This item was submitted to [Loughborough's Research Repository](#) by the author.
Items in Figshare are protected by copyright, with all rights reserved, unless otherwise indicated.

Longitudinal bulk strain solitons in a hyperelastic rod with quadratic and cubic nonlinearities

PLEASE CITE THE PUBLISHED VERSION

<https://doi.org/10.1134/S0040577920030046>

PUBLISHER

Springer Verlag

VERSION

AM (Accepted Manuscript)

PUBLISHER STATEMENT

This is a post-peer-review, pre-copyedit version of an article published in Theoretical and Mathematical Physics. The final authenticated version is available online at: <https://doi.org/10.1134/S0040577920030046>

LICENCE

CC BY-NC-ND 4.0

REPOSITORY RECORD

Garbuzov, FE, YM Beltukov, and Karima Khusnutdinova. 2020. "Longitudinal Bulk Strain Solitons in a Hyperelastic Rod with Quadratic and Cubic Nonlinearities". Loughborough University.
<https://hdl.handle.net/2134/10043207.v1>.

Longitudinal bulk strain solitons in a hyperelastic rod with quadratic and cubic nonlinearities

F.E. Garbuzov¹, Y.M. Beltukov¹, K.R. Khusnutdinova^{2,*}

¹ Ioffe Institute, 26 Polytekhnicheskaya, St. Petersburg 194021, Russia

² Department of Mathematical Sciences, Loughborough University,
Loughborough LE11 3TU, United Kingdom

* Corresponding author: K.Khusnutdinova@lboro.ac.uk

Abstract

In this paper we study nonlinear longitudinal bulk strain waves in a hyperelastic rod of circular cross section within the scope of the general weakly-nonlinear model with quadratic and cubic nonlinearities. We derive the extended Boussinesq and Korteweg - de Vries - type equations and construct a family of approximate weakly-nonlinear soliton solutions with the help of the near-identity transformations. These solutions are compared with the results of direct numerical simulations of the original fully nonlinear problem formulation.

MSC: 35Q51, 35Q53

PACS: 62.30, 43.25

Keywords: Hyperelastic rod, Korteweg - de Vries - type equation, near-identity transformation, soliton

1 Introduction

Solitons have been a subject of a huge body of theoretical and experimental research in such areas as fluids and nonlinear optics, largely because of the compact form of the governing equations and availability of a large amount of experimental and observational data (see [1, 2, 3, 4] and references therein). In contrast to that, the studies of solitary waves in solids is a relatively recent area of research, generally requiring greater efforts because of the complexity and great variability of the properties of solids reflected in their constitutive relations, as well as significant experimental challenges (for example, [5, 6, 7, 8, 9, 10, 11] and references therein). Considerable progress has been made in the studies of bulk strain solitons in hyperelastic rods, starting with the works of G.A. Nariboli and A. Sedov [12] and L.A. Ostrovsky and A.M. Sutin [13], and significantly advanced by A.M. Samsonov and his group (see [14, 15, 16, 17, 18, 19] and references therein). Theoretical studies were based on the Boussinesq and Korteweg-de Vries-type models developed within the scope of the weakly-nonlinear elasticity theory (Murnaghan's 5 constant model for elastic energy [20]), with differing degree of rigour. A systematic asymptotic analysis has been developed by H.-H. Dai and X. Fan [21] and K.R. Khusnutdinova et al. [22], within the scope of nonlinear elasticity and lattice modelling, respectively. The models have been used to study, in particular, the scattering of long longitudinal bulk strain solitary waves by delamination (see [23, 24, 25], and for related experiments see [26, 27]). The derivations within

the scope of elasticity theory have been simplified and generalised to include surface loading and longitudinal pre-stretch in [28], resulting in the forced Boussinesq-type models.

In the present paper we aim to study both weakly- and strongly-nonlinear elastic solitons, and therefore we extend the derivation of nonlinear two-directional long wave models for longitudinal waves to hyperelastic materials described by the 9 constant model including both quadratic and cubic nonlinearities. We account for both geometrical and physical sources of nonlinearity and develop a systematic asymptotic analysis. The derivations are performed using symbolic computations with MATHEMATICA [29]. We then derive a uni-directional extended Korteweg - de Vries (KdV) - type model and study its solitary wave solutions both analytically, with the help of the near-identity transformations [30, 31] (see also the review [32] and references therein) and direct numerical simulations of the full problem formulation.

2 Problem formulation

We consider a rod of circular cross section with the radius R and use cylindrical coordinates (x, r, φ) with the axial coordinate x , radial coordinate r and angular coordinate φ . We use the Lagrangian description and denote the displacement vector by $\underline{U} = (U, V, W)$, where U is the axial displacement, V is the radial displacement and W is the torsion.

Figure 1: Rod of circular cross section.

We use the fourth-order Landau-Lifshits expression [33] for the energy of elastic deformation (equivalent to the Murnaghan 9 constant model [7]), which can be written as follows:

$$\begin{aligned} \Pi = \frac{\lambda}{2}(\text{tr } \underline{\underline{\mathcal{E}}})^2 + \mu \text{tr } \underline{\underline{\mathcal{E}}}^2 + \frac{A}{3} \text{tr } \underline{\underline{\mathcal{E}}}^3 + B \text{tr } \underline{\underline{\mathcal{E}}} \text{tr } \underline{\underline{\mathcal{E}}}^2 + \frac{C}{3}(\text{tr } \underline{\underline{\mathcal{E}}})^3 \\ + D \text{tr } \underline{\underline{\mathcal{E}}} \text{tr } \underline{\underline{\mathcal{E}}}^3 + F(\text{tr } \underline{\underline{\mathcal{E}}})^2 \text{tr } \underline{\underline{\mathcal{E}}}^2 + G(\text{tr } \underline{\underline{\mathcal{E}}}^2)^2 + H(\text{tr } \underline{\underline{\mathcal{E}}})^4, \end{aligned} \quad (1)$$

where $\underline{\underline{\mathcal{E}}} = (\nabla \underline{U}^T + \nabla \underline{U} + \nabla \underline{U}^T \cdot \nabla \underline{U})/2$ is Cauchy-Green strain tensor. We now consider an exact reduction of the full equations of motion describing solutions with no torsion, and where the longitudinal and transverse displacements U and V are independent of φ :

$$U = U(x, r, t), \quad V = V(x, r, t), \quad W = 0. \quad (2)$$

The equations of motion take the form

$$\rho \frac{\partial^2 U(x, r, t)}{\partial t^2} - \frac{\partial P_{xx}}{\partial x} - \frac{\partial P_{xr}}{\partial r} - \frac{P_{xr}}{r} = 0, \quad (3)$$

$$\rho \frac{\partial^2 V(x, r, t)}{\partial t^2} - \frac{\partial P_{rx}}{\partial x} - \frac{\partial P_{rr}}{\partial r} - \frac{P_{rr} - P_{\varphi\varphi}}{r} = 0, \quad (4)$$

while the third equation is identically satisfied. Here, $P_{\alpha\beta}$ denotes components of the first Piola-

Kirchhoff stress tensor.

We assume that the rod is not subjected to any external loading, i.e. the stress has to vanish at the surface of the rod

$$P_{rr} = P_{xr} = 0 \quad \text{at} \quad r = R \quad (5)$$

Since the component $P_{\varphi r} \equiv 0$, the third boundary condition $P_{\varphi r} = 0$ at $r = R$ is identically satisfied.

We consider long longitudinal waves in a symmetric rod, hence we add symmetry conditions which require the longitudinal displacement to be an even function of r and the radial displacement to be an odd function of r (e.g., [28]).

3 Extended Boussinesq-type equation

We extend the approach developed in our previous paper [28]. We look for a solution of the problem in the form of power series expansions of the displacements in the radial coordinate:

$$U(x, r, t) = U_0(x, t) + r^2 U_2(x, t) + r^4 U_4(x, t) + r^6 U_6(x, t) + \dots, \quad (6)$$

$$V(x, r, t) = r V_1(x, t) + r^3 V_3(x, t) + r^5 V_5(x, t) + r^7 V_7(x, t) + \dots, \quad (7)$$

which follow from the symmetry conditions. We consider the waves of small amplitude and large length compared to the radius of the rod. Hence we non-dimensionalise the variables as follows:

$$\tilde{t} = \frac{t}{L/c}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{r} = \frac{r}{\delta L}, \quad \tilde{U} = \frac{U}{\varepsilon L}, \quad \tilde{V} = \frac{V}{\varepsilon \delta L}, \quad (8)$$

which yields $\tilde{U}_n = \frac{L^n U_n}{\varepsilon L}$, $\tilde{V}_n = \frac{L^n V_n}{\varepsilon L}$ for $n \geq 0$, assuming that L is the characteristic wavelength, c is the linear wave speed, E is the Young modulus, ε is the small amplitude parameter (characterising the longitudinal strain), and $\delta = \frac{R}{L}$ is the second small parameter (long wavelength parameter). Here, the tilde denotes dimensionless variables and tractions. In the following we will use the expressions for the Young modulus and the Poisson ratio in terms of the Lamé coefficients:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}. \quad (9)$$

Then, the expansions (6) and (7) take the form

$$\tilde{U}(\tilde{x}, \tilde{r}, \tilde{t}) = \tilde{U}_0(\tilde{x}, \tilde{t}) + \tilde{r}^2 \tilde{U}_2(\tilde{x}, \tilde{t}) + \tilde{r}^4 \tilde{U}_4(\tilde{x}, \tilde{t}) + O(\tilde{r}^6), \quad (10)$$

$$\tilde{V}(\tilde{x}, \tilde{r}, \tilde{t}) = \tilde{r} \tilde{V}_1(\tilde{x}, \tilde{t}) + \tilde{r}^3 \tilde{V}_3(\tilde{x}, \tilde{t}) + \tilde{r}^5 \tilde{V}_5(\tilde{x}, \tilde{t}) + O(\tilde{r}^7). \quad (11)$$

In what follows we omit the tildes.

Substituting (10) and (11) into the equations of motion (3) and (4) we obtain

$$\begin{aligned} & \rho c^2 U_{0tt} - (\lambda + 2\mu) U_{0xx} - 2(\lambda + \mu) V_{1x} - 4\mu U_2 + \Phi_{1,1}\varepsilon + \Phi_{1,2}\varepsilon^2 \\ & + [\rho c^2 U_{2tt} - (\lambda + 2\mu) U_{2xx} - 4(\lambda + \mu) V_{3x} - 16\mu U_4 + \Phi_{1,3}\varepsilon] r^2 \\ & + [\rho c^2 U_{4tt} - (\lambda + 2\mu) U_{4xx} - 6(\lambda + \mu) V_{5x} - 36\mu U_6] r^4 + O(\varepsilon^3, \varepsilon^2 r^2, \varepsilon r^4, r^6) = 0, \end{aligned} \quad (12)$$

$$\begin{aligned}
& r(\rho c^2 V_{1tt} - \mu V_{1xx} - 2(\lambda + \mu)U_{2x} - 8(\lambda + 2\mu)V_3 + \Phi_{2,1}\varepsilon + \Phi_{2,2}\varepsilon^2 \\
& - [\rho c^2 V_{3tt} - \mu V_{3xx} - 4(\lambda + \mu)U_{4x} - 24(\lambda + 2\mu)V_5 + \Phi_{2,3}\varepsilon] r^2 \\
& - [\rho c^2 V_{3tt} - \mu V_{5xx} - 6(\lambda + \mu)U_{6x} - 48(\lambda + 2\mu)V_7] r^4 + O(\varepsilon^3, \varepsilon^2 r^2, \varepsilon r^4, r^6) = 0,
\end{aligned} \tag{13}$$

Here, the subscripts x and t denote partial derivatives and $\Phi_{i,1}$, $\Phi_{i,2}$, $\Phi_{i,3}$ denote all nonlinear terms with coefficients ε , ε^2 and εr^2 respectively. The functions U_2 , V_3 , U_4 , V_5 , U_6 can be obtained using the expansions in power series of ε :

$$U_2 = U_2^{(0)} + \varepsilon U_2^{(1)} + \varepsilon^2 U_2^{(2)} + \dots$$

Equating to zero the coefficients at different powers of ε and r in (12) and (13) results in

$$U_2 = \frac{1}{4\mu} [\rho c^2 U_{0tt} - (\lambda + 2\mu)U_{0xx} - 2(\lambda + \mu)V_{1x}] + \varepsilon U_2^{(1)}(x, t) + \varepsilon^2 U_2^{(2)}(x, t) + O(\varepsilon^3), \tag{14}$$

$$V_3 = \frac{1}{8(\lambda + 2\mu)} [\rho c^2 V_{1tt} - 2(\lambda + \mu)U_{2x} - \mu V_{1xx}] + \varepsilon V_3^{(1)}(x, t) + \varepsilon^2 V_3^{(2)}(x, t) + O(\varepsilon^3), \tag{15}$$

$$U_4 = \frac{1}{16\mu} [\rho c^2 U_{2tt} - (\lambda + 2\mu)U_{2xx} - 4(\lambda + \mu)V_{3x}] + \varepsilon U_4^{(1)}(x, t) + O(\varepsilon^2), \tag{16}$$

$$V_5 = \frac{1}{24(\lambda + 2\mu)} (\rho c^2 V_{3tt} - 4(\lambda + \mu)U_{4x} - \mu V_{3xx}) + \varepsilon V_5^{(1)}(x, t) + O(\varepsilon^2), \tag{17}$$

$$U_6 = \frac{1}{36\mu} [\rho c^2 U_{4tt} - (\lambda + 2\mu)U_{4xx} - 6(\lambda + \mu)V_{5x}] + O(\varepsilon). \tag{18}$$

The expressions for the functions $U_2^{(1)}$, $U_2^{(2)}$, $V_3^{(1)}$, $V_3^{(2)}$, $U_4^{(1)}$, $V_5^{(1)}$ are cumbersome and are not shown here. Next, substituting the functions U_2 , V_3 , U_4 , V_5 , U_6 into the boundary conditions (5) we obtain the equations

$$\begin{aligned}
& 2(\lambda + \mu)V_1 + \lambda U_{0x} + \varepsilon \Psi_{1,1} + \varepsilon^2 \Psi_{1,2} + \delta^2 \left[d_1 U_{0xxx} + \rho c^2 d_2 U_{0xtt} + \rho c^2 d_3 V_{1tt} + d_4 V_{1xx} \right] \\
& + \delta^4 \left[(d_5 V_{1xx} + \rho c^2 d_6 V_{1tt})_{xx} + \rho^2 c^4 (d_7 U_{0x} + d_8 V_1)_{tttt} + (d_9 U_{0xx} + \rho c^2 d_{10} U_{0tt})_{xxx} \right] \\
& + \varepsilon \delta^2 \Psi_{1,3} + O(\varepsilon^3, \varepsilon^2 \delta^2, \varepsilon \delta^4, \delta^6) = 0,
\end{aligned} \tag{19}$$

$$\begin{aligned}
& \rho c^2 U_{0tt} - 2\lambda V_{1x} - (\lambda + 2\mu)U_{0xx} + \varepsilon \Psi_{2,1} + \varepsilon^2 \Psi_{2,2} + \varepsilon \delta^2 \Psi_{2,3} \\
& + \delta^2 \left[e_1 U_{0xxxx} + \rho^2 c^4 e_2 U_{0tttt} + \rho c^2 e_3 U_{0xxtt} + e_4 V_{1xxx} + \rho c^2 e_5 V_{1xtt} \right] \\
& + \delta^4 \left[(e_6 V_{1xx} + \rho c^2 e_7 V_{1tt})_{xxx} + \rho^2 c^4 (e_8 V_{1x} + e_9 U_{0xx} + \rho c^2 e_{10} U_{0tt})_{tttt} \right. \\
& \left. + (e_{11} U_{0xx} - \rho c^2 U_{0tt})_{xxx} \right] + O(\varepsilon^3, \varepsilon^2 \delta^2, \varepsilon \delta^4, \delta^6) = 0.
\end{aligned} \tag{20}$$

Here the coefficients d_i , e_i depend on the Lamé elastic moduli, and $\Psi_{i,j}$ denote nonlinear terms. Elimination of the function V_1 from the equations (19) and (20) can be done by expanding it into power series in ε and δ^2 . Unknown terms in this expansion can be found by equating to zero the coefficients of ε , δ^2 , ε^2 , δ^4 and $\varepsilon \delta^2$ in (19):

$$\begin{aligned}
V_1(x, t) = & -\frac{\lambda}{2(\lambda + \mu)} U_{0x} + \varepsilon f(x, t) + \delta^2 g(x, t) + \varepsilon^2 \tilde{f}(x, t) + \delta^4 \tilde{g}(x, t) + \varepsilon \delta^2 \tilde{h}(x, t) \\
& + O(\varepsilon^3, \varepsilon^2 \delta^2, \varepsilon \delta^4, \delta^6).
\end{aligned} \tag{21}$$

Here we do not show the expressions for the functions f , g , $\tilde{f}$, $\tilde{g}$, $\tilde{h}$, for brevity. Then, the substitution

of V_1 into (20) results in the following equation for U_0 :

$$U_{0tt} - U_{0xx} + \varepsilon \frac{\beta_1}{E} (U_{0x}^2)_x + \varepsilon^2 \frac{\beta_2}{E^2} (U_{0x}^3)_x + \delta^2 \left[\alpha_1 U_{0tttt} + \alpha_2 U_{0xxtt} + \alpha_3 U_{0xxxx} + \varepsilon \tilde{J}_0(U_0) \right] + \delta^4 (\alpha_4 U_{0ttttt} + \alpha_5 U_{0xxttt} + \alpha_6 U_{0xxxxt} + \alpha_7 U_{0xxxxx}) + O(\varepsilon^3, \varepsilon^2 \delta^2, \varepsilon \delta^4, \delta^6) = 0, \quad (22)$$

where the coefficients α_i , β_i and the nonlinear function $\tilde{J}_0$ are given in Appendix A. Then, assuming the balance between the nonlinear and dispersive terms $\varepsilon \sim \delta^2$, and truncating this equation, we obtain an extended Boussinesq-type equation:

$$U_{0tt} - U_{0xx} + \varepsilon \left(\frac{\beta_1}{E} (U_{0x}^2)_x + \alpha_1 U_{0tttt} + \alpha_2 U_{0xxtt} + \alpha_3 U_{0xxxx} \right) + \varepsilon^2 \left[\frac{\beta_2}{E^2} (U_{0x}^3)_x + \alpha_4 U_{0ttttt} + \alpha_5 U_{0xxttt} + \alpha_6 U_{0xxxxt} + \alpha_7 U_{0xxxxx} + \tilde{J}_0(U_0) \right] = 0. \quad (23)$$

4 Extended Korteweg - de Vries - type equation

We introduce the characteristic variables $\xi = x - t$, $\eta = x + t$, $\tau = \varepsilon t$ and expand the unknown function U_0 into the power series in ε . In order to derive a unidirectional model here we simply look for a solution in the form

$$U_0(x, t) = U_0^{(0)}(\xi, \tau) + \varepsilon U_0^{(1)}(\xi, \tau) + \varepsilon^2 U_0^{(2)}(\xi, \tau) + \dots, \quad (24)$$

which can be justified by allowing the dependence of the higher-order terms on both characteristic variables and requiring the absence of secular terms in the asymptotic expansion (e.g. [4]). Substitution of U_0 into the equation (23) yields:

$$\left(U_{0\tau}^{(0)} - \frac{\beta_1}{2E} U_{0\xi}^{(0)2} - \frac{q_1}{2} U_{0\xi\xi\xi}^{(0)} \right)_\xi + \varepsilon \left[\left(U_{0\tau}^{(1)} - \frac{\beta_1}{2E} U_{0\xi}^{(0)} U_{0\xi}^{(1)} - \frac{q_1}{2} U_{0\xi\xi\xi}^{(1)} \right)_\xi + (2\alpha_1 + \alpha_2) U_{0\xi\xi\xi\tau}^{(0)} - \frac{1}{2} U_{0\tau\tau}^{(0)} - \frac{\beta_2}{2E^2} (U_{0\xi}^{(0)3})_\xi - \frac{q_2}{2E} U_{0\xi\xi}^{(0)} U_{0\xi\xi\xi}^{(0)} - \frac{q_3}{2E} U_{0\xi}^{(0)} U_{0\xi\xi\xi\xi}^{(0)} - \frac{q_4}{2} U_{0\xi\xi\xi\xi\xi}^{(0)} \right] + O(\varepsilon^2) = 0, \quad (25)$$

where $q_1 = \alpha_1 + \alpha_2 + \alpha_3$, $q_2 = 3\gamma_1 + 2\gamma_2 + 3\gamma_3 + 6\gamma_4 + \gamma_5 + 6\gamma_6 + 2\gamma_7$, $q_3 = \gamma_1 + \gamma_3 + 2\gamma_4 + \gamma_5 + 2\gamma_6$, $q_4 = \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$. The τ -derivatives in $O(\varepsilon)$ in (25) can be eliminated using the first order asymptotic expression: $U_{0\tau}^{(0)} = \frac{\beta_1}{2E} U_{0\xi}^{(0)2} + \frac{q_1}{2} U_{0\xi\xi\xi}^{(0)} + O(\varepsilon)$. Then, introduction of the new function $u = U_{0\xi}^{(0)} + \varepsilon U_{0\xi}^{(1)}$ yields:

$$u_\tau - \frac{\beta_1}{2E} (u^2)_\xi - \frac{q_1}{2} u_{\xi\xi\xi} - \varepsilon \left[\frac{3\beta_2 + \beta_1^2}{6E^2} (u^3)_\xi + \frac{2q_2 + 3(q_1 - 4(2\alpha_1 + \alpha_2))\beta_1}{4E} u_\xi u_{\xi\xi} + \frac{q_3 + (q_1 - 2(2\alpha_1 + \alpha_2))\beta_1}{2E} u u_{\xi\xi\xi} + \left(\frac{q_4}{2} - \frac{q_1(2\alpha_1 + \alpha_2)}{2} + \frac{q_1^2}{8} \right) u_{\xi\xi\xi\xi} \right] + O(\varepsilon^2) = 0. \quad (26)$$

Note that since U is a longitudinal displacement the function u can be treated as a longitudinal strain. The equation (26) has been derived and studied mainly in the context of waves in fluids (see [34, 35, 36, 37, 39, 40, 41, 42] and references therein), and is often referred to as an extended Korteweg - de Vries (eKdV) equation. To the best of our knowledge, this is the first derivation of this equation in the context of waves in solids.

Studying solitary wave solutions of the derived equation directly is a complicated task, and therefore here we aim to reduce the eKdV equation (26) to the Gardner equation using direct and

inverse near-identity transformations of the form

$$\hat{u} = u + \varepsilon(a_1 u_{\xi\xi} + a_2 \xi u_\tau + a_3 u_\xi \int_{\xi_0}^{\xi} u d\xi), \quad (27)$$

$$u = \hat{u} - \varepsilon(a_1 \hat{u}_{\xi\xi} + a_2 \xi \hat{u}_\tau + a_3 \hat{u}_\xi \int_{\xi_0}^{\xi} \hat{u} d\xi), \quad (28)$$

up to $O(\varepsilon^2)$ corrections. We note that the general near-identity transformations discussed in [30, 31, 32] contain also the $\varepsilon a_4 u^2$ term which we do not need here since we wish to retain both the quadratic and cubic nonlinearities in the equation (26). It is well-known from the studied in fluids that the Gardner equation has a rich family of solitons reducing to KdV solitons in the case of small amplitude [39]. We would like to use these known solutions and to compare with the results of direct numerical simulations. Various near-identity transformations have been used to study nonlinear waves in two- and three-layered fluids (e.g., [43, 44, 45, 46]). We note that in the context of solids there are 9 free parameters (constants characterising elastic properties of various materials), and generally there is more freedom in the choice of coefficients of the equation (26) than in the known fluid contexts.

The appropriate choice of coefficients allows us to eliminate all higher-order dispersive terms from (26):

$$a_1 = \frac{1}{12} \left[\frac{10q_4}{q_1} + q_1 + 2(2\alpha_1 + \alpha_2) + \frac{3(q_3 - q_2)}{\beta_1} \right],$$

$$a_2 = \frac{4q_4 - 4q_1(2\alpha_1 + \alpha_2) + q_1^2}{6q_1^2}, \quad a_3 = \frac{\beta_1 q_1(4\alpha_1 + 2\alpha_2 + q_1) + 3q_1 q_3 - 8\beta_1 q_4}{9E q_1^2}.$$

The resulting Gardner equation takes the form:

$$\hat{u}_\tau - \frac{\beta_1}{2E} (\hat{u}^2)_\xi - \frac{q_1}{2} \hat{u}_{\xi\xi\xi} - \varepsilon \frac{\hat{\beta}_2}{2E^2} (\hat{u}^3)_\xi + O(\varepsilon^2) = 0, \quad (29)$$

where $\hat{\beta}_2 = \beta_2 + (\beta_1^2(1 - 2a_2) - a_3 E \beta_1)/3$. The equation (29) has a family of solitary wave solutions parametrised by the amplitude M (e.g. [39]):

$$\hat{u}(\xi, t) = \frac{M}{1 + N \cosh K\theta}, \quad N = \sqrt{1 + \frac{3\varepsilon \hat{\beta}_2 M}{2\beta_1 E}}, \quad K = \sqrt{\frac{\beta_1 M}{3E q_1}}, \quad v = -\frac{\beta_1 M}{6E}, \quad (30)$$

where $\theta = \xi - vt$. We use this solution to create an asymptotic solution with $O(\varepsilon^2)$ accuracy for the extended KdV (26) using the inverse near-identity transformation (28):

$$u(\xi, t) = \frac{M}{1 + N \cosh K\theta} \left[1 - \frac{\varepsilon a_1 N K^2 [N(\cosh 2K\theta - 3) - 2 \cosh K\theta]}{2(1 + N \cosh K\theta)^2} \right. \\ \left. + \frac{\varepsilon N K \sinh K\theta}{1 + N \cosh K\theta} \left(-a_2 \xi v + \frac{2a_3 M \operatorname{atanh} \left(\sqrt{\frac{1-N}{1+N}} \tanh \frac{K}{2} \theta \right)}{\sqrt{1-N^2}} \right) \right]. \quad (31)$$

We note that the term which contains ξ explicitly is not secular, because it is of the same order as $\xi / \cosh K\xi$, which decays to 0 as $\xi \rightarrow \infty$ (this term can be removed by a phase shift).

It is now interesting to compare the performance of the simple Gardner soliton (30) and the formula (31) for the solution of the original higher-order KdV equation both within the range of

its formal asymptotic validity (i.e. small amplitude), as well as the case when cubic and quadratic nonlinear terms become comparable ($\varepsilon\beta_2 \sim \beta_1$, large amplitude), and to check whether the formula (31) is a better approximation to the full numerical solution of the problem than the simple Gardner soliton (30). We note that, strictly speaking, the second case is beyond the range of validity of the asymptotic expansion, however it is interesting to test whether the asymptotic formula can still be useful in one way or another. We note that weakly-nonlinear solutions have been compared with the results of direct numerical simulations in several settings relevant to oceanic studies (see [46, 47, 48, 49, 50, 51] and references therein). To the best of our knowledge there were no comparisons for solids.

We consider two hyperelastic materials with elastic moduli given in Table 1. Here for brevity we study only solitons of negative polarity (solitons of compression), hence we choose the moduli A, B and C so that the coefficient of quadratic nonlinearity is negative. Material 1 has negative coefficient of cubic nonlinearity (GE− case), therefore corresponding family of small-amplitude solitons contains “table-top” solitons. This coefficient in Material 2 is positive (GE+ case). We note that $q_1 = -\nu^2/2 < 0$, hence the dispersive coefficient in the Gardner equation (29) is always positive. Examples of solitons in both materials are given in Figure 2.

Table 1: Elastic moduli.

	Young’s m. E , GPa	Poisson’s ratio, ν	Landau moduli, GPa						Density ρ , kg/m ³	Coefficients	
			A	B	C	D	F	G		β_1/E	β_2/E^2
Material 1	5	0.34	−5.85			−2.93			1000	1	13.3
Material 2	5	0.34	−5.85			14.18			1000	1	−13.3

Figure 2: Comparison of Gardner solitons (30) and eKdV asymptotic solutions (31) for two materials: (a) Material 1, $M = -0.04$ and $M = -0.049982$; (b) Material 2, $M = -0.05$ and $M = -0.2$.

In order to compare the derived asymptotic solutions with the results of direct numerical simulations of the full equations (3) – (5) a multidomain pseudospectral method is applied [52]. A set of Legendre polynomials in both x and r variables are used for the spatial discretization of the problem: $\underline{U}(x, r, t) = \sum_{n,m} \hat{\underline{U}}_{nm}(t) \Phi_n(x) \Psi_m(r)$, where $\underline{U}$ is the displacement vector. The multidomain method allows us to compute the solution relatively quickly on a fine mesh with 650 points in x and 5 points in r (the rod we consider is thin compared to the wavelength, hence we do not need a lot of points in r coordinate).

In Figures 3, 4, 5 the numerical results are obtained using the initial condition in the form of the eKdV asymptotic solution (31). The initial soliton and the KdV soliton are plotted for comparison. The data for the soliton’s velocity and shape is summarised in Figure 6. From these comparisons we can see that overall the solution (31) performs better than the solution of the KdV or Gardner equation, although all three solutions work very well for the case of a very small amplitude, and the Gardner soliton is also a reasonable approximation in the case of the small but finite amplitude. From our experiments we can conclude that, at finite amplitude, the constructed weakly-nonlinear solution can

be used as least as a very good initial guess in order to generate strongly-nonlinear soliton solutions.

Figure 3: Material 1 (GE- case). Left plot $\varepsilon = 0.005$, right plot $\varepsilon = 0.027$.

Figure 4: Material 1 (GE- case). Amplitude of the initial soliton is close to the critical one (when $N = 0$), $\varepsilon = 0.05$ on both plots.

Figure 5: GE+ case. Left plot $\varepsilon = 0.02$, right plot $\varepsilon = 0.05$.

Figure 6: Relations between soliton parameters. The left plots correspond to GE- case, while the right ones to GE+. All solitons are approximated by the function $\frac{M}{1 + N \cosh K(x - vt)}$. Curves for the full equations (blue) are obtained from a set of numerical simulations.

5 Conclusions

In this paper we derived the extended Boussinesq and Korteweg - de Vries equations describing long nonlinear longitudinal bulk strain waves in generic weakly-nonlinear hyperelastic materials with the accuracy up to an including the cubic terms in the equations. The extended Korteweg - de Vries equation was then reduced to the Gardner equation with the help of a near-identity transformation in order to make use of the known family of soliton solutions of this equation. The inverse near-identity transformation was used to obtain the solution of the derived extended Korteweg - de Vries equation. The solutions were compared with each other and with the results of direct numerical simulations of the original fully nonlinear problem formulation, showing very good agreement for the waves of small amplitude, but also reasonably extending their relevance to the waves of moderate amplitude. In particular, the weakly-nonlinear solution has allowed us to generate and observe a stably propagating strongly-nonlinear longitudinal “table-top” soliton.

6 Acknowledgements and comments

We would like to thank O. Kurkina, A. Mikhailov, D. Pelinovsky, Y. Stepanyants and T. Talipova for useful references and discussions. We are also grateful to the organisers of the conference SCT-19 in honour of V.E. Zakharov’s 80th Birthday where some of these discussions have taken place. F. Garbuzov and Y. Beltukov acknowledge the financial support from the Russian Science Foundation under the grant no. 17-72-20201. The authors declare that they have no conflicts of interests.

Appendix A. Coefficients.

Coefficients in the equation (22) expressed in terms of Young’s modulus E and Poisson’s ratio ν instead of Lamé moduli:

$$\begin{aligned}
\alpha_1 &= \alpha_3 = \frac{1+\nu}{4}, \quad \alpha_2 = -\frac{1+\nu+\nu^2}{2}, \\
\alpha_4 &= \frac{(1+\nu)^2}{48}, \quad \alpha_5 = -\frac{5+3\nu+10\nu^3-4\nu^4-12\nu^5}{48(1-\nu)} \\
\alpha_6 &= \frac{14+5\nu+16\nu^2-8\nu^3-24\nu^4}{96(1-\nu)}, \quad \alpha_7 = -\frac{6+13\nu+14\nu^2+6\nu^3}{96(1+\nu)}, \\
\beta_1 &= -\left(\frac{3}{2}E + A(1-2\nu^3) + 3B(1+2\nu^2)(1-2\nu) + C(1-2\nu)^3\right), \\
\beta_2 &= 4(B+C)^2 - E\left(2A+6B+2C+4(D+F+G+H) + \frac{E}{2}\right) \\
&\quad + 4\nu(-5B^2-14BC-9C^2+E(3B+3C+2D+4F+8H)) \\
&\quad + 4\nu^2(18B^2+44BC+30C^2+2A(B+C)-E(3B+6C+6F+4G+24H)) \\
&\quad + 4\nu^3(-32B^2-76BC-40C^2-6AB-10AC+E(A+6B+4C+2D+8F+32H)) \\
&\quad + 4\nu^4(A^2+28B^2+40BC+12A(B+C)-4E(D+2F+G+4H)) \\
&\quad - 4\nu^5(A^2+4A(B-2C)-4(3B^2+20BC+12C^2))-8\nu^6(A+6B+4C)^2 \\
\tilde{J}_0 &= \frac{1}{E}[\gamma_1(U_{0x}U_{0tt})_{tt} + \gamma_2(U_{0tt}^2)_x + \gamma_3(U_{0x}U_{0tt})_{xx} + \gamma_4(U_{0x}^2)_{xtt} + \gamma_5(U_{0x}U_{0xtt})_x + \gamma_6(U_{0x}^2)_{xxx} \\
&\quad + \gamma_7(U_{0xx}^2)_x],
\end{aligned}$$

$$\begin{aligned}
\gamma_1 &= -\frac{1+\nu}{4} [2E + A(1-\nu^2) + 2B(1+\nu)(1-2\nu)], \quad \gamma_2 = \frac{\gamma_1}{2}, \quad \gamma_3 = -4\gamma_1, \\
\gamma_4 &= -\frac{1+\nu}{8} [E + A + 2B + 2\nu(B+2C) - \nu^2(A+20B+24C-2E) + 4\nu^3(A+10B+12C) \\
&\quad - 8\nu^4(A+6B+4C)], \\
\gamma_5 &= -\frac{1+\nu}{4} [5E + 3A + 10B + 4C - 2\nu(9B+10C) + \nu^2(A+12B+24C+2E) \\
&\quad - 4\nu^3(A+2B-4C) - 8\nu^4(A+6B+4C)], \\
\gamma_6 &= -\frac{1}{8} [4E + 3A + 10B + 4C + \nu(3A-12B-20C+4E) + \nu^2(A+6B+24C+2E) \\
&\quad - \nu^3(7A+20B-16C) - 8\nu^4(A+6B+4C)], \\
\gamma_7 &= -\frac{1}{8} [8E + 5A + 18B + 8C + \nu(A-36B-44C+2E) + \nu^2(3A+42B+72C+2E) \\
&\quad - \nu^3(13A+60B+16C) - 8\nu^4(A+6B+4C)],
\end{aligned}$$

References

- [1] G.B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York, 1974.
- [2] M.J. Ablowitz, H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, 1981.
- [3] S. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, *Theory of solitons: The Inverse Scattering Method*, Springer, US, 1984.
- [4] A.C. Newell, *Solitons in Mathematics and Physics*, SIAM, US, 1985.
- [5] G.A. Maugin, *Nonlinear waves in elastic crystals*, Oxford University Press, Oxford, 1999.
- [6] H.-H. Dai, Z. Cai, Phase transition in a slender cylinder composed of an incompressible elastic material. I. Asymptotic model equation, *Proc. Roy. Soc. A* 462 (2006) 419-438.
- [7] A. Mayer, Nonlinear surface acoustic waves: Theory, *Ultrasonics* 48 (2008) 478-481.
- [8] P. Hess, A.M. Lomonosov, Solitary surface acoustic waves and bulk solitons in nanosecond and picosecond laser ultrasonics, *Ultrasonics* 50 (2010) 167-171.
- [9] J. Engelbrecht, A. Salupere and K. Tamm, Waves in microstructured solids and the Boussinesq paradigm, *Wave Motion* 48 (2011) 717-726.
- [10] A. Pau, F. Lanza di Scalea, Nonlinear guided wave propagation in prestressed plates, *J. Acoust. Soc. Am.* 137 (2015) 1529-1540.
- [11] T. Peets, K. Tamm, J. Engelbrecht, On the role of nonlinearities in the Boussinesq-type wave equations, *Wave Motion* 71 (2017) 113-119.
- [12] G.A. Nariboli, A. Sedov, Burgers-Korteweg de Vries equation for viscoelastic rods and plates, *J. Math. Anal. Appl.* 32(3) (1970) 661-677.
- [13] L.A. Ostrovsky, A.M. Sutin, Nonlinear elastic waves in rods, *PMM* 41 (1977) 531-537.

- [14] A.M. Samsonov, Structural optimization in nonlinear wave propagation problems. In: *Structural Optimization under Dynamical Loading. Seminar and Workshop for Junior Scientists*, U. Lepik ed., Tartu University Press, 75-76 (1982).
- [15] A.M. Samsonov, Soliton evolution in a rod with variable cross section, *Sov. Physics - Doklady* 29 (1984) 586-587.
- [16] A.M. Samsonov, A.V. Porubov, Refinement of the model for the propagation of longitudinal strain waves in a rod with nonlinear elasticity, *Tech. Phys. Lett.* 19(6) (1993) 365-366.
- [17] A.V. Porubov, M.G. Velarde, Dispersive - dissipative solitons in nonlinear solids, *Wave Motion* 31(3) (2000) 197-207.
- [18] A.M. Samsonov, Strain solitons in solids and how to construct them, Chapman & Hall/CRC, Boca Raton, 2001.
- [19] A.V. Porubov, Amplification of nonlinear strain waves in solids, World Scientific, Singapore, 2003.
- [20] F.D. Murnaghan, Finite deformation of an elastic solid, John Wiley and Sons, New York, 1951.
- [21] H.-H. Dai, X. Fan, Asymptotically approximate model equations for weakly nonlinear long waves in compressible elastic rods and their comparisons with other simplified model equations, *Maths. Mech. Solids* 9 (2004) 61-79.
- [22] K.R. Khusnutdinova, A.M. Samsonov, A.S. Zakharov, Nonlinear layered lattice model and generalized solitary waves in imperfectly bonded structures, *Phys. Rev. E* 79(5) (2009) 056606.
- [23] K.R. Khusnutdinova, A.M. Samsonov, Fission of a longitudinal strain solitary wave in a delaminated bar, *Phys. Rev. E* 77 (2008) 066603.
- [24] K.R. Khusnutdinova, M.R. Tranter, Modelling of nonlinear wave scattering in a delaminated elastic bar, *Proc. R. Soc. A* 471 (2015) 20150584.
- [25] K.R. Khusnutdinova, M.R. Tranter, On radiating solitary waves in bi-layers with delamination and coupled Ostrovsky equations, *Chaos* 27 (2017) 013112.
- [26] G.V. Dreiden, K.R. Khusnutdinova, A.M. Samsonov, and I.V. Semenova, Splitting induced generation of soliton trains in layered waveguides, *J. Appl. Phys.* 107 (2010) 034909.
- [27] G.V. Dreiden, K.R. Khusnutdinova, A.M. Samsonov, and I.V. Semenova, Bulk strain solitary waves in bonded layered polymeric bars with delamination, *J. Appl. Phys.* 112 (2012) 063516.
- [28] F.E. Garbuzov, K.R. Khusnutdinova, I.V. Semenova, On Boussinesq-type models for long longitudinal waves in elastic rods, *Wave Motion* 88 (2019) 129-143.
- [29] MATHEMATICA and WOLFRAM MATHEMATICA are registered trademarks of Wolfram Research Inc. (www.wolfram.com)
- [30] Y. Kodama, Normal forms for weakly dispersive wave equations, *Phys. Lett. A* 112 (1985) 193-196.
- [31] A.S. Fokas, Q.M. Liu, Asymptotic integrability of water waves, *Phys. Rev. Lett.* 77 (1996) 2347-2351.

- [32] Y. Hiraoka and Y. Kodama, Normal forms and solitons, In A.V. Mikhailov ed. *Integrability*, Lecture Notes in Physics, Springer-Verlag, Berlin, Heidelberg, 2009.
- [33] L.D. Landau and E.M. Lifshitz, Theory of Elasticity, Pergamon Press, Oxford, 1986.
- [34] D.J. Benney, Long non-linear waves in fluid flows, *J. Math. Phys.* 45 (1966) 52-63.
- [35] C.-Y. Lee and R.C. Beardsley, The generation of long nonlinear internal waves in weakly stratified shear flows, *J. Geophys. Res.* 79 (1974) 453-457.
- [36] C. Koop and G. Butler, An investigation of internal solitary waves in a two-fluid system, *J. Fluid Mech.* 112 (1981) 225-251.
- [37] K. Lamb and L. Yan, The evolution of internal wave undular bores: comparisons of a fully nonlinear numerical model with weakly nonlinear theory, *J. Phys. Oceanography* 26 (1996) 2712-2734,
- [38] T.R. Marchant and N.F. Smyth, The extended Korteweg - de Vries equation and the resonant flow of a fluid over topography, *J. Fluid Mech.* 221 (1990) 263-288.
- [39] R. Grimshaw, E. Pelinovsky, and O. Poloukhina, Higher-order Korteweg - de Vries models for internal solitary waves in a stratified shear flow with a free surface, *Nonlinear Processes Geophys.* 9 (2002) 221-235.
- [40] A.R. Giniatullin, A.A. Kurkin, O.E. Kurkina, and Y.A. Stepanyants, Generalised Korteweg - de Vries equation for internal waves in two-layer fluid, *Fundam. Appl. Hydrophysics* 7 (2014) 16-28.
- [41] A. Karczewska, P. Rozmej, and E. Infeld, Shallow-water soliton dynamics beyond the Korteweg - de Vries equation, *Phys. Rev. E* 90 (2014) 012907.
- [42] K.R. Khusnutdinova, Y.A. Stepanyants, M.R. Tranter, Soliton solutions to the fifth-order Korteweg - de Vries equation and their applications to surface and internal water waves, *Phys. Fluids* 30 (2018) 022104.
- [43] T.R. Marchant and N.F. Smyth, Soliton interaction for the extended Korteweg - de Vries equation, *J. Appl. Math.* 56 (1996) 157-176.
- [44] T.R. Marchant and N.F. Smyth, An undular bore solution for the higher-order Korteweg - de Vries equation, *J. Phys. A: Math. Gen.* 39 (2006) L563-L569.
- [45] O.E. Poloukhina, A.V. Slyunyaev, Corrected evolution model on the basis of Gardner equation for internal waves in a stratified fluid, *Izvestiya A.M. Prokhorov AIN, Appl. Maths and Mechs.* 18 (2006) 82 - 91 (in Russian).
- [46] E.A. Ruvinskaya, O.E. Kurkina, A.A. Kurkin, Dynamics of nonlinear internal gravitational waves in layered fluids, State Technical University of Nizhni Novgorod, 2014 (in Russian).
- [47] E.N. Pelinovskii, O. Polukhina, and K. G. Lamb, *Oceanology* (Engl. Transl.) 40, 757, 2000.
- [48] K.G. Lamb, O. Polukhina, T. Talipova, E. Pelinovsky, W. Xiao, A. Kurkin, Breather generation in fully nonlinear models of a stratified fluid, *Phys. Rev. E* 75 (2007) 046306.

- [49] V. Maderich, T. Talipova, R. Grimshaw, E. Pelinovsky, B.H. Choi, I. Brovchenko, K. Terletska, and D.C. Kim, The transformation of an interfacial solitary wave of elevation at a bottom step, *Nonlin. Processes Geophys.* 16 (2009) 33-42.
- [50] V. Maderich, T. Talipova, R. Grimshaw, K. Terletska, I. Brovchenko, E. Pelinovsky and B.H. Choi, Interaction of a large amplitude interfacial solitary wave of depression with a bottom step, *Phys. Fluids* 22 (2010) 076602.
- [51] T. Talipova, K. Terletska, V. Maderich, I. Brovchenko, K. T. Jung et al., Internal solitary wave transformation over a bottom step: Loss of energy, *Phys. Fluids* 25 (2013) 032110.
- [52] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods. Evolution to Complex Geometries and Applications to Fluid Dynamics, Springer-Verlag, Berlin, 2007.