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Model predictive control: terminal region and terminal weighting matrix

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The manuscript was received on 25 October 2007 and was accepted after revision for publication on 5 December 2007.

DOI: 10.1243/09596518JSCE509

Abstract: Many efforts have been made for model predictive control MPC to enlarge terminal region where the stability is guaranteed by including a proper terminal weighting. This paper investigates how to maximize the ellipsoidal terminal region by using a more general method to design MPC: calculating terminal region and terminal weighting separately. With Jordan canonical form, the new ellipsoid-based method can result in terminal regions which are infinite (in certain directions) even in the case of open-loop unstable systems. Some most popular methods are just special cases of this new method. Simulation results illustrate the effectiveness of the proposed algorithms.

Keywords: model predictive control (MPC), terminal region, terminal weighting matrix, linear matrix inequalities (LMI)

1 INTRODUCTION

Model predictive control (MPC), also referred to as receding horizon control (RHC), has been broadly adopted in industry, especially as a promising means to deal with multivariable constrained control problems [1–4]. Stability has been one of the main problems in MPC since early MPC was criticized for its loss of stability [4]. After over two decades' study, stability of MPC is now reaching its pre-mature stage and many methods have been presented. Terminal penalty techniques are widely used to address the stability issue of MPC [5–17]. As discussed in reference [3], the core idea behind most of these conditions is to add a terminal weighting term in the performance index and impose constraints on the state in the end of the horizon, i.e. the terminal state, within a region, referred to as a terminal region, to address the stability and feasibility. This idea has also been extended from linear systems to non-linear systems, e.g. [18–20]. Besides terminal penalty techniques, stability-enforcing constraints are often used to develop stable MPC algorithms, where

stability is achieved by adding extra constraints into the on-line optimization problem in order to enforce the state to contract to the origin in each step; see e.g. [21–23]. Recently, the stability of MPC has been established by utilizing an appropriately designed backup controller and coupling it with MPC implementation [24] as well as Lyapunov-based predictive control designs that guarantee feasibility from an explicitly characterized set (not restricted to the terminal region) of initial conditions [25], and these results were extended to the cases where state constraints [26], uncertainty [27], and rate constraints [28] need to be considered.

This paper aims to shed a little more light on the widely used terminal penalty techniques. As is well known, the introduction of the terminal penalty and the terminal region have boosted the research work on the stability of MPC for systems with input constraints. How to expand the terminal region as large as possible is an important issue. In the last decade, many research works have considered the terminal region as either an ellipsoidal region [5–11] or a polyhedral region [12–14].

The ellipsoidal terminal region of MPC is usually determined by a positive definite matrix, and it is usually calculated simultaneously with the terminal penalty in existing literature. One early and popular

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ellipsoid-based method in reference [5] adopts the terminal weighting matrix to define the terminal region. The terminal region is then maximized by solving the offline optimization problem formulated in linear matrix inequalities (LMI) in terms of the terminal weighting matrix and stabilizing constant state feedback gains. In the terminal region, when the terminal weighting matrix is applied, the performance index of online optimization is non-increasing, and, when the state feedback gains are applied, the input constraints are satisfied. To enlarge the terminal region further, an extra parameter μ is introduced as a tuning knob in reference [6], where the terminal region is determined by μ times the terminal weighting matrix. The ellipsoid-based methods are widely used in the research work on MPC; e.g. see [5–10], even for non-linear systems [6], [9] and robust cases [10]. In the literature, similarly, the terminal region is defined with either of the above two strong relationships to the terminal weighting matrix. Although these relationships might be useful to handle system uncertainties, they are very likely to result in conservativeness when maximizing the terminal region. Furthermore, because the terminal weighting matrix must be positive definite, the associated ellipsoidal terminal regions are unlikely to be infinite.

This paper attempts to remove the above relationships between terminal region and terminal weighting, and investigate the potential of maximizing the terminal region of MPC by calculating terminal region and terminal weighting matrix separately. Although some papers require no such relationships [3], they focus more on developing general MPC algorithms rather than studying the potential of maximizing the terminal region of MPC. They fail to identify the idea of calculating terminal region and terminal weighting matrix separately as a most important factor in the maximization of the terminal region. Actually, to the authors' knowledge no paper has explicitly studied or even mentioned this idea.

The work reported in reference [11] is an interesting attempt to remove the above relationships. It defines the stability region and terminal region as two different ellipsoids with no explicit relationship, and every state in the stability region must be able to be driven into the terminal region within just one predictive horizon by the MPC controller. However, the paper [11] does not point out the importance of removing the above relationships. The size of the stability region in reference [11] mainly depends on the length of the predictive horizon, and, as with other ellipsoid-based methods,

it also seems unlikely to achieve an infinite stability region.

In many other works [12–14], the terminal region is alternatively expressed as a polyhedral region. The result given in reference [12] is very attractive, where the terminal region is defined as a polyhedron by using closed-loop system eigenvectors. For any given stabilizing constant state feedback gains, an appropriate polyhedral terminal region is derived in conjunction with input constraints. Stability is guaranteed by applying an adequate finite terminal weight corresponding to the terminal region. It is reported that such a polyhedral terminal region could be infinite (in certain directions) if the system has some stable modes. However, the stabilizing feedback gains need to be pre-determined, while no details are available in reference [12] about whether it is possible or how to calculate the stabilizing feedback gains in terms of the maximization of the terminal region. Polyhedron-based methods are also extended to both non-linear systems [13] and robust cases [14]. A common problem to polyhedron-based methods is that the terminal region becomes very complicated when the plant is a high-order system.

This paper follows a common MPC practice and employs a terminal penalty in conjunction with an ellipsoidal terminal region, but calculates them separately in order to allow a higher degree of freedom with which to maximize the terminal region. The terminal region is determined by a semi-positive definite terminal region matrix, which is calculated simultaneously with a constant terminal state feedback gain. When the terminal state feedback gain is available, a positive definite terminal weighting matrix is determined such that, when it is applied to the online optimization, the performance index is non-increasing and then the closed-loop stability can be guaranteed. To weaken the influence of numerical computation errors, a system transformation into its Jordan canonical form may be necessary to de-couple its stable and unstable modes. Then, the above calculations become relatively easier to solve, and the maximized terminal region is infinite in the direction of stable modes of the open-loop system.

2 PROBLEM FORMULATION

Consider the following linear discrete-time system

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \quad (1)$$

with control constraints

$$|u_i| \leq \bar{u}_i, \quad i = 1, \dots, m \quad (2)$$

where k is the discrete time index, $\mathbf{x} \in R^n$ represents the system state, $\mathbf{u} \in R^m$ is the input vector, and $\bar{u}_i$, $i = 1, \dots, m$ is the input constraints. The MPC aims, at state $\mathbf{x}(k)$ and time instant k , to solve a minimization problem formulated as

$$\min_{\mathbf{u}(k|k), \dots, \mathbf{u}(k+N-1|k)} J(k) \quad (3)$$

subject to equations (1) and (2), where

$$J(k) = \mathbf{x}(k+N|k)^T \mathbf{P} \mathbf{x}(k+N|k) + \sum_{i=0}^{N-1} \left[\mathbf{x}(k+i|k)^T \mathbf{Q} \mathbf{x}(k+i|k) + \mathbf{u}(k+i|k)^T \mathbf{R} \mathbf{u}(k+i|k) \right], \quad (4)$$

is the performance index, $\mathbf{Q} > 0$ (or $\mathbf{Q} \geq 0$ and $[\mathbf{A}, \mathbf{Q}^{1/2}]$ is detectable) and $\mathbf{R} > 0$ are state and control weighting matrices respectively, $\mathbf{P} > 0$ is the terminal weighting matrix, and N is the length of the predictive horizon. When an optimized control sequence $\mathbf{u}^*(k|k), \dots, \mathbf{u}^*(k+N-1|k)$ is obtained, the MPC law is then determined by

$$\mathbf{u}(k) = \mathbf{u}^*(k|k) \quad (5)$$

As is well known, the terminal penalty term $\mathbf{x}(k+N|k)^T \mathbf{P} \mathbf{x}(k+N|k)$ in the performance index (4) is introduced for stability purposes. For any initial state $\mathbf{x}_0$, if it is feasible for MPC to steer the terminal state $\mathbf{x}(N|0)$ to a certain region, usually called terminal region, then the stability of MPC is guaranteed for any $k > 0$ when the terminal penalty is applied. The terminal region, denoted as ν in this paper, is defined as a region where, once the terminal state $\mathbf{x}(k+N|k)$ under the control sequence $\mathbf{u}^*(k|k), \dots, \mathbf{u}^*(k+N-1|k)$ yielded by minimization of the cost (4), arrives, there exists a control sequence $\mathbf{u}(k+N+i|k)$, $i = 0, 1, \dots, \infty$, satisfying the constraints (2), which can steer the state to the origin. In general, the terminal control law is described as

$$\mathbf{u}(k+N|k) = \mathbf{K}_{\text{term}} \mathbf{x}(k+N|k) \quad (6)$$

where $\mathbf{K}_{\text{term}}$ is the terminal feedback gain.

As in most relevant literature this paper regards the terminal region as a stability region. Owing to the input constraints (2), the terminal region is usually just a subset of the whole state space. It is very difficult, especially for complicated systems, to tell what exactly is the theoretical maximum terminal

region, but it is possible and practicable to find a subset of it. The ellipsoid-based methods suppose the terminal region is an ellipsoidal region, and usually formulate it as following

$$\nu = \{ \mathbf{x} \in R^n | \mathbf{x}^T \mathbf{Z} \mathbf{x} \leq 1 \} \quad (7)$$

where $0 < \mathbf{Z} \in R^{n \times n}$. Obviously, this ellipsoidal terminal region is determined by the positive definite matrix $\mathbf{Z}$. In order to distinguish from the terminal weighting matrix (TWM) $\mathbf{P}$, $\mathbf{Z}$ is referred to as terminal region matrix (TRM). How to find an ellipsoidal terminal region as large as possible and how to determine the corresponding $\mathbf{P}$ is the focus of this paper. Many results have now been developed based on the introduction of some artificial relationships between $\mathbf{P}$ and $\mathbf{Z}$, e.g. $\mathbf{P} = \mathbf{Z}$ or $\mathbf{P} = \mu \mathbf{Z}$. With either of these relationships, the terminal region can be easily maximized by optimizing $\mathbf{P}$. However, the cost for introducing these artificial relationships is conservativeness. Consequently, these ellipsoidal terminal regions are usually very small, and according to the relevant literature, it seems impossible to achieve an infinite terminal region even if the system itself is stable.

In this paper, these artificial relationships between $\mathbf{P}$ and $\mathbf{Z}$ are to be removed so that the ellipsoidal terminal region could be maximized with a higher degree of freedom. As will be illustrated later, the attempt is successful, the terminal region is significantly enlarged, and, furthermore, an infinite terminal region is achievable as long as the system has stable modes.

3 NEW MPC ALGORITHM

The design of the new MPC algorithm is described in this section, where no explicit relationship is needed between the terminal weighting matrix and the terminal region of the MPC. Therefore, the new MPC is referred to as MPC separating weighting and region (MPCSWR).

3.1 Minimization problem 1 (MP1)

This problem aims to maximize the terminal region, and it can be formulated as a convex optimization problem

$$\min_{\mathbf{S}, \mathbf{S}^{-1}} \log(\det(\mathbf{S}^{-1})) \quad (8)$$

subject to

$$\begin{bmatrix} \mathbf{S} & (\mathbf{AS} + \mathbf{B}\bar{\mathbf{S}})^T \\ \mathbf{AS} + \mathbf{B}\bar{\mathbf{S}} & \mathbf{S} \end{bmatrix} > 0 \quad (9)$$

$$\begin{bmatrix} \mathbf{Y} & \bar{\mathbf{S}} \\ \bar{\mathbf{S}}^T & \mathbf{S} \end{bmatrix} \geq 0, \quad Y_{jj} \leq \bar{u}_j^2, \quad j=1, \dots, m \quad (10)$$

where $0 < \mathbf{S} \in R^{n \times n}$ and $\bar{\mathbf{S}} \in R^{m \times n}$. Let the TRM be

$$\mathbf{Z} = \mathbf{S}^{-1} \quad (11)$$

then the terminal region is determined by

$$\mathcal{V} = \{ \mathbf{x} \in R^n \mid \mathbf{x}^T \mathbf{Z} \mathbf{x} \leq 1 \} = \{ \mathbf{x} \in R^n \mid \mathbf{x}^T \mathbf{S}^{-1} \mathbf{x} \leq 1 \} \quad (12)$$

and the associated terminal control gain is

$$\mathbf{K}_{\text{term}} = \bar{\mathbf{S}} \mathbf{S}^{-1} = \bar{\mathbf{S}} \mathbf{Z} \quad (13)$$

Remark 1

In effect, MP1 just minimizes $\mathbf{Z}$, but has nothing to do with $\mathbf{P}$. Since a large $\mathbf{P}$ will probably degrade the performance of MPC, for the optimality purpose, $\mathbf{P}$ should be as small as possible. Therefore, another minimization problem needs to be solved not to determine $\mathbf{Z}$ but to minimize $\mathbf{P}$.

3.2 Minimization problem 2 (MP2)

Suppose MP1 is completed and consequently $\mathbf{K}_{\text{term}}$ is already available. Then, based on $\mathbf{K}_{\text{term}}$, this problem aims to find the minimal $\mathbf{P}$. It is formulated also as a convex optimization problem

$$\min_{\mathbf{W}} \log(\det(\mathbf{W}^{-1})) \quad (14)$$

subject to

$$\begin{bmatrix} \mathbf{W} & \mathbf{W}^T(\mathbf{A} + \mathbf{B}\mathbf{K}_{\text{term}})^T & (\mathbf{Q}^{1/2}\mathbf{W})^T & (\mathbf{K}_{\text{term}}\mathbf{W})^T \\ (\mathbf{A} + \mathbf{B}\mathbf{K}_{\text{term}})\mathbf{W} & \mathbf{W} & \mathbf{0} & \mathbf{0} \\ \mathbf{Q}^{1/2}\mathbf{W} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{K}_{\text{term}}\mathbf{W} & \mathbf{0} & \mathbf{0} & \mathbf{R}^{-1} \end{bmatrix} \geq 0 \quad (15)$$

where $0 < \mathbf{W} \in R^{n \times n}$. Then, the TWM is determined by

$$\mathbf{P} = \mathbf{W}^{-1} \quad (16)$$

Remark 2

By solving MP1 and then MP2, it is possible to achieve not only a large terminal region but also a relatively

small terminal penalty. This means that both stability and optimality of MPC could be improved when the link between $\mathbf{P}$ and $\mathbf{Z}$ is cut off.

4 STABILITY AND FEASIBILITY

Theorem 1

Suppose there is a solution to MP1, i.e. there exist matrices $\mathbf{S}$ and $\bar{\mathbf{S}}$ such that conditions (9) and (10) hold. Then MP2 is solvable, i.e. there exists at least one matrix $\mathbf{W}$ such that condition (15) is satisfied.

Proof

By using the transform (11), (13), and (16), it can be proved that conditions (9) and (15) are equivalent to

$$\mathbf{Z} > (\mathbf{A} + \mathbf{B}\mathbf{K}_{\text{term}})^T \mathbf{Z} (\mathbf{A} + \mathbf{B}\mathbf{K}_{\text{term}}) \quad (17)$$

$$\mathbf{P} \geq (\mathbf{A} + \mathbf{B}\mathbf{K}_{\text{term}})^T \mathbf{P} (\mathbf{A} + \mathbf{B}\mathbf{K}_{\text{term}}) + \mathbf{Q} + \mathbf{K}_{\text{term}}^T \mathbf{R} \mathbf{K}_{\text{term}} \quad (18)$$

respectively. MP1 and MP2 are solvable if there exist $\mathbf{Z}$, $\mathbf{K}_{\text{term}}$, and $\mathbf{P}$ such that conditions (17) and (18) hold.

If equation (17) holds, since $\mathbf{Q} \geq 0$, $\mathbf{R} > 0$, and $\mathbf{K}_{\text{term}}$ are all finite constants, there always exists a scalar $0 < \mu \in R$ to make

$$\mu \mathbf{Z} \geq \mu (\mathbf{A} + \mathbf{B}\mathbf{K}_{\text{term}})^T \mathbf{Z} (\mathbf{A} + \mathbf{B}\mathbf{K}_{\text{term}}) + \mathbf{Q} + \mathbf{K}_{\text{term}}^T \mathbf{R} \mathbf{K}_{\text{term}} \quad (19)$$

hold. This implies that $\mathbf{P} = \mu \mathbf{Z}$ is a solution to MP2. So, MP2 is solvable if MP1 is solvable.

QED

Remark 3

Theorem 1 shows that the idea of separating the TWM and the TRM is feasible. As long as a maximum terminal region is determined by solving MP1, there surely exists a feasible TWM, and furthermore, it can be minimized by solving MP2 for the benefit of the performance of MPC.

Theorem 2

Consider a discrete-time linear system (1) subject to the input constraints (2). Suppose there exist matrix $0 < \mathbf{S} \in R^{n \times n}$, $\bar{\mathbf{S}} \in R^{m \times n}$, and $0 < \mathbf{W} \in R^{n \times n}$ such that conditions (9), (10), and (15) hold. Let the terminal

region and the TWM be determined according to equations (12) and (16) respectively, then the MPC optimization problem (3) subject to (1), (2), and $\mathbf{x}(k+N|k) \in \nu$ is feasible for all $k \geq 0$ and all initial states $\mathbf{x}_0 \in \nu$. Moreover, the MPC stemming from this optimization problem exponentially stabilizes the system for all initial states $\mathbf{x}_0 \in \nu$ while satisfying the constraints (2).

Proof

In the same fashion as in reference [5], it can be demonstrated that the satisfaction of conditions (9) and (10) guarantees the feasibility of MPCSWR, i.e. the input constraints (2) and $\mathbf{x}(k+N|k) \in \nu$ can always be satisfied, while condition (15) guarantees the exponential stability.

QED

Remark 4

From Theorem 2, it can be seen that no explicit link between $\mathbf{W}$ and $\mathbf{S}$, i.e. $\mathbf{P}$ and $\mathbf{Z}$, is necessary. Therefore, when those artificial links such as $\mathbf{P} = \mathbf{Z}$ in reference [5] and $\mathbf{P} = \mu\mathbf{Z}$ in reference [6] are cut off, a higher degree of freedom is achieved to maximize terminal region. Actually, the work reported in references [5] and [6] can be considered as two special cases of the general algorithm presented in the current paper. All results reported in references [5] and [6] can easily be derived by adding new conditions $\mathbf{P} = \mathbf{Z}$ and $\mathbf{P} = \mu\mathbf{Z}$ respectively, which, as will be shown by simulation results later, result in conservative terminal regions.

Remark 5

Since conditions (9), (10), and (15) are all in LMI format, it is possible to extend the results given by Theorems 1 and 2 to non-linear systems by utilizing linear differential inclusion (LDI) techniques. Basically, a minimum convex hull needs to be defined to cover the LDI of the original non-linear system, and this convex hull is known as the relaxed LDI of the non-linear system. All vertices of the convex hull are then used to design the MPC algorithm. As every trajectory of the non-linear system is also a trajectory of the relaxed LDI, a MPC that can stabilize the relaxed LDI can also stabilize the original non-linear system [29].

5 JORDAN CANONICAL FORM

Consider a system described by equations (1) and (2), where $(\mathbf{A}, \mathbf{B})$ is stabilizable. Let $\mathbf{x}(k) = \mathbf{Y}\bar{\mathbf{x}}(k)$ be

the transformation which brings the system (1) into its Jordan canonical form, so that

$$\begin{cases} \bar{\mathbf{x}}(k+1) = \mathbf{Y}^{-1}\mathbf{A}\mathbf{Y}\bar{\mathbf{x}}(k) + \mathbf{Y}^{-1}\mathbf{B}\mathbf{u}(k) \\ \quad = \begin{bmatrix} \bar{\mathbf{A}}_s & 0 \\ 0 & \bar{\mathbf{A}}_u \end{bmatrix} \bar{\mathbf{x}}(k) + \begin{bmatrix} \bar{\mathbf{B}}_s \\ \bar{\mathbf{B}}_u \end{bmatrix} \mathbf{u}(k) \\ \bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0 \end{cases} \quad (20)$$

where $\bar{\mathbf{A}}_s \in R^{n_s \times n_s}$ has all its eigenvalues strictly inside the unit circle, whereas $\bar{\mathbf{A}}_u \in R^{n_u \times n_u}$ has all its eigenvalues on or outside the unit circle and $n_s + n_u = n$. The assumption of stabilizability implies that $(\bar{\mathbf{A}}_u, \bar{\mathbf{B}}_u)$ is controllable.

Suppose the ellipsoidal terminal region is determined by

$$\nu = \{\mathbf{x} \in R^n | \mathbf{x}^T \bar{\mathbf{Z}} \mathbf{x} \leq 1\} \quad (21)$$

where $\bar{\mathbf{Z}}$ is the TRM and is a semi-positive definite matrix such that the terminal region could be infinite.

Because $(\bar{\mathbf{A}}_s, \bar{\mathbf{B}}_s)$ is stable, one can choose the terminal gain and the TRM respectively as

$$\bar{\mathbf{K}}_{\text{term}} = [\bar{\mathbf{K}}_s \quad \bar{\mathbf{K}}_u] = [0 \quad \bar{\mathbf{K}}_u] \quad (22)$$

$$\bar{\mathbf{Z}} = \begin{bmatrix} 0 & 0 \\ 0 & \bar{\mathbf{Z}}_u \end{bmatrix} \quad (23)$$

where $\bar{\mathbf{K}}_u$ and $\bar{\mathbf{Z}}_u$ are determined by solving MP1. Then, for the constrained system (1), the terminal gain is

$$\mathbf{K}_{\text{term}} = \bar{\mathbf{K}}_{\text{term}} \mathbf{Y}^{-1} = [0 \quad \bar{\mathbf{K}}_u] \mathbf{Y}^{-1} \quad (24)$$

and the corresponding terminal region is $\nu = \{\mathbf{x} \in R^n | \mathbf{x}^T \mathbf{Z} \mathbf{x} \leq 1\}$ where

$$\mathbf{Z} = (\mathbf{Y}^{-1})^T \bar{\mathbf{Z}} \mathbf{Y}^{-1} \quad (25)$$

Based on $\mathbf{K}_{\text{term}}$, $\mathbf{P}$ can be optimized by solving MP2.

Remark 6

Condition (23) implies that the ellipsoidal terminal region is infinite in certain directions, and these directions are related to the stable mode $(\bar{\mathbf{A}}_s, \bar{\mathbf{B}}_s)$. According to condition (9) in MP1

$$(\bar{\mathbf{A}}_u + \bar{\mathbf{B}}_u \bar{\mathbf{K}}_u)^T \bar{\mathbf{Z}}_u (\bar{\mathbf{A}}_u + \bar{\mathbf{B}}_u \bar{\mathbf{K}}_u) < \bar{\mathbf{Z}}_u \quad (26)$$

which implies

$$\begin{aligned}
& \begin{bmatrix} 0 & 0 \\ 0 & (\bar{\mathbf{A}}_u + \bar{\mathbf{B}}_u \bar{\mathbf{K}}_u)^T \bar{\mathbf{Z}}_u (\bar{\mathbf{A}}_u + \bar{\mathbf{B}}_u \bar{\mathbf{K}}_u) \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ 0 & \bar{\mathbf{Z}}_u \end{bmatrix} \\
& \Leftrightarrow \left(\begin{bmatrix} \bar{\mathbf{A}}_s & 0 \\ 0 & \bar{\mathbf{A}}_u \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_s \\ \bar{\mathbf{B}}_u \end{bmatrix} \bar{\mathbf{K}}_{\text{term}} \right)^T \bar{\mathbf{Z}} \left(\begin{bmatrix} \bar{\mathbf{A}}_s & 0 \\ 0 & \bar{\mathbf{A}}_u \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_s \\ \bar{\mathbf{B}}_u \end{bmatrix} \bar{\mathbf{K}}_{\text{term}} \right) \leq \bar{\mathbf{Z}} \\
& \Leftrightarrow (\mathbf{A} + \mathbf{B} \mathbf{K}_{\text{term}})^T \mathbf{Z} (\mathbf{A} + \mathbf{B} \mathbf{K}_{\text{term}}) \leq \mathbf{Z} \quad (27)
\end{aligned}$$

Although condition (27) is different from condition (17), it is easy to prove that the feasibility in Theorem 2 is still guaranteed by equation (27). The stability in Theorem 2 depends on whether MP2 is solvable under equation (26). Since $\bar{\mathbf{A}}_s$ is stable, one can find a matrix $\bar{\mathbf{Z}}_s > 0$ such that

$$\bar{\mathbf{A}}_s^T \bar{\mathbf{Z}}_s \bar{\mathbf{A}}_s < \bar{\mathbf{Z}}_s \quad (28)$$

Then, it follows from equations (22), (26), and (28) that

$$\mathbf{P} = \mu \begin{bmatrix} \bar{\mathbf{Z}}_s & 0 \\ 0 & \bar{\mathbf{Z}}_u \end{bmatrix} \quad (29)$$

is a feasible solution to MP2, as long as μ is large enough. Therefore, Theorem 2 still holds.

6 NUMERICAL EXAMPLES

There are two objectives in the simulation study. First, the new method is compared with some other MPC algorithms to find out whether or not the terminal region is significantly enlarged after introducing the idea of designing the TWM and the TRM separately. Second, it is necessary to investigate the gap between the terminal region the new MPC achieves and the possible maximum stability region in order to see how much room is left further to enlarge the terminal region of MPC.

6.1 Compared with some other MPC methods

In this sub-section, the system used in reference [12] is borrowed so that comparison could be made between the terminal regions resulting from different MPC methods. The system matrix and the control matrix are respectively

$$\mathbf{A} = \begin{bmatrix} 0.8750 & 1.1250 \\ 0.3750 & 1.1625 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (30)$$

and the input constraint is $|u| \leq 1$. This is an unstable but controllable system. To apply the MPC scheme to stabilize this system, the prediction horizon is chosen as $N=3$ and the weighting

matrices in the performance index (4) are chosen as

$$\mathbf{Q} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \quad \mathbf{R} = 1 \quad (31)$$

First, two popular ellipsoid-based methods are tried; one requires $\mathbf{P} = \mathbf{Z}$ [5] and the other introduces the parameter μ so that $\mathbf{P} = \mu \mathbf{Z}$ [6]. Then, the MPCSWR method is tried twice, one time with the original system and the other time with the corresponding Jordan canonical form. The terminal regions are plotted in Fig. 1, where Fig. 1(a) shows the state space around the origin, while in order better to illustrate finite terminal regions and infinite ones, Fig. 1(b) and Fig. 1(c) show two areas that are far away from the origin of the state space. The result of the polyhedron-based method in reference [12] is also included in Fig. 1. Table 1 gives the volume of each terminal region.

The following deductions can be made from Fig. 1:

- compared with previous ellipsoid-based methods, the new method proposed in this paper significantly enlarges the terminal region;
- based on the corresponding Jordan canonical form, the new method results in a terminal region, i.e. TR_5 , infinite in the direction of stable mode;
- the new method is even more effective than the polyhedron-based method reported in reference [12].

In fact, unless it is possible to pre-determine the stabilizing constant gains in terms of the maximization of the terminal region, the result in reference [12] is even worse locally than TR_2 , as illustrated in Fig. 1(a).

As discussed before, MPCSWR achieves a large terminal region at the cost of control performance, as illustrated in Fig. 2, where for the sake of fair comparison, the summing up cost, i.e. $\sum_{i=0}^k [x(i)^T Q x(i) + u(i)^T R u(i)]$, $k = 1, \dots, \infty$ is introduced to assess control performances. Following the idea of calculating TWM and TRM separately, the maximization of the terminal region often leads to a large TWM, as shown in Table 1, where MPCSWR uses a TWM thousands of times larger than the TWM adopted by the MPC in reference [5]. Since TWM defines an upper bound for control performance, MPCSWR with a large TWM could end up with poor control performance.

However, the actual control performance achieved by MPCs depends not only on TWM, but also on the online optimization process. From Fig. 2, it can be seen that MPCSWR still achieves a satisfactory

Fig. 1 Terminal regions determined by different MPC algorithms

Table 1 K_{term} , TRM, TWM and volume of terminal regions

	TR_1 [5]	TR_2 [6]	TR_3 [12]	TR_4	TR_5
K_{term}	$-[1.8885 \ 5.5671]$	$-[1.0002 \ 3.0007]$	$-[1.2000 \ 3.6000]$	$-[1.0009 \ 3.0029]$	$-[1.0008 \ 3.0024]$
TRM	$\begin{bmatrix} 32.25 & 91.35 \\ 91.35 & 390.97 \end{bmatrix}$	$\begin{bmatrix} 1.0007 & 3.0018 \\ 3.0018 & 9.0063 \end{bmatrix}$	(polyhedral region)	$\begin{bmatrix} 1.0024 & 3.0070 \\ 3.0070 & 9.0204 \end{bmatrix}$	$\begin{bmatrix} 1.0016 & 3.0048 \\ 3.0048 & 9.0144 \end{bmatrix}$
TWM	$\begin{bmatrix} 32.25 & 91.35 \\ 91.35 & 390.97 \end{bmatrix}$	$10\,000^* \begin{bmatrix} 0.4204 & 1.2544 \\ 1.2544 & 3.7566 \end{bmatrix}$	$\begin{bmatrix} 69.97 & 201.58 \\ 201.58 & 713.09 \end{bmatrix}$	$10\,000^* \begin{bmatrix} 1.0944 & 3.2808 \\ 3.2808 & 9.8499 \end{bmatrix}$	$100\,000^* \begin{bmatrix} 0.1391 & 0.4171 \\ 0.4171 & 1.2520 \end{bmatrix}$
Volume	0.05	74.02	∞	194.82	∞

Fig. 2 Control performances ($x_0 = [-6.9 \ 2.2]'$)

control performance. Actually, different MPCs achieve very similar control performances in this simulation. Further to value MPCSWR, when the system state is far away from the origin or on the boundary of the terminal region, stability other than performance should be of higher priority for a controller, as illustrated in Fig. 3, where the MPCs in references [5] and [12] become unstable although they have relatively small TWMs. In practice, time-varying terminal penalty can also be used in order dynamically to trade off between stability and performance, e.g. see references [30] and [31]. Therefore, MPCSWR should have a role to play in real implementations of MPC.

6.2 How far away from the possible maximum stability region?

Much work has been done to enlarge the stability/terminal region of MPC for systems with input constraints, but how close the estimated region of MPC is to the possible maximum stability region still remains an unsolved problem. The main objective of this subsection is to investigate whether the estimated stability region of the MPC algorithm developed in this paper is close to the possible maximum stability region.

In this subsection, the possible maximum stability region of the following system is investigated

Fig. 3 Control performances ($x_0 = [-150 \ 50]'$)

Fig. 4 An inverted pendulum system mounted on a cart

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 10 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ -0.015 \end{bmatrix} u(t) \quad (32)$$

which is derived from a simplified inverted pendulum system under some assumptions. The angle is represented by θ and $u(t)$ is the external force, as depicted in Fig. 4. More details about this inverted pendulum system can be found in reference [32]. However, in the simulation study, the physical background and those assumptions for simplifying the inverted pendulum system are ignored, in order better to concentrate on the objective of studying the possible maximum stability region. The input constraint on the system (32) is

$$|u(t)| \leq 100 \quad (33)$$

According to equation (32), one has that, because of the input constraint (33), any initial state $[\theta_0 \ 0]^T$ with

$|\theta_0| \geq 1.5$ can never be driven back to the origin. For any initial state $[\theta_0 \ \dot{\theta}_0]^T$, suppose a proper extreme input (± 100) is applied, if θ reaches ± 1.5 at the right moment when $\dot{\theta}$ is reduced to 0 from $\dot{\theta}_0$, then all these initial states compose the boundary of the possible maximum stability region of the system (32). The whole boundary is tested out through extensive simulation study, as shown in Fig. 5. For example, $[0 \ \pm 0.4743]^T$ are the intersections of the boundary with the $\dot{\theta}$ axis. It is obvious that the boundary has nothing to do with control strategies but only depends on the system dynamics (32) and the input constraint (33). From a mathematical point of view, the possible maximum stability region of the system (32) is infinite in a certain direction, as shown in Fig. 5, regardless of the physical meaning of θ and $\dot{\theta}$. To make it much clearer, some state trajectories are also plotted in Fig. 5. Since these state trajectories are somehow symmetrical with respect to the origin, only some initial states (represented by '*' in Figure 5) from certain areas are chosen to start simulation. The thick solid lines represent those state trajectories starting from initial states within the stability region, while the thick dot-and-dash lines for those starting from the outside of the stability region. Clearly, the possible maximum stability region of the system (32) is infinite in a certain direction.

Based on the corresponding discrete-time system of (32) with a sampling time of 0.1 s, some existing MPC algorithms [5], [6], [12] and the MPCSWR proposed in the current paper are tested to estimate their ability to maximize terminal region. For the polyhedron-based method in reference [12], the stabilizing constant gains are predetermined as linear quadratic regulators (LQR) gains. For the MPCSWR, the Jordan canonical form of the system

Fig. 5 Possible maximum stability region

Fig. 6 MPC terminal regions

(32) is adopted. Their terminal regions are given in Fig. 6 along with the possible maximum stability region. It can clearly be seen that the terminal region determined by the MPCSWR is the largest one among all MPC terminal regions, and more importantly, it is close to the possible maximum stability region in this case. This means that the idea introduced in this paper to design the TWM and the TRM separately is very successful in terms of the size of terminal region.

7 CONCLUSIONS

The present paper clearly demonstrates, then proves that, for MPC, the terminal region can be significantly enlarged by separating the terminal region matrix and terminal weighting matrix. Following this idea, a new MPC algorithm with the terminal region defined as an ellipsoid is proposed, where no explicit relationship exists between the terminal region matrix and the terminal weighting matrix, which allows a higher degree of freedom to expand the terminal region. Furthermore, by using the Jordan canonical form of the system, the new algorithm can result in a terminal region, which is infinite in the direction of stable modes of the system. In effect, some existing methods are just special cases of the new MPC proposed in this paper. The effectiveness of the proposed MPC is illustrated by numerical examples. The reported approach has a potential of being extended to non-linear systems by utilizing LDI techniques, which is worth further investigation.

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APPENDIX

Notation

A	system state matrix
B	input matrix
<i>J</i>	performance index
<i>k</i>	discrete-time index
K_{term}	terminal control gain
<i>N</i>	length of discrete-time moving horizon
P	terminal weighting matrix
Q	state weighting matrix
R	input weighting matrix
<i>t</i>	continuous time (s)
u	input vector
<i>v</i>	terminal region
x	system state vector
Y	Jordan canonical transforming matrix
Z	terminal region matrix
<i>θ</i>	angle in simplified inverted pendulum system (d)