, as:

$$\mathbf{M}_u = \begin{bmatrix} 44.42 & 0 & -19.32 & 0 & 0 & 0 \\ 0 & 44.42 & 9.66 & 0 & 0 & 0 \\ -19.32 & 9.66 & 736.59 & 0 & 0 & 0 \\ 0 & 0 & 0 & 41.42 & 0 & -19.32 \\ 0 & 0 & 0 & 0 & 41.42 & 9.66 \\ 0 & 0 & 0 & -19.32 & 9.66 & 650.49 \end{bmatrix} \cdot 10^3 \quad (24)$$

$$\mathbf{K}_u = \begin{bmatrix} 15.31 & 0 & 0 & -6.74 & 0 & 0 \\ 0 & 4.10 & 0 & 0 & -2.09 & 0 \\ 0 & 0 & 205.93 & 0 & 0 & -96.73 \\ -6.74 & 0 & 0 & 4.49 & 0 & 0 \\ 0 & -2.09 & 0 & 0 & 1.69 & 0 \\ 0 & 0 & -96.73 & 0 & 0 & 70.63 \end{bmatrix} \cdot 10^7 \quad (25)$$

Furthermore, the damping ratio is set at 2% for each mode. The resulting frequencies and the percentages of the effective modal mass γ at each mode are listed in Table 6.

Table 6: Frequencies and percentage of effective modal mass for the i -th mode.

Mode	f_i [Hz]	γ_x [%]	γ_y [%]
1	1.70	0.004	93.257
2	2.62	53.166	0.072
3	2.94	35.726	0.053
4	5.55	0.000	6.613
5	9.31	2.778	0.005
6	10.68	8.326	0.001

The equations of motion of the combined system are given by Eqs. (4) - (10), in which $n = 6$, $p = 2$, and $\mathbf{X}$, $\mathbf{L}$ and $\boldsymbol{\tau}$ are defined as:

$$\mathbf{X} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{\phi1} \\ u_{x2} \\ u_{y2} \\ u_{\phi2} \\ u_{xN} \\ u_{yN} \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \boldsymbol{\tau} = \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \\ \cos \theta \\ \sin \theta \\ 0 \\ \cos \theta \\ \sin \theta \end{bmatrix} \quad (26)$$

being θ the epicentral direction of the ground motion.

The design of the two NESs has been based on the results of the preliminary modal analysis on the uncontrolled structure. In particular, the NES acting in the y direction has been designed by considering a dummy SDOF structure with natural frequency equal to the 1st modal frequency of the building, i.e. $\omega_1 = 10.67$ rad/s. The second one acting in x direction has been designed with respect to a dummy SDOF structure having natural frequency equal to the 2nd modal frequency ($\omega_2 = 16.43$ rad/s).

In both cases, the damping ratio of the equivalent SDOF has been set to 2% and the NES masses have been calculated as 5% of the effective modal masses for the first two modes, respectively, i.e.:

$$m_{N,y} = 0.05 m_{1e,y} = 0.05 \gamma_y^{(1)} \tau_y^T \mathbf{M}_u \tau_y \quad (27)$$

$$m_{N,x} = 0.05 m_{1e,x} = 0.05 \gamma_x^{(2)} \tau_x^T \mathbf{M}_u \tau_x$$

in which τ_x and τ_y are the influence vectors for the cases of ground motion acting in the x and y directions respectively, obtained by substituting $\theta = 0^\circ$ for x -direction and $\theta = 90^\circ$ for y -direction in the general expression Eq. (27); $\gamma_y^{(1)}$ and $\gamma_x^{(2)}$ are the percentages of effective modal masses at the fundamental modes in y -direction and in x -direction, respectively. The parameters of the two NESs are reported in Table 7.

Figure 16: J_3 and J_4 functions and optimal NES parameters obtained by numerical optimisations (black dot) and by using the empirical formulae (red cross): (a)-(b) $\theta = 0^\circ$; (c)-(d) $\theta = 90^\circ$.

In order to evaluate the accuracy in determining the design parameters of the NES, the latter have been compared to those obtained by numerical optimisations. For this purpose, two objective functions have been selected. These are the mean value, over 10000 simulated samples of the structural response, of the dimensionless ratios between the variances of the magnitude of the inter-storey displacements in its

Table 7: NES parameters for the two NESs obtained using the proposed method.

NES along x -axis			NES along y -axis		
$m_{N,x}$	$k_{N,x}$	$c_{N,x}$	$m_{N,y}$	$k_{N,y}$	$c_{N,y}$
[kg]	[kN/m ³]	[Ns/m]	[kg]	[kN/m ³]	[Ns/m]
2282	684540	10203	4002	198170	11624

controlled and uncontrolled configurations:

$$J_3 = E \left[\frac{\sigma_{d_1}^2}{\sigma_{d_{1,u}}^2} \right] \quad J_4 = E \left[\frac{\sigma_{d_2}^2}{\sigma_{d_{2,u}}^2} \right] \quad (28)$$

with:

$$d_1 = \sqrt{u_{x1}^2 + u_{y1}^2} \quad d_2 = \sqrt{(u_{x2} - u_{x1})^2 + (u_{y2} - u_{y1})^2} \quad (29)$$

$$d_{1,u} = \sqrt{u_{x1,u}^2 + u_{y1,u}^2} \quad d_{2,u} = \sqrt{(u_{x2,u} - u_{x1,u})^2 + (u_{y2,u} - u_{y1,u})^2} \quad (30)$$

where the subscript u stands for uncontrolled configuration.

For the cases of $\theta = 0^\circ$ and $\theta = 90^\circ$, J_3 and J_4 values obtained by using parameters optimised with the empirical formulae (marked with red crosses) versus the present minima of J_3 and J_4 obtained by numerical optimisations (marked with black dots) are reported in Figures 16a-d. The relative error committed on the reduction of the dynamic response of the primary structure is about 5.1%. This confirms that although the empirical formulae have been obtained for SDOF structures, they can be used also for more complex structures.

In order to understand the actual performance of the two NESs, the response of the uncontrolled and controlled configurations have been compared in terms of PSDs of the displacements of the two storeys. Figures 17a-d depict the PSDs of the displacements in the x and y directions, respectively for $\theta = 0^\circ$ and $\theta = 90^\circ$. The effects of the two NESs are prevalent on the first two modes of the structure, which are the modes considered for the design of the NES parameters.

Finally, Figure 18 shows the performance of the system of two orthogonal NESs in terms of J_3 and J_4 for different values of the epicentral direction θ showing a reduction of the structural response varying from 30% to 60% of the uncontrolled case.

4.4. Validation of NES optimisation procedure for historic seismic records

In the previous sections, the proposed empirical formulae have been validated against MCSs for both SDOF and MDOF systems, as shown in Figures 8, 13 and 16, respectively. However, both empirical formulae and MCS results rely on the same assumption of stationary excitations, implicit in the methodology used to numerically generate ground motions compatible with EC8 response spectra. This also entails the same initial conditions are used for all generated ground accelerations, e.g. all signals have the same duration. Conversely, historic earthquakes are inherently non-stationary and have random duration.

Given these limitations, one could wonder if the design of the NES performed using the empirical formulae is actually optimum when the main structure is excited by historical earthquake records rather than numerically generated samples of a stationary process (provided the former are still compatible with

Figure 17: Comparison between the uncontrolled and controlled configurations of the system in terms of: (a)-(b) PSD of the displacements of the first and second floor, respectively, in x direction for $\theta = 0^\circ$; (c)-(d) PSD of the displacements of the first and second floor, respectively, in y direction for $\theta = 90^\circ$.

the selected design response spectrum). Hence, the dynamic response to a set of spectrum-compatible historic earthquake records is herein reported for the two storeys two bays frame introduced in section 4.3.

Seven ground motion accelerations compatible with the EC8 elastic response spectrum used in section 4.3 ($a_g = 0.25$, ground type A) are listed in Table 8. The latter have been selected using the REXEL computer program, version 3.5 [43], by properly setting a magnitude interval (from 6.0 to 7.0) and a distance from the epicentre less than 30 km. Individual ground motion records have been scaled and the spectrum compatibility has been achieved by imposing a $\pm 10\%$ tolerance with respect to the target response spectrum in the period range between 0.10s and 2.0s.

Each ground motion has been used for two different scenarios, with each earthquake acting either along the x or the y direction. The frame has been analysed in its uncontrolled and controlled configuration, in the latter coupled with two NESs, as already described in section 4.3.

In first instance, the NESs parameters have been set equal to the values reported in Table 7, obtained using the proposed empirical formulae. Then, for each scenario, a parametric analysis has been performed varying in turn the nonlinear stiffness and damping of the NES acting parallel to the ground motion (since the other NES, acting orthogonal to the ground motion, has minor influence on the structural response in

Figure 18: J_3 and J_4 functions versus the incidence angle of the ground motion.

Event	Station ID	Earthquake name	Date	Mw	Fault mechanism	Original PGA [g]	Scaled PGA [g]	Scale Factor
1	ST64	Montenegro	15/04/1979	6.9	thrust	0.181	0.25	1.382
2	ST98	Campano Lucano	23/11/1980	6.9	normal	0.060	0.25	4.173
3	ST98	Campano Lucano	23/11/1980	6.9	normal	0.060	0.25	4.174
4	ST161	Golbasi	05/05/1986	6.0	oblique	0.055	0.25	4.558
5	ST2496	South Iceland	21/06/2000	6.4	strike slip	0.116	0.25	2.147
6	ST539	Bingol	01/05/2003	6.3	strike slip	0.298	0.25	0.841
7	ST3311	Avej	22/06/2002	6.5	thrust	0.446	0.25	0.561

Table 8: Characteristic of selected ground motion records.

this case).

In order to maintain consistency with the optimisation criteria introduced in previous sections, the variances of the magnitude of the inter-storey displacement of the second storey of the frame, Eq. (30), subjected to the full set of earthquake records in Table 8, have been determined for each configuration. Hence, the objective functions J_3 and J_4 have been computed according to Eq. (28).

Results of these parametric analyses are shown in Figures 19, for both cases of ground motions acting parallel to the x and y directions. As it can be easily observed, the configuration with NES parameters determined by empirical formulae (indicated with a red dot in Figure 19) is quasi-optimal for both nonlinear stiffness and damping, despite the non-stationarity of the excitation.

The analysis has been repeated for 5 different earthquake set (each set consistent of seven historic earthquake records scaled to be compatible with the same EC8 elastic response spectrum), confirming the results shown in Figure 19. Hence, it can be concluded that, with respect to the chosen optimisation criterion, the empirical formulae proposed in Eqs. (15) and (16) maintain their validity for the design of structures in realistic scenarios, when actual non-stationary historic ground motion must be considered, despite been derived under the simplifying assumptions introduced in the previous sections.

Finally, in order to show the effects of the optimised NES on the structure, the deterministic response to the Golbasi 1986 earthquake is reported (Table 8, seismic record #4), assuming the ground excitation to be

Figure 19: Objective functions versus NES nonlinear stiffness for the 3D frame excited by spectrum compatible historical earthquake records: (a) J_3 and (b) J_4 for ground motion in the x direction; (c) J_3 and (d) J_4 for ground motion in the y direction.

acting parallel to the y direction.

The second floor inter-storey displacements d_2 for the uncontrolled and controlled frame configurations (optimised according to the empirical formulae) are compared in Figure 20a. Further, in order to show resonance phenomena between the structure and the control device, Figures 20b-d show time-frequency dependency of the NES displacement response and of second floor inter-storey displacements d_2 in its uncontrolled and controlled configuration, respectively, performed by wavelet analysis. The latter is very useful to study the dynamic behaviour of a nonlinear system since it highlights energy distribution in the frequency domain for short time windows, rather than averaging on the entire time history like the Fourier transform does. [44].

In each wavelet plot, darkly shaded areas correspond to regions where the amplitude of the wavelet transform is high, which reveals the presence of a significant frequency component, whereas lightly shaded regions correspond to low amplitudes. A Morlet wavelet transform was applied to the time domain signals with sampling frequency equal to 100 Hz and frequency of the mother wavelet equal to 1 Hz.

As shown in Figure 20b, the NES mass is strongly excited by the input seismic wave and it resonates

Figure 20: 3D frame response to Golbasi 1986 earthquake: (a) interstorey drift in uncontrolled and controlled configuration; wavelet transform of (b) NES displacements and interstorey drift in (c) uncontrolled and (d) controlled configurations.

in a wider frequency range than the linear principal frequency of the structure (equal to 1.70 Hz, see Table 6). Further, Figures 20c and 20d emphasize how the amplitude of the inter-storey displacements are much lower in the controlled configuration, particularly in the time window between 4 and 14 seconds, when the intensity of the seismic wave is the largest.

5. Conclusions

Although several studies investigated the complex behaviour of the Nonlinear Energy Sink (NES), its optimal design in several scenarios is still a challenge because of its sensitivity to loading perturbations. When the NES is used to mitigate the dynamic response of a structure induced by seismic actions, its optimal parameters are generally determined using optimisation analyses which require a considerable computational effort.

In this paper, an approximate formulation able to estimate the design parameters of a type-I NES configuration subjected to seismic loads has been developed for pre-design purposes. Optimal configurations

of the NES are defined by minimising the statistics of the structural response when subjected to a stationary process consistent with elastic Eurocode 8 response spectra. In particular, the NES efficiency is evaluated as the dimensionless ratio between the RMS of displacements of the primary structure in its controlled (i.e. with attached NES) and uncontrolled (i.e. without NES) configurations in a stochastic framework.

Accurate empirical formulae for the optimal NES stiffness and damping have been derived by a curve fitting procedure of the numerical results.

Based on these assumptions, the following conclusions can be drawn:

- It has been proven that despite the complex and strong nonlinear behaviour of the NES, for given NES efficiency the optimal nonlinear stiffness is inversely proportional to the square of the peak ground acceleration, differently from the case of white noise excitation in which it is inversely proportional to the PSD amplitude. Conversely, the optimal NES damping is almost independent of the load amplitude, as in the case of white noise.
- With respect to the other parameters, the optimal nonlinear stiffness increases with the frequency and damping of the primary structure and with the mass ratio ε , as already seen for white noise excitations. However, the relationships among these parameters are here nonlinear. On the other hand, the optimal damping ratio increases with the structural frequency and mass ratio and decreases for increasing structural damping ratio, showing a trend similar to the case of white noise excitations.
- To determine the optimal value of the NES damping, using the expression proposed for the case of white noise excitations does not induce significant errors.
- The proposed approach can be applied also to the case of MDOF structure controlled by one or multiple NESs. The NESs design can be carried out by considering dummy SDOF systems whose dynamic characteristics depend on the fundamental modes of the structure. Using this approach delivers the optimal design parameters of the NES with the same computational effort required to estimate the modal parameters of the structure, while, otherwise, a considerably longer computational effort would be required to implement a Monte Carlo Simulation using time-domain analysis.
- Despite being derived under the assumption of stationary excitations, the empirical formulae are valid for NES design when the main structure is subjected to non-stationary historic ground motions, provided that the historic records are compatible with the same design spectrum.

The proposed formulation allows practitioner engineers to properly design NESs able to reduce the seismic response of a structure for a specific site, once selected the ground type and the design ground acceleration of the site in which the structure arises, with a maximum relative error committed in the evaluation of the NES performance below 5%.

Appendix A. Supplementary Design Charts

Herein, the design charts obtained directly using the Eqs. (15) and (17) are reported for the case of $\varepsilon = 0.05$ and EC8 ground types B to E.

Figure A.21: Design chart to determine the optimal parameters values for different values of f_1 and ζ_1 for $\varepsilon = 0.05$: (a) soil type B; (b) soil type C.

Figure A.22: Design chart to determine the optimal parameters values for different values of f_1 and ζ_1 for $\varepsilon = 0.05$: (a) soil type D; (b) soil type E.

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