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L^p RESOLVENT ESTIMATES FOR MAGNETIC SCHRÖDINGER OPERATORS WITH UNBOUNDED BACKGROUND FIELDS

JEAN-CLAUDE CUENIN AND CARLOS E. KENIG

ABSTRACT. We prove L^p and smoothing estimates for the resolvent of magnetic Schrödinger operators. We allow electromagnetic potentials that are small perturbations of a smooth, but possibly unbounded background potential. As an application, we prove an estimate on the location of eigenvalues of magnetic Schrödinger and Pauli operators with complex electromagnetic potentials.

1. INTRODUCTION

Resolvent estimates for Schrödinger operators play a decisive role in numerous areas in spectral and scattering theory, as well as partial differential equations. In particular, resolvent estimates which are uniform in the spectral parameter are intimately connected with dispersive and smoothing estimates for the corresponding (time-dependent) Schrödinger equation, as observed by Kato [16].

As a general rule, resolvent estimates that hold up to the spectrum (usually called a limiting absorption principle) are associated with global in time Strichartz and smoothing estimates for the Schrödinger flow. Results in this category are usually obtained by considering a decaying electromagnetic potential as a perturbation of the free Laplacian, see for example [14] for small perturbations and [5, 6, 8, 3, 12, 13] for large perturbations.

On the other hand, resolvent estimates that are uniform only up to a $\mathcal{O}(1)$ distance to the spectrum are associated with local in time estimates. This is usually due to the presence of eigenvalues or resonances that prevent the dispersion of the flow. Potentials in this situation are usually unbounded. Prominent examples here are the harmonic oscillator (quadratic electric potential) and the constant magnetic field (linear vector potential). We mention [11, 32, 33, 4, 28, 2] for estimates involving unbounded potentials.

There is a big gap in the regularity and decay conditions for the electromagnetic potential between the two scenarios. In the first, the potentials can usually be quite rough, but have sufficient decay at infinity. In the second case, unbounded potentials are allowed but they are usually assumed to be smooth. Very little is known in the intermediate case. Our resolvent estimate is a step in this direction.

The paper is organized as follows. In Section 2 we state the assumptions on the potentials and the resolvent estimate in the simplest case, with a uniform bound with respect to the spectral parameter. In Section 3 we prove the resolvent estimate for the unperturbed operator. In Section 4 we use a perturbative argument to prove the estimate in the general case. In the final Section 5 we state a more precise version of the resolvent estimate and give an application to eigenvalue bounds for

Schrödinger operators with complex-valued potentials.

Notation

- $\langle x \rangle = (1 + |x|^2)^{1/2}$.
- $\mathcal{D}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$, and $\mathcal{D}'(\mathbb{R}^n)$ is the space of distributions.
- $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space, and $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions.
- $\mathcal{B}(X, Y)$ is the space of bounded linear operators between Banach spaces X and Y .
- $A \lesssim B$ if there exist a constant $C > 0$ (depending only on fixed quantities) such that $A \leq CB$.
- $\langle u, v \rangle = \int_{\mathbb{R}^n} u(x) \bar{v}(x) dx$ for $u, v \in \mathcal{S}(\mathbb{R}^n)$.
- If X is a Banach space densely and continuously embedded in $L^2(\mathbb{R}^n)$, we identify $L^2(\mathbb{R}^n)$ with a dense subspace of X' . Thus, the duality pairing $\langle \cdot, \cdot \rangle_{X, X'}$ extends the L^2 -scalar product $\langle \cdot, \cdot \rangle$. This is meant when we write $X \subset L^2(\mathbb{R}^n) \subset X'$.
- $\sigma(P)$ is the spectrum of P .
- $\text{dom}(P)$ is the domain of P .

2. ASSUMPTIONS AND MAIN RESULT

We consider the Schrödinger operator

$$(2.1) \quad P = (-i\nabla + A(x))^2 + V(x), \quad \text{dom}(P) = \mathcal{D}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n), \quad n \geq 2.$$

Here, $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the vector potential and $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is the electric potential. The vector potential is unique only up to a change of gauge $A \rightarrow A + d\phi$ (when considered as a 1-form); the physically observable, gauge-invariant quantity is the magnetic field $B = dA$. In the following, $\epsilon > 0$ is a yet undetermined constant that will later be chosen sufficiently small, and $\delta > 0$ is arbitrary but fixed.

Assumptions on the potentials

- $A = A_0 + A_1$, where

$$(2.2) \quad A_0 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n), \quad |\partial_x^\alpha A_0| \leq C_\alpha \langle x \rangle^{1-|\alpha|},$$

$$(2.3) \quad A_1 \in C^1(\mathbb{R}^n, \mathbb{R}^n), \quad \sup_{x \in \mathbb{R}^n} \langle x \rangle^{1+\delta} (|A_1(x)| + |\nabla A_1(x)|) \leq \epsilon.$$

- $V = \widetilde{V}_0 + V_1$, where

$$(2.4) \quad \widetilde{V}_0 = V_0 + W, \quad V_0 \in C^\infty(\mathbb{R}^n, \mathbb{R}), \quad |\partial_x^\alpha V_0| \leq C_\alpha \langle x \rangle^{2-|\alpha|}, \quad W \in L^\infty(\mathbb{R}^n, \mathbb{R})$$

$$(2.5) \quad V_1 \in L^r(\mathbb{R}^n, \mathbb{R}), \quad \|V_1\|_{L^r} \leq \epsilon.$$

Here, $r \in (1, \infty]$ for $n = 2$ and $r \in [n/2, \infty]$ for $n \geq 3$.

Remark 2.1. The regularity assumption on A_1 in (2.3) could perhaps be relaxed, e.g. as in [6].

We will consider P as a small perturbation of the Schrödinger operator

$$(2.6) \quad \widetilde{P}_0 = (-i\nabla + A_0(x))^2 + \widetilde{V}_0(x), \quad D(\widetilde{P}_0) = \mathcal{D}(\mathbb{R}^n).$$

In the case $W = 0$, we also write

$$(2.7) \quad P_0 = (-i\nabla + A_0(x))^2 + V_0(x), \quad \text{dom}(P_0) = \mathcal{D}(\mathbb{R}^n).$$

Denote by $E_{1/2}$ the Weyl quantization of the symbol $e_{1/2}(x, \xi) := (\langle \xi \rangle^2 + \langle x \rangle^2)^{1/4}$, i.e. for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$E_{1/2}u(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} e_{1/2}\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

Our resolvent estimate involves the following spaces. Let X be the completion of $\mathcal{D}(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_X := \|u\|_{L^2} + \|\langle x \rangle^{-\frac{1+\mu}{2}} E_{1/2}u\|_{L^2} + \|u\|_{L^q}$$

where $0 < \mu \leq \delta$ is fixed. Then its topological dual X' is the space of distributions $f \in \mathcal{D}'(\mathbb{R}^n)$ such that the norm

$$\|f\|_{X'} := \inf_{f=f_1+f_2+f_3} \left(\|f_1\|_{L^2} + \|\langle x \rangle^{\frac{1+\mu}{2}} E_{-1/2}f_2\|_{L^2} + \|f_3\|_{L^{q'}} \right)$$

is finite. Here, $q = 2r'$, i.e.

$$\begin{cases} q \in [2, \infty) & \text{if } n = 2, \\ q \in [2, 2n/(n-2)] & \text{if } n \geq 3, \end{cases}$$

Our main result is the following theorem.

Theorem 2.2. *Assume that A, V satisfy Assumptions (2.2)–(2.5) with $A_0, \tilde{V}_0$ fixed. Moreover, let $a > 0$ be fixed. Then there exists $\epsilon_0 > 0$ such that whenever A_1, V_1 satisfy (2.3), (2.5) with $\epsilon < \epsilon_0$, then we have the estimate*

$$(2.8) \quad \|u\|_X \leq C \|(P - z)u\|_{X'}$$

for all $z \in \mathbb{C}$ with $|\operatorname{Im} z| \geq a$ and for all $u \in \mathcal{D}(\mathbb{R}^n)$. The constants ϵ_0, C depend on $n, q, \mu, \delta, a, \|W\|_{L^\infty}$ and on finitely many seminorms C_α in (2.2) and (2.4).

Remark 2.3. In Section 5 we will state a more precise version of the estimate (2.8) that takes into account the dependence of C on $|\operatorname{Im} z|$ for large values of $|\operatorname{Im} z|$. We will also allow V_1, A_1 to be complex-valued.

3. $L^{q'} \rightarrow L^q$ AND SMOOTHING ESTIMATES

In this section, we will establish the $L^{q'} \rightarrow L^q$ and the smoothing estimate for P_0 . These will be the main ingredients in the proof of Theorem 2.2. We follow the general approach of Koch and Tataru in [22, Section 4], where a version of the resolvent estimate (2.8) was proved for the Hermite operator. In fact, the bound (2.8) follows in this special case by combining Proposition 4.2, Proposition 4.6 and Proposition 4.7 in [22].

We start with the $L^{q'} \rightarrow L^q$ estimate. The proof of Theorem 3.1 below follows the same arguments as that of Proposition 4.6 in [22] for the Hermite operator, see also [21, Section 2].

Theorem 3.1 ($L^{q'} \rightarrow L^q$ estimate). *Let $q \in [2, \infty)$ if $n = 2$ and $q \in [2, 2n/(n-2)]$ if $n \geq 3$, and let $a > 0$ be fixed. Then there exists $C_0 > 0$ such that for all $z \in \mathbb{C}$ with $|\operatorname{Im} z| \geq a$ and for all $u \in \mathcal{D}(\mathbb{R}^n)$, we have the estimate*

$$(3.1) \quad |\operatorname{Im} z|^{1/2} \|u\|_{L^2} + \|u\|_{L^q} \leq C_0 \|(P_0 - z)u\|_{L^{q'}}.$$

The constant C_0 depends on n, q, a , and on finitely many seminorms C_α in (2.2) and (2.4).

Proof. By [24, Theorem 4], P_0 is essentially selfadjoint.¹ By abuse of notation we continue to denote the selfadjoint extension of P_0 by the same symbol. It then follows that $\sigma(P_0) \subset \mathbb{R}$ and

$$(3.2) \quad \|(P_0 - z)^{-1}\|_{\mathcal{B}(L^2)} \leq \frac{1}{|\operatorname{Im} z|},$$

where $(P_0 - z)^{-1}$ is the L^2 -resolvent. For $k \in \mathbb{Z}$, the spectral projection of P_0 corresponding to the interval $[k, k+1]$ is denoted by $\Pi_{[k, k+1]}$. Let $P_0(x, \xi)$ be the symbol of the operator P_0 , i.e.

$$P_0(x, \xi) = (\xi + A_0(x))^2 + V_0(x), \quad (x, \xi) \in \mathbb{R}^{2n}.$$

Under the assumptions (2.2) and (2.4) on A_0 and V_0 , we have the symbol bounds

$$(3.3) \quad |\partial_\xi^\alpha \partial_x^\beta P_0(x, \xi)| \leq C_{\alpha, \beta}, \quad |\alpha| + |\beta| \geq 2.$$

As in the proof of [21, Thm. 1] this implies the short-time dispersive estimate

$$(3.4) \quad \|e^{itP_0}\|_{\mathcal{B}(L^1, L^\infty)} \lesssim |t|^{-n/2}, \quad t \leq T \ll 1.$$

The proof is based on a phase-space representation [20, Prop. 4.3] of the fundamental solution of the Schrödinger equation for P_0 ,

$$(3.5) \quad i\partial_t u - P_0 u = f, \quad u|_{t=0} = u_0.$$

By now standard abstract arguments [18], the full range of Strichartz estimates for the Schrödinger equation holds: If (p_1, q_1) , (p_2, q_2) are sharp Schrödinger-admissible, i.e. if

$$(3.6) \quad \frac{2}{p_i} + \frac{n}{q_i} = \frac{n}{2}, \quad p_i \in [2, \infty], \quad q_i \in [2, 2n/(n-2)], \quad (n, p_i, q_i) \neq (2, 2, \infty),$$

then

$$(3.7) \quad \|u\|_{L_t^{p_1}([0, T]; L_x^{q_1})} \lesssim \|u_0\|_{L_x^2} + \|f\|_{L_t^{p'_2}([0, T]; L_x^{q'_2})}.$$

We now fix a pair (p, q) as in (3.6) and with q as in the theorem for the rest of the proof. Let $\Pi_{[k, k+1]}$ be the spectral projection of P_0 onto the interval $[k, k+1]$, $k \in \mathbb{Z}$. For any $u \in L^2(\mathbb{R}^n)$, the function $v(x, t) := e^{-itk} \Pi_{[k, k+1]} u(x)$ satisfies the estimate

$$(3.8) \quad \|(i\partial_t - P_0)v\|_{L_t^\infty([0, T]; L_x^2)} \leq \|u\|_{L_x^2}.$$

Applying the Strichartz estimates (3.7) with $(p_1, q_1) = (p, q)$ and $(p'_2, q'_2) = (\infty, 2)$ to $v(x, t)$, it follows that

$$(3.9) \quad \|\Pi_{[k, k+1]} u\|_{L^q} \lesssim \|u\|_{L^2}.$$

This argument can be found in [21], see Corollary 2.3 there. Since $\Pi_{[k, k+1]}^* = \Pi_{[k, k+1]}^2$, the dual as well as the TT^* version of (3.9) yield

$$(3.10) \quad \|\Pi_{[k, k+1]} u\|_{L^2} \lesssim \|u\|_{L^{q'}}, \quad \|\Pi_{[k, k+1]} u\|_{L^q} \lesssim \|u\|_{L^{q'}}.$$

¹Actually, the hypothesis in [24, Theorem 4] is that $V_0(x) \leq c|x|^2$, but this may be achieved by adding a bounded perturbation that we can absorb into W . Since our potentials are smooth, essential self-adjointness also follows from the proof of [34, C.2.12].

Using the first inequality in (3.10), orthogonality of the spectral projections and the spectral theorem, we see that for any $u \in L^2(\mathbb{R}^n) \cap L^{q'}(\mathbb{R}^n)$, we have

$$(3.11) \quad \begin{aligned} \|(P_0 - z)^{-1}u\|_{L^2}^2 &= \sum_{k \in \mathbb{Z}} \|(P_0 - z)^{-1}\Pi_{[k, k+1]}u\|_{L^2}^2 \\ &\leq \sum_{k \in \mathbb{Z}} \sup_{\lambda \in [k, k+1]} |\lambda - z|^{-2} \|\Pi_{[k, k+1]}u\|_{L^2}^2 \lesssim |\operatorname{Im} z|^{-1} \|u\|_{L^{q'}}^2. \end{aligned}$$

Here, we estimated the sum by

$$\sum_{k \in \mathbb{Z}} \sup_{\lambda \in [k, k+1]} |\lambda - z|^{-2} \leq \sum_{|k - \lfloor \operatorname{Re} z \rfloor| \leq 3} \frac{1}{a|\operatorname{Im} z|} + \sum_{|k - \lfloor \operatorname{Re} z \rfloor| > 3} \frac{1}{(\lambda - \operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$$

where $\lfloor \operatorname{Re} z \rfloor$ is the integer part of $\operatorname{Re} z$. Setting $k' := k - \lfloor \operatorname{Re} z \rfloor \in \mathbb{Z}$ in the second sum, we have $|k'| > 3$, and

$$|\lambda - \operatorname{Re} z| \geq |k'| - |\lambda - k| - |\operatorname{Re} z - \lfloor \operatorname{Re} z \rfloor| \geq |k'| - 2 > 1.$$

Changing variables $k \rightarrow k'$, it follows that the second sum is $\mathcal{O}(|\operatorname{Im} z|^{-1})$. Since $L^2(\mathbb{R}^n) \cap L^{q'}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, the BLT theorem [27, Theorem I.7] implies that

$$(3.12) \quad \|(P_0 - z)^{-1}\|_{\mathcal{B}(L^{q'}, L^2)} \lesssim |\operatorname{Im} z|^{-1/2}.$$

This proves the first half of (3.1).

We now apply the Strichartz estimates (3.7) to $v(x, t) = e^{-itz}u(x)$. Note that v satisfies the Schrödinger equation (3.5) with

$$f(x, t) = e^{-itz}(z - P_0)u(x).$$

Applying the Strichartz estimates (3.7) with $(p_1, q_1) = (p_2, q_2) = (p, q)$, we obtain

$$(3.13) \quad \|u\|_{L^q} \lesssim \|u\|_{L^2} + \|(P_0 - z)u\|_{L^{q'}}.$$

Combining (3.12) and (3.13), we arrive at

$$\|u\|_{L^q} \lesssim \|(P_0 - z)u\|_{L^{q'}}.$$

This proves the second half of (3.1). $\square$

We now establish the smoothing estimate, following Robbiano-Zuily [28]. For this purpose, we introduce the Hörmander metric

$$g = \frac{dx^2}{\langle x \rangle^2} + \frac{d\xi^2}{\langle \xi \rangle^2}.$$

Let m be a temperate weight function, that is $m \in C^\infty(\mathbb{R}^{2n})$ satisfies

$$m(x + y, \xi + \eta) \lesssim m(x, \xi)(1 + |y| + |\eta|)^s$$

for some $s > 0$. In particular, all weight functions of the form $m(x, \xi) = \langle x \rangle^{s_1} \langle \xi \rangle^{s_2}$ with $s_1, s_2 \in \mathbb{R}$ are temperate, by Petree's inequality. The symbol class $S(m, g)$ is defined as

$$\begin{aligned} S(m, g) &= \{a \in C^\infty(\mathbb{R}^{2n}) : \forall \alpha, \beta \in \mathbb{N}^n \exists C_{\alpha, \beta} > 0 \text{ s.t.} \\ &\quad |\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq C_{\alpha, \beta} m(x, \xi) \langle \xi \rangle^{-|\alpha|} \langle x \rangle^{-|\beta|}\}, \end{aligned}$$

see [15] or [26] for details. We denote the corresponding class of (Weyl) pseudo-differential operators by $Op^w(S(m, g))$. We will only need the following basic fact

about the calculus in $S(m, g)$, see e.g. [26, Theorem 1.2.17]. If $a \in S(m_1, g)$ and $b \in S(m_2, g)$, then $Op^w(a)Op^w(b) = Op^w(c) \in Op^w(S(m_1 m_2, g))$, where

$$(3.14) \quad c(x, \xi) \sim \sum_{\alpha, \beta} (-1)^{|\beta|} (\alpha! \beta!)^{-1} 2^{-|\alpha+\beta|} \partial_\xi^\alpha D_x^\beta a(x, \xi) \partial_\xi^\beta D_x^\alpha b(x, \xi),$$

in the sense that the difference between $c(x, \xi)$ and the truncated sum with $|\alpha+\beta| < N$ is in $S(m_1 m_2 \langle x \rangle^{-N} \langle \xi \rangle^{-N}, g)$. In particular,

$$(3.15) \quad [Op^w(a), Op^w(b)] \in Op^w(S(m_1 m_2 \langle x \rangle^{-1} \langle \xi \rangle^{-1}, g)).$$

Note that $\langle x \rangle^{-\frac{1+\mu}{2}} (\langle \xi \rangle^2 + \langle x \rangle^2)^{1/4} \in S(\langle \xi \rangle^{1/2}, g) \subset S(\langle x \rangle \langle \xi \rangle, g)$, so that we can use (3.15) in the proof of (3.22) and in the proof of Lemma C.1.

Moreover, $S(1, g) \subset S_{1,0}^0$ where $S_{1,0}^0$ denotes the symbol class

$$S_{1,0}^0 = \{a \in C^\infty(\mathbb{R}^{2n}) : \forall \alpha, \beta \in \mathbb{N}^n \exists C_{\alpha, \beta} > 0 \text{ s.t. } |\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{-|\alpha|}\}.$$

In particular, if $a \in Op^w(S(1, g))$, then $Op^w(a)$ is bounded in L^p for all $p \in (1, \infty)$, see e.g. [31, Proposition VI.5.4]. We will use this in the proof of Theorem 2.2. For the proof of the subsequent theorem, we only use the L^2 -boundedness.

It will be convenient to introduce the spaces $Y \supset X$ and $Y' \subset X'$ with norms

$$(3.16) \quad \|u\|_Y := \|u\|_{L^2} + \|\langle x \rangle^{-\frac{1+\mu}{2}} E_{1/2} u\|_{L^2},$$

$$(3.17) \quad \|f\|_{Y'} := \inf_{f=f_1+f_2} \left(\|f_1\|_{L^2} + \|\langle x \rangle^{\frac{1+\mu}{2}} E_{-1/2} f_2\|_{L^2} \right).$$

Note that $X = Y \cap L^q$ and $X' = Y' + L^{q'}$.

Theorem 3.2 (Smoothing estimate). *Let $\mu > 0$, and let $a > 0$ be fixed. Then for all $z \in \mathbb{C}$ with $|\text{Im} z| \geq a$ and for all $u \in \mathcal{D}(\mathbb{R}^n)$, we have the estimate*

$$(3.18) \quad \|u\|_Y \leq C_0 \|(P_0 - z)u\|_{Y'}.$$

The constant C_0 depends on n, μ, a , and on finitely many seminorms C_α in (2.2) and (2.4).

Proof. Inequality (3.18) follows from the following four inequalities: For all $u \in \mathcal{D}(\mathbb{R}^n)$, we have

$$(3.19) \quad \|u\|_{L^2} \leq |\text{Im} z|^{-1} \|(P_0 - z)u\|_{L^2},$$

$$(3.20) \quad \|u\|_{L^2} \lesssim |\text{Im} z|^{-1/2} \|\langle x \rangle^{\frac{1+\mu}{2}} E_{-1/2} (P_0 - z)u\|_{L^2},$$

$$(3.21) \quad \|\langle x \rangle^{-\frac{1+\mu}{2}} E_{1/2} u\|_{L^2} \lesssim |\text{Im} z|^{-1/2} \|(P_0 - z)u\|_{L^2},$$

$$(3.22) \quad \|\langle x \rangle^{-\frac{1+\mu}{2}} E_{1/2} u\|_{L^2} \lesssim \|\langle x \rangle^{\frac{1+\mu}{2}} E_{-1/2} (P_0 - z)u\|_{L^2}.$$

Again, (3.19) immediately follows from (3.2). Inequality (3.20) follows from (3.21) by a duality argument that we shall postpone to the end of the proof. It remains to prove (3.21) and (3.22).

Lemma 2.2 in [28] implies that there exists a real-valued symbol $\lambda \in S(1, g)$, satisfying the positive commutator estimate

$$(3.23) \quad \|\langle x \rangle^{-\frac{1+\mu}{2}} E_{1/2} u\|_{L^2}^2 \lesssim \langle -i[P_0, Op^w(\lambda)]u, u \rangle + \|u\|_{L^2}^2.$$

This estimate is not stated in [28] in the exact form given here, but it easily follows from their estimate (2.16), together with [28, Lemma 2.2 (iii)] and the L^2 -boundedness of $S(1, g)$ symbols. Note that we may replace P_0 by $P_0 - z$ in (3.23)

since the commutator with $z \in \mathbb{C}$ is zero. Then (3.23), Cauchy-Schwartz inequality and the L^2 -boundedness of $Op^w(\lambda)$ yield the estimate

$$\begin{aligned} \|u\|_Y^2 &\lesssim \langle -i[P_0 - z, Op^w(\lambda)]u, u \rangle + \|u\|_{L^2}^2 \\ &\leq 2\|Op^w(\lambda)u\|_{L^2} (\|(P_0 - z)u\|_{L^2} + |\operatorname{Im} z| \|u\|_{L^2}) + \|u\|_{L^2}^2 \\ &\leq (4C_\lambda |\operatorname{Im} z|^{-1} + |\operatorname{Im} z|^{-2}) \|(P_0 - z)u\|_{L^2}^2 \\ &\lesssim |\operatorname{Im} z|^{-1} \|(P_0 - z)u\|_{L^2}^2 \end{aligned}$$

where $C_\lambda := \|Op^w(\lambda)\|_{\mathcal{B}(L^2)}$. Note that $Op^w(\lambda)$ is self-adjoint. In the last inequality, we also used (3.2) again. This proves (3.21).

Similarly, (3.23) and duality of Y, Y' yield

$$\begin{aligned} \|u\|_Y^2 &\lesssim \langle -i[P_0 - z, Op^w(\lambda)]u, u \rangle + \|u\|_{L^2}^2 \\ &\leq 2\|Op^w(\lambda)u\|_Y \|(P_0 - z)u\|_{Y'} + 2|\operatorname{Im} z| \|Op^w(\lambda)u\|_{L^2} \|u\|_{L^2} + \|u\|_{L^2}^2 \\ &\leq \epsilon \|Op^w(\lambda)u\|_Y^2 + \epsilon^{-1} \|(P_0 - z)u\|_{Y'}^2 + (1 + 2C_\lambda |\operatorname{Im} z|) \|u\|_{L^2}^2 \\ &\leq \epsilon (C'_\lambda)^2 \|u\|_Y^2 + \epsilon^{-1} \|(P_0 - z)u\|_{Y'}^2 + (1 + 2C_\lambda |\operatorname{Im} z|) \|u\|_{L^2}^2 \end{aligned}$$

for any $\epsilon > 0$. In addition to the L^2 -boundedness of $Op^w(\lambda)$, we used that the commutator $[\langle x \rangle^{-\frac{1+\mu}{2}} E_{1/2}, Op^w(\lambda)]$ is L^2 -bounded, being in $Op^w(S(1, g))$ by (3.15). This implies that $C'_\lambda := \|Op^w(\lambda)\|_{\mathcal{B}(Y)} < \infty$. Hiding the term with ϵ in the above inequality on the left, we get

$$(3.24) \quad \|u\|_Y^2 \lesssim \|(P_0 - z)u\|_{Y'}^2 + |\operatorname{Im} z| \|u\|_{L^2}^2.$$

Combining (3.20) and (3.24), we get (3.22).

We now provide the details of the duality argument leading to (3.20). From (3.19), (3.21), we see that

$$\|(P_0 - z)^{-1}f\|_Y \lesssim |\operatorname{Im} z|^{-1/2} \|f\|_{L^2}, \quad \text{for all } f \in (P_0 - z)\mathcal{D}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n).$$

Since $(P_0 - z)^{-1}$ is L^2 -bounded and $\mathcal{D}(\mathbb{R}^n)$ is a core² for P_0 , the set $(P_0 - z)\mathcal{D}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ [17, Problem III.5.19]. Therefore,

$$(P_0 - z)^{-1} \in \mathcal{B}(L^2, Y) \text{ with } \|(P_0 - z)^{-1}\|_{\mathcal{B}(L^2, Y)} \lesssim |\operatorname{Im} z|^{-1/2}.$$

Let $f \in L^2(\mathbb{R}^n)$, $g \in \mathcal{D}(\mathbb{R}^n)$. Then

$$|\langle f, g \rangle| \leq \|(P_0 - \bar{z})^{-1}f\|_Y \|(P_0 - z)g\|_{Y'} \lesssim |\operatorname{Im} z|^{-1/2} \|f\|_{L^2} \|(P_0 - z)g\|_{Y'}$$

Taking the supremum over all $f \in L^2(\mathbb{R}^n)$ with $\|f\|_{L^2} = 1$, we arrive at

$$\|g\|_{L^2} \lesssim |\operatorname{Im} z|^{-1/2} \|(P_0 - z)g\|_{Y'},$$

proving (3.20). □

4. PROOF OF THEOREM 2.2

We first prove Theorem 2.2 for P_0 and for $\tilde{P}_0$.

²This is equivalent to the essential selfadjointness of P_0 on $\mathcal{D}(\mathbb{R}^n)$.

Proof of Theorem 2.2 for P_0 . Since $X = Y \cap L^q$ and $X' = Y' + L^{q'}$, inequality (2.8) is equivalent to the following four inequalities: For all $u \in \mathcal{D}(\mathbb{R}^n)$, we have

$$(4.1) \quad \|u\|_Y \lesssim \|(P_0 - z)u\|_{Y'},$$

$$(4.2) \quad \|u\|_Y \lesssim \|(P_0 - z)u\|_{L^{q'}},$$

$$(4.3) \quad \|u\|_{L^q} \lesssim \|(P_0 - z)u\|_{Y'},$$

$$(4.4) \quad \|u\|_{L^q} \lesssim \|(P_0 - z)u\|_{L^{q'}}.$$

We have already proved (4.1) and (4.4) in Theorem 3.2 and Theorem 3.1, respectively. Moreover, (4.3) follows from (4.2) by a duality argument. Since it is a bit more involved than the previous one, we relegate its proof to Appendix C. It remains to prove (4.2). Here we use that $Op^w(\lambda) \in \mathcal{B}(L^q)$ [31, Proposition VI.5.4] since $1 < q < \infty$. Denoting $C_{\lambda,q} := \|Op^w(\lambda)\|_{\mathcal{B}(L^q)}$, it then follows from (3.23) that

$$\begin{aligned} \|u\|_Y^2 &\lesssim \langle -i[P_0 - z, Op^w(\lambda)]u, u \rangle + \|u\|_{L^2}^2 \\ &\leq 2\|Op^w(\lambda)u\|_{L^q} \|(P_0 - z)u\|_{L^{q'}} + 2|\operatorname{Im} z| \|Op^w(\lambda)u\|_{L^2} \|u\|_{L^2} + \|u\|_{L^2}^2 \\ &\leq \|Op^w(\lambda)u\|_{L^q}^2 + \|(P_0 - z)u\|_{L^{q'}}^2 + (1 + 2C_\lambda |\operatorname{Im} z|) \|u\|_{L^2}^2 \\ &\leq C_{\lambda,q}^2 \|u\|_{L^q}^2 + \|(P_0 - z)u\|_{L^{q'}}^2 + (1 + 2C_\lambda |\operatorname{Im} z|) \|u\|_{L^2}^2 \\ &\lesssim \|(P_0 - z)u\|_{L^{q'}}^2 \end{aligned}$$

where we used (3.1) in the last step. $\square$

Proof of Theorem 2.2 for $\tilde{P}_0$. An inspection of the previous proofs shows that we may add the perturbation $W \in L^\infty$ to the operator P_0 , without any smallness assumption on the norm. This is because our estimates control L^2 -norms.

For the reader's convenience, we provide the argument for the first part of the proof of Theorem 3.1, i.e. for the spectral projection estimate (3.9). To this end, we observe that the analogue of the energy estimate (3.8),

$$\|(\partial_t - \tilde{P}_0)e^{-itk}\tilde{\Pi}_{[k,k+1]}u\|_{L_t^\infty([0,T];L_x^2)} \leq \|u\|_{L_x^2},$$

holds for the spectral projections $\tilde{\Pi}_{[k,k+1]}$ of $\tilde{P}_0$, by selfadjointness. We write

$$(i\partial_t - P_0)e^{-itk}\tilde{\Pi}_{[k,k+1]}u = (i\partial_t - \tilde{P}_0)e^{-itk}\tilde{\Pi}_{[k,k+1]}u + e^{-itk}W\tilde{\Pi}_{[k,k+1]}u$$

and apply the Strichartz estimates (3.7) with $(p_1, q_1) = (p, q)$ and $(p_2, q_2) = (\infty, 2)$. This yields

$$\|\tilde{\Pi}_{[k,k+1]}u\|_{L^q} \lesssim (1 + \|W\|_{L^\infty})\|u\|_{L^2}.$$

Similarly, one can show that all the previous inequalities for P_0 continue to hold for $\tilde{P}_0$ with the same modification of the constant. $\square$

To treat the general case, we write $P = \tilde{P}_0 + L$, where

$$(4.5) \quad L = -2iA_1 \cdot \nabla - i(\nabla \cdot A_1) + 2A_0 \cdot A_1 + A_1^2 + V_1.$$

Lemma 4.1. *Assume (2.2), (2.3) and (2.5). Then $L : X \rightarrow X'$ is bounded with $\|Lu\|_{X \rightarrow X'} \leq C_L \epsilon$. The constant C_L depends on n, μ, δ and on $\|\langle x \rangle^{-1}A_0\|_{L^\infty}$.*

Proof. We start with the estimate

$$\begin{aligned} \|Lu\|_{X'} &\leq 2\|\langle x \rangle^{\frac{1+\mu}{2}} E_{-1/2} A_1 \cdot \nabla u\|_{L^2} + \|(\nabla \cdot A_1)u\|_{L^2} \\ &\quad + 2\|A_0 \cdot A_1 u\|_{L^2} + \|A_1^2 u\|_{L^2} + \|V_1 u\|_{L^{q'}}. \end{aligned}$$

Using the assumptions (2.2)–(2.5) on A and V , we immediately see from Hölder's inequality that

$$\begin{aligned} & \|(\nabla \cdot A_1)u\|_{L^2} + 2\|A_0 \cdot A_1 u\|_{L^2} + \|A_1^2 u\|_{L^2} + \|V_1 u\|_{L^{q'}} \\ & \leq (3 + 2\|\langle x \rangle^{-1} A_0\|_{L^\infty}) \epsilon \|u\|_X. \end{aligned}$$

It remains to prove

$$(4.6) \quad \|\langle x \rangle^{\frac{1+\mu}{2}} E_{-1/2} A_1 \cdot \nabla u\|_{L^2} \leq \epsilon C_{n,\mu,\delta} \|u\|_X.$$

We bound the left hand side from above by

$$\|\langle x \rangle^{\frac{1+\mu}{2}} A_1 \cdot E_{-1/2} \nabla u\|_{L^2} + \|\langle x \rangle^{\frac{1+\mu}{2}} [E_{-1/2}, A_1] \cdot \nabla u\|_{L^2} =: I + II.$$

By (2.3), the smoothing estimate in Theorem 3.2 and the L^2 -boundedness of operators in $Op^w(S(1, g) \subset Op^w(S_{1,0}^0))$, we have the estimate

$$I \leq \epsilon \|\langle x \rangle^{-\frac{1+\mu}{2}} E_{-1/2} \nabla u\|_{L^2} \leq \epsilon \|E_{-1} \nabla u\|_{L^2} \lesssim \epsilon \|u\|_{L^2}.$$

In the first inequality we used that $\delta \geq \mu$ (with δ as in (2.3)). In order to prove $II \lesssim \epsilon \|u\|_X$, we define

$$T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n), \quad T := \langle x \rangle^{\frac{1+\mu}{2}} [E_{-1/2}, A_1] \langle x \rangle^{\frac{1+\mu}{2}}.$$

Then we can write

$$(4.7) \quad II = \|T \langle D \rangle^{1/2} \langle D \rangle^{-1/2} \langle x \rangle^{-\frac{1+\mu}{2}} \nabla E_{-1/2} \langle x \rangle^{\frac{1+\mu}{2}} \langle x \rangle^{-\frac{1+\mu}{2}} E_{1/2} u\|_{L^2}.$$

We claim that $T \langle D \rangle^{1/2}$ extends to a bounded operator on L^2 with norm $\lesssim \epsilon$. From (4.7) we then see that

$$II \lesssim \epsilon \|\langle D \rangle^{-1/2} \langle x \rangle^{-\frac{1+\mu}{2}} \nabla E_{-1/2} \langle x \rangle^{\frac{1+\mu}{2}}\|_{\mathcal{B}(L^2)} \|\langle x \rangle^{-\frac{1+\mu}{2}} E_{1/2} u\|_{L^2} \lesssim \epsilon \|u\|_X.$$

In the last inequality we used the L^2 -boundedness of operators in $Op^w(S(1, g))$ again. It remains to prove the L^2 -boundedness of $T \langle D \rangle^{1/2}$. We will show that

$$(4.8) \quad \begin{aligned} |T(x, y)| + |\nabla_x T(x, y)| + |\nabla_y T(x, y)| & \lesssim \epsilon |x - y|^{-n+\nu} \mathbf{1}\{|x - y| \leq 1\} \\ & + \epsilon |x - y|^{-n-\nu} \mathbf{1}\{|x - y| > 1\} \end{aligned}$$

for some $\nu > 0$. By Schur's inequality, this implies $\|T^*\|_{\mathcal{B}(L^2, W^{1,2})} \lesssim \epsilon$ and hence the claim, by duality. We split the kernel as

$$T(x, y) = T_1(x, y) + T_2(x, y),$$

where

$$T_1(x, y) := T(x, y) \mathbf{1}\{|x - y| \leq 1\}, \quad T_2(x, y) := T(x, y) \mathbf{1}\{|x - y| > 1\}.$$

Estimate for $T_1(x, y)$: The distributional kernel of $E_{-1/2}$ is

$$E_{-1/2}(x, y) = \lim_{\lambda \rightarrow \infty} \int e^{i(x-y) \cdot \xi} e_{-1/2} \left(\frac{x+y}{2}, \xi \right) \gamma(\xi/\lambda) d\xi$$

where $\gamma \in \mathcal{S}(\mathbb{R}^n)$ is such that $\gamma(0) = 1$. For $|x - y| \neq 0$, $E_{-1/2}(x, y)$ coincides with a smooth function, satisfying the bound

$$(4.9) \quad |E_{-1/2}(x, y)| \lesssim |x - y|^{1/2-n}, \quad |x - y| \neq 0,$$

see [31, Proposition VI.4.1]. By assumption (2.3) on A and Petree's inequality

$$\langle x \rangle^s \leq \langle x - y \rangle^{|s|} \langle y \rangle^s, \quad s \in \mathbb{R},$$

we get for $\delta \geq \mu$ and $0 < |x - y| \leq 1$,

$$|T_1(x, y)| \lesssim \epsilon \langle x \rangle^{-1-\delta} \langle x \rangle^{\frac{1+\mu}{2}} \langle y \rangle^{\frac{1+\mu}{2}} |x - y|^{1/2-n} \lesssim \epsilon |x - y|^{1/2-n}.$$

Similarly, using the assumption (2.3) on ∇A_1 , we get the same estimate for all terms of $\nabla_x T_1(x, y)$, except when the gradient falls on $E_{-1/2}(x, y)$. Since $\nabla E_{-1/2} \in Op^w(S_{1,0}^{1/2}(\mathbb{R}^n))$, we have $|\nabla_x E_{-1/2}(x, y)| \lesssim |x - y|^{-1/2-n}$, see [31, Proposition VI.4.1] again. Using the mean-value theorem, we then find for $\delta \geq \mu$ and $0 < |x - y| \leq 1$,

$$\begin{aligned} |\nabla_x T_1(x, y)| &\lesssim \left| \frac{A_1(x) - A_1(y)}{x - y} \right| \langle x \rangle^{\frac{1+\mu}{2}} \langle y \rangle^{\frac{1+\mu}{2}} |x - y|^{1/2-n} \\ &\lesssim \sup_{z \in \{x+t(y-x): 0 \leq t \leq 1\}} |\nabla A_1(z)| \langle x \rangle^{\frac{1+\mu}{2}} \langle y \rangle^{\frac{1+\mu}{2}} |x - y|^{1/2-n} \\ &\lesssim \epsilon \sup_{z \in \{x+t(y-x): 0 \leq t \leq 1\}} \langle z \rangle^{-1-\delta} \langle x \rangle^{\frac{1+\mu}{2}} \langle y \rangle^{\frac{1+\mu}{2}} |x - y|^{1/2-n} \\ &\lesssim \epsilon \langle x \rangle^{-1-\delta} \langle x \rangle^{1+\mu} |x - y|^{1/2-n} \lesssim \epsilon |x - y|^{1/2-n}. \end{aligned}$$

In the penultimate inequality, we used Petree's inequality again. The estimate for $|\nabla_y T_1(x, y)|$ is analogous.

Estimate for $T_2(x, y)$: For $|x - y| \geq 1$, we have $|E_{-1/2}(x, y)| \leq C_N |x - y|^{-N}$ for any $N > 0$, see again [31, Proposition VI.4.1]. The same is of course true for $\nabla_x E_{-1/2}(x, y)$ and $\nabla_y E_{-1/2}(x, y)$. Using Petree's inequality once more, together with the bounds (2.3), we thus obtain

$$\begin{aligned} |T_2(x, y)| + |\nabla_x T_2(x, y)| + |\nabla_y T_2(x, y)| &\leq C_N \epsilon |x - y|^{-N} \langle x \rangle^{-1-\delta} \langle x \rangle^{1+\mu} \langle x - y \rangle^{2+\delta+\mu} \\ &\leq C_N \epsilon |x - y|^{-N+2+\delta+\mu}. \end{aligned}$$

Choosing $N > n + 2 + \delta + \mu$ yields the desired bound (4.8). $\square$

Proof of Theorem 2.2 in the general case. By what we have already proved, Theorem 2.2 holds for $\tilde{P}_0$. Lemma 4.1 then yields that for $u \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\|(P - z)u\|_{X'} \geq \|(P_0 - z)u\|_{X'} - \|Lu\|_{X'} \geq \left(\frac{1}{C_0} - C_L \epsilon \right) \|u\|_X \geq \frac{1}{2C_0} \|u\|_X,$$

provided $\epsilon \leq (2C_L C_0)^{-1}$; here, C_0, C_L are the constants in Theorem 2.2 and Lemma 4.1, respectively. $\square$

5. APPLICATION TO COMPLEX-VALUED POTENTIALS

In this Section we use the resolvent estimate to find upper bounds on the location of (complex) eigenvalues for Schrödinger operators with complex-valued potentials. For Schrödinger operators $-\Delta + V$ with decaying but singular potentials V , results of this type have been established e.g. in [1] in one dimension and in [9, 10, 7] in higher dimensions. Estimates for sums of eigenvalues of the Schrödinger operator with constant magnetic field perturbed by complex electric potentials were obtained in [29]. The conditions on the potential there are much more restrictive than ours and the results (when applied to a single eigenvalue) are considerably weaker. To our knowledge, our eigenvalue estimates are the first with a complex-valued magnetic potential.

For definiteness, we assume that P_0 is either the harmonic oscillator or the Schrödinger operator with constant magnetic field (called the Landau Hamiltonian

in quantum mechanics), but other examples could easily be accommodated. Hence, from now on, either

$$(5.1) \quad P_0 = -\Delta + |x|^2, \quad x \in \mathbb{R}^n \quad (\text{Harmonic Oscillator}),$$

or, for n even and $B_0 > 0$,

$$(5.2) \quad P_0 = \sum_{j=1}^{n/2} \left[\left(-i\partial_{x_j} - \frac{B_0}{2}y_j \right)^2 + \left(-i\partial_{y_j} + \frac{B_0}{2}x_j \right)^2 \right] \quad (\text{Landau Hamiltonian}),$$

where in the case (5.2) we denoted the independent variable by $(x, y) = z \in \mathbb{R}^n$. In the mathematical literature, (5.1) is also called the Hermite operator and (5.2) is known as the twisted Laplacian. The spectra of these operators can be computed explicitly to be

$$\sigma(P_0) = 2\mathbb{N} + m(n),$$

where $m(n) = n$ in the case of (5.1) and $m(n) = n/2$ in the case of (5.2).

In the notation of (2.7), the harmonic oscillator (5.1) corresponds to P_0 with $V_0(x) = |x|^2$, $A_0(x) = 0$, while the Landau Hamiltonian (5.2) corresponds to P_0 with $V_0(z) = 0$ and

$$A_0(z) = \frac{B_0}{2}(-y_1, x_1, \dots, -y_{n/2}, x_{n/2}), \quad z = (x, y) \in \mathbb{R}^n.$$

As before, we allow a real-valued bounded potential $W \in L^\infty(\mathbb{R}^n, \mathbb{R})$ in the definition of the unperturbed operator $\tilde{P}_0 = P_0 + W$. We now consider a perturbation of $\tilde{P}_0$ by a *complex-valued* electromagnetic potential (A_1, V_1) , that is we consider the Schrödinger operator

$$P = (-i\nabla + A)^2 + V = \tilde{P}_0 + L, \quad \text{dom}(P) = \mathcal{D}(\mathbb{R}^n).$$

Here, L is given by (4.5). We only require a smallness assumption on A_1 , not on V_1 . To be clear, we repeat the assumptions on A_1, V_1 at this point.

$$(5.3) \quad A_1 \in C^1(\mathbb{R}^n, \mathbb{C}^n), \quad \sup_{x \in \mathbb{R}^n} \langle x \rangle^{1+\delta} (|A_1(x)| + |\nabla A_1(x)|) \leq \epsilon.$$

$$(5.4) \quad V_1 \in L^r(\mathbb{R}^n, \mathbb{C}), \quad r \in (1, \infty] \text{ for } n = 2 \text{ and } r \in [n/2, \infty] \text{ for } n \geq 3.$$

Let $Q(P_0)$ be the form domain³ of P_0 ,

$$(5.5) \quad Q(P_0) = \{u \in L^2(\mathbb{R}^n) : \nabla_{A_0} u \in L^2(\mathbb{R}^n), V_0^{1/2} u \in L^2(\mathbb{R}^n)\}.$$

By Lemma A.1 there exists a unique m -sectorial extension of P with the property $\text{dom}(P) \subset Q(P_0)$. By abuse of notation we will still denote this extension by P in the following theorem.

Theorem 5.1. *Assume that P_0 is either the harmonic oscillator (5.1) or the Landau Hamiltonian (5.2) (when n is even). Assume that A_1, V_1 satisfy (5.3)–(5.4). For $a > 0$ fixed, there exists $\epsilon_0 > 0$ such that whenever A_1 satisfies (2.3) with $\epsilon < \epsilon_0$, then every eigenvalue z of P with $|\text{Im}z| \geq a$ satisfies the estimate*

$$(5.6) \quad |\text{Im}z|^{1-\frac{n}{2r}} \leq \frac{C_0(1 + \|W\|_{L^\infty})}{1 - \epsilon/\epsilon_0} \|V_1\|_{L^r}.$$

Here, C_0 is the constant in the estimate (B.6), and ϵ_0 depends on $n, \delta, \mu, a, B_0, C_0$.

³ $Q(P_0) = D(q_0)$ in the notation of Appendix A. We equip $Q(P_0)$ with the form norm (A.3).

Remark 5.2. Since the left hand side of (5.6) is $\geq a^{1-\frac{n}{2r}}$ by assumption, Theorem 5.1 implies that for any $a > 0$ and $0 \leq \epsilon < \epsilon_0$, there exists $v_0 = v_0(a, \epsilon)$ such that for $\|V\|_{L^r} \leq v_0$, all eigenvalues $z \in \sigma(P)$ are contained in $\{z \in \mathbb{C} : |\operatorname{Im} z| \leq a\}$.

Proof of Theorem 5.1. Assume that $z \in \mathbb{C}$, with $|\operatorname{Im} z| \geq a$, is an eigenvalue of P , i.e. there exists $u \in \operatorname{dom}(P)$, with $\|u\|_{L^2} = 1$, such that $Pu = zu$. Since $P_0 \in \mathcal{B}(Q(P_0), Q(P_0)')$ and $L \in \mathcal{B}(Q(P_0), Q(P_0)')$ by Lemma A.1 ii), we have

$$(5.7) \quad (P_0 - z)u = -Lu \quad \text{in } Q(P_0)'.$$

Since $Q(P_0) \subset X(z)$ densely and continuously, by Lemma B.2, we have $u \in X(z)$, $\|u\|_{X(z)} \neq 0$, and (5.7) implies that $(P_0 - z)u + Lu = 0$ in $X(z)'$. By Lemma 4.1, $L \in \mathcal{B}(X(z), X(z)')$, and thus $(P_0 - z)u \in X(z)'$. This means that

$$(5.8) \quad (P_0 - z)u = -Lu \quad \text{in } X(z)'.$$

Then (5.8) and (B.5) yield

$$(5.9) \quad \frac{1}{C_0(1 + \|W\|_{L^\infty})} \|u\|_{X(z)} \leq \|(P_0 - z)u\|_{X(z)'} = \|Lu\|_{X(z)'}. \quad \square$$

We estimate the right hand side from above, as in the proof of the general case of Theorem 2.2 in Section 4, by

$$\begin{aligned} \|Lu\|_{X(z)'} &\leq \left(\epsilon C_{n,\mu,\delta} + 2\epsilon(1 + B_0) |\operatorname{Im} z|^{-1} + |\operatorname{Im} z|^{n(\frac{1}{2}-\frac{1}{q})-1} \|V_1\|_{L^r} \right) \|u\|_{X(z)} \\ &\leq \left(\epsilon C_{n,\mu,\delta,a,B_0} + |\operatorname{Im} z|^{n(\frac{1}{2}-\frac{1}{q})-1} \|V_1\|_{L^r} \right) \|u\|_{X(z)} \end{aligned}$$

Together with (5.9), this implies that

$$\|u\|_{X(z)} \leq C_0(1 + \|W\|_{L^\infty}) \left(\epsilon C_{n,\mu,\delta,a,B_0} + |\operatorname{Im} z|^{n(\frac{1}{2}-\frac{1}{q})-1} \|V_1\|_{L^r} \right) \|u\|_{X(z)}.$$

Dividing both sides by $\|u\|_{X(z)} \neq 0$, it follows that any eigenvalue z of P with $|\operatorname{Im} z| \geq a$, must satisfy the inequality

$$1 \leq C_0(1 + \|W\|_{L^\infty}) \left(\epsilon C_{n,\mu,\delta,a,B_0} + |\operatorname{Im} z|^{n(\frac{1}{2}-\frac{1}{q})-1} \|V_1\|_{L^r} \right).$$

If we set $\epsilon_0 = 1/(C_0(1 + \|W\|_{L^\infty})C_{n,\mu,\delta,a,B_0})$, then this estimate is equivalent to (5.6). $\square$

Instead of the Landau Hamiltonian (5.2), we can also consider the Pauli operator with constant magnetic field. For simplicity, we assume that $n = 2$ here, but the general case when n is even can be handled with no additional difficulty. On $\mathcal{D}(\mathbb{R}^2, \mathbb{C}^2)$, the Pauli operator is given by

$$(5.10) \quad P = \begin{pmatrix} (-i\nabla + A(x))^2 + B(x) & 0 \\ 0 & (-i\nabla + A(x))^2 - B(x) \end{pmatrix} + W(x) + V_1(x).$$

Here, $A = (A^1, A^2) = (A_0^1, A_0^2) + (A_1^1, A_1^2)$ and $B = \partial_1 A^2 - \partial_2 A^1$. We choose $A_0(z) = \frac{B_0}{2}(-y, x)$ for $z = (x, y) \in \mathbb{R}^2$ and $B_0 > 0$. Although we could easily allow W and V_1 to be matrix-valued potentials, we assume that they are scalar multiples of the identity in $\mathbb{C}^2$.

Corollary 5.3. *Assume that $n = 2$ and that P is the Pauli operator (5.10). Assume also that A_1, V_1 satisfy (5.3)–(5.4). For $a > 0$ fixed, there exists $\epsilon_0 > 0$ such that*

whenever A_1 satisfies (2.3) with $\epsilon < \epsilon_0$, then every eigenvalue z of P with $|\operatorname{Im} z| \geq a$ satisfies the estimate

$$|\operatorname{Im} z|^{1-\frac{n}{2r}} \leq \frac{C_0(1+B_0+\|W\|_{L^\infty})}{1-\epsilon/\epsilon_0} \|V_1\|_{L^r}.$$

Here, C_0 is the constant in the estimate (B.6), and ϵ_0 depends on $n, \delta, \mu, a, B_0, C_0$.

Proof. In view of the direct sum structure of (5.10), the proof reduces to proving (5.6) for eigenvalues z of the Schrödinger operators

$$P_\pm = (-i\nabla + A(x))^2 \pm B(x) + W(x) + V_1(x).$$

The proof is analogous to the one of Theorem 5.1. $\square$

Remark 5.4. Note that the reason we were able to remove the smallness assumption on V_1 , but not on A_1 is that the smoothing part of the $X(z)$ norm is z -independent (see Appendix B).

APPENDIX A. DEFINITION OF P AS AN m -SECTORIAL OPERATOR

Here we provide the operator theoretic details that were omitted in Section 5. For simplicity, we assume in addition to (2.2)–(2.5) that $V_0 \geq 0$. This ensures that the quadratic form of the unperturbed operator is nonnegative and makes the definition of the form sum easier. Note that the assumption is satisfied in the applications in Section 5. Writing L in (4.5) in the form

$$L = -2iA_1 \cdot \nabla_{A_0} - i(\nabla_{A_0} \cdot A_1) - A_0 \cdot A_1 + A_1^2 + V_1 \equiv L_A + V_1,$$

we define the quadratic forms

$$(A.1) \quad \mathfrak{p}_0(u) = \|\nabla_{A_0} u\|_{L^2}^2 + \|V_0^{1/2} u\|^2,$$

$$(A.2) \quad \mathfrak{l}(u) = \langle L_A u, u \rangle + \langle |V_1|^{1/2} u, V_1^{1/2} u \rangle,$$

with domains

$$\begin{aligned} D(\mathfrak{p}_0) &= \{u \in L^2(\mathbb{R}^n) : \nabla_{A_0} u \in L^2(\mathbb{R}^n), V_0^{1/2} u \in L^2(\mathbb{R}^n)\}, \\ D(\mathfrak{l}) &= \{u \in L^2(\mathbb{R}^n) : \nabla_{A_0} u \in L^2(\mathbb{R}^n), |V_1|^{1/2} u \in L^2(\mathbb{R}^n)\}, \end{aligned}$$

Here, $\nabla_{A_0} = \nabla + iA_0(x)$ is the covariant derivative, and $V_1^{1/2} = e^{i\varphi}|V_1|^{1/2}$ for $V_1 = |V_1|e^{i\varphi}$. We also use the magnetic Sobolev spaces

$$H_{A_0}^1(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \nabla_{A_0} u \in L^2(\mathbb{R}^n)\}, \quad \|u\|_{H_{A_0}^1} = \|u\|_{L^2} + \|\nabla_{A_0} u\|_{L^2}.$$

We have the continuous and dense⁴ embedding $D(\mathfrak{p}_0) \subset H_{A_0}^1(\mathbb{R}^n)$ when $D(\mathfrak{p}_0)$ is equipped with the norm

$$(A.3) \quad \|u\|_{+1} := (\|u\|_{L^2}^2 + \mathfrak{p}_0(u))^{1/2}.$$

The content of the following lemma is standard. In order to be self-contained, we give a proof.

Lemma A.1. *Assume (2.2)–(2.5) and that $V_0 \geq 0$. Then the following hold.*

- i) $\mathfrak{p}_0$ is a closed and nonnegative form.
- ii) $\mathfrak{l}$ is relatively bounded with respect to $\mathfrak{p}_0$. with relative bound zero.
- iii) $\mathfrak{p}_0 + \mathfrak{l}$, with $D(\mathfrak{p}_0 + \mathfrak{l}) = D(\mathfrak{p}_0)$, is a closed sectorial form.

⁴By [25, Theorem 7.22] $\mathcal{D}(\mathbb{R}^n)$ is dense in $H_{A_0}^1(\mathbb{R}^n)$.

- iv) *There exists a unique m -sectorial operator P associated to the form $\mathfrak{p}_0 + \mathfrak{l}$, with the property that $\text{dom}(P) \subset D(\mathfrak{p}_0)$.*

Proof. i) $\mathfrak{p}_0$ is clearly nonnegative. To prove that it is closed, let $(u_n)_n \subset D(\mathfrak{p}_0)$ be a Cauchy sequence with respect to the form norm $\|\cdot\|_{+,1}$. Since $L^2(\mathbb{R}^n)$ is complete, it follows that there exist $u, v, w \in L^2(\mathbb{R}^n)$ such that

$$u_n \rightarrow u, \quad \nabla_{A_0} u_n \rightarrow v, \quad V_0^{1/2} u_n \rightarrow w \quad \text{in } L^2(\mathbb{R}^n).$$

Since ∇_{A_0} and $V_0^{1/2}$ are continuous from $\mathcal{D}'(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$, it follows that $\nabla_{A_0} u = v \in L^2(\mathbb{R}^n)$ and $V_0^{1/2} u = w \in L^2(\mathbb{R}^n)$. This shows that $D(\mathfrak{p}_0)$ is complete, i.e. $\mathfrak{p}_0$ is closed.

ii) In view of the assumption $V_0 \geq 0$ it is sufficient to prove the Lemma for $V_0 = 0$; in this case, $D(\mathfrak{p}_0) = H_{A_0}^1(\mathbb{R}^n)$. Also, since W is bounded, it does not affect the relative bound, so we may assume $W = 0$ as well. Fix $\epsilon > 0$ arbitrarily small. By Sobolev embedding and the diamagnetic inequality

$$(A.4) \quad \|u\|_{H^1} \leq \|u\|_{H_{A_0}^1},$$

we have the continuous (and dense) embedding

$$(A.5) \quad H_{A_0}^1(\mathbb{R}^n) \subset L^q(\mathbb{R}^n).$$

Hölder's inequality and (A.5) then yield the estimate

$$(A.6) \quad \| |V_1|^{1/2} u \|_{L^2} \leq C \|V_1\|_{L^{\frac{q}{q-2}}}^{1/2} \|u\|_{H_{A_0}^1} = C \|V_1\|_{L^r}^{1/2} \|u\|_{H_{A_0}^1}.$$

By decomposing $V_1 = V_{1,R} + (V_1 - V_{1,R})$, with $V_{1,R} = V_1 \mathbf{1}\{|V_1(x)| \leq R\}$ and absorbing $V_{1,R} \in L^\infty(\mathbb{R}^n)$ into W , we can assume that $C \|V_1\|_{L^r} \leq \epsilon/2$.

Next, we observe that since $A_1 \in L^\infty(\mathbb{R}^n)$, we have

$$(A.7) \quad |\langle L_A u, u \rangle| \leq \frac{\epsilon}{2} \|\nabla_{A_0} u\|_{L^2}^2 + C_\epsilon \|A_1\|_{L^\infty}^2 \|u\|_{L^2}^2$$

for some $C_\epsilon > 0$. Altogether, (A.6)–(A.7) imply that $D(\mathfrak{l}) \subset D(\mathfrak{p}_0)$, and that we have the estimate

$$\mathfrak{l}(u) \leq C \|u\|_{L^2}^2 + \epsilon \mathfrak{p}_0(u),$$

with a constant C depending on ϵ , $\|A_1\|_{L^\infty}$ and $\|V_1\|_{L^r}$.

iii) follows from [16, Theorem VI.3.4] and i)-ii).

iv) follows from the first representation theorem [16, Theorem VI.2.1]. $\square$

APPENDIX B. A MORE PRECISE VERSION OF THEOREM 2.2

We prove a more precise version of Theorem 2.2 for P_0 that takes into account the z -dependence of the constant. It is most convenient to include this dependence in the definition of the spaces X and X' , compare Section 4 in [22]. The weighted spaces carry the norms

$$\begin{aligned} \|u\|_{X(z)} &= |\text{Im} z|^{\frac{1}{2}} \|u\|_{L^2} + \|\langle x \rangle^{-\frac{1+\mu}{2}} E_{1/2} u\|_{L^2} + |\text{Im} z|^{\frac{1}{2} - \frac{n}{2}(\frac{1}{2} - \frac{1}{q})} \|u\|_{L^q}, \\ \|f\|_{X'(z)} &= \inf_{f=f_1+f_2+f_3} \left(|\text{Im} z|^{-\frac{1}{2}} \|f_1\|_{L^2} + \|\langle x \rangle^{\frac{1+\mu}{2}} E_{-1/2} f_2\|_{L^2} \right. \\ &\quad \left. + |\text{Im} z|^{\frac{n}{2}(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2}} \|f_3\|_{L^{q'}} \right). \end{aligned}$$

Proposition B.1. *$X(z)$ and $X'(z)$ are Banach spaces.*

Proof. The dual of a normed space is always complete, so we only need to show that $X(z)$ is complete. We may as well prove this for $|\operatorname{Im} z| = 1$, i.e. for the case $X(z) = X$. We recall that $X = Y \cap L^{q'}$, where Y was defined in (3.16). The spaces Y and $L^{q'}$ are compatible in the sense that whenever $(u_n)_n \subset X$, $u \in Y$, $v \in L^{q'}$ satisfy $\|u_n - u\|_Y \rightarrow 0$ and $\|v_n - u\|_{L^{q'}} \rightarrow 0$, then $u = v \in X$. This is true because $u = v$ in $\mathcal{D}'$. It follows that X is complete if and only if Y is complete (since $L^{q'}$ is complete). Moreover, Y is complete if and only if the operator $T = \langle x \rangle^{-\frac{1+\mu}{2}} E_{1/2} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ with $\operatorname{dom}(T) = \mathcal{D}(\mathbb{R}^n)$ is closable. To prove the latter, let $(u_n) \subset \mathcal{D}(\mathbb{R}^n)$ be a sequence such that $u_n \rightarrow 0$ and $Tu_n \rightarrow v$ in $L^2(\mathbb{R}^n)$. We have to show that $v = 0$. Indeed, since $T : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is continuous [34, Theorem 4.16], it follows from the assumptions that $Tu_n \rightarrow 0$ in $\mathcal{S}'(\mathbb{R}^n)$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, it follows that $v = 0$ in $L^2(\mathbb{R}^n)$. $\square$

Lemma B.2. *Assume (2.2)–(2.5) and that $V_0 \geq 0$. Let $\mathfrak{p}_0$ be given by (5.5). Then we have the dense and continuous embedding $D(\mathfrak{p}_0) \subset X(z)$, where $D(\mathfrak{p}_0)$ is equipped with the form norm $\|\cdot\|_{+1}$.*

Proof. Since $D(\mathfrak{p}_0) \subset H_{A_0}^1(\mathbb{R}^n)$ continuously and densely, it is sufficient to prove that the embedding $H_{A_0}^1(\mathbb{R}^n) \subset X(z)$ is continuous and dense. In view of (A.5) and the fact that $\mathcal{D}(\mathbb{R}^n)$ is dense in $H_{A_0}^1(\mathbb{R}^n)$ [25, Theorem 7.22], it remains⁵ to prove the estimate

$$(B.1) \quad \|\langle x \rangle^{-\frac{1+\mu}{2}} E_{1/2} u\|_{L^2} \lesssim \|u\|_{H_{A_0}^1}, \quad u \in \mathcal{D}(\mathbb{R}^n).$$

Since the conditions of Lemma 2.2 in [28] are satisfied for $-\Delta_{A_0}$ (i.e. for P_0 with $V_0 = 0$), we have an analogue of the commutator estimate (3.23), namely⁶

$$(B.2) \quad \|\langle x \rangle^{-\frac{1+\mu}{2}} E_{1/2} u\|_{L^2}^2 \lesssim \langle -i[-\Delta_{A_0}, Op^w(\lambda)]u, u \rangle + \|u\|_{L^2}^2.$$

Using the L^2 -boundedness of $Op^w(\lambda) \in Op^w(S(1, g))$ and of $[\nabla_{A_0}, Op^w(\lambda)] \in Op^w(S(1, g))$, we estimate

$$(B.3) \quad \langle -i[-\Delta_{A_0}, Op^w(\lambda)]u, u \rangle \leq 2\|\nabla_{A_0} u\|_{L^2} \|\nabla_{A_0} Op^w(\lambda)u\|_{L^2} \lesssim \|u\|_{H_{A_0}^1}.$$

Combining (B.2)–(B.3), we get (B.1). $\square$

For technical reasons, we assume the following condition on the spectral projections $\Pi_{[k, k+1]}$ of P_0 in $n = 2$ dimensions,

$$(B.4) \quad \|\Pi_{[k, k+1]}\|_{\mathcal{B}(L^2, L^\infty)} \lesssim 1, \quad n = 2.$$

This is only used in an interpolation argument, see (4.23) in [22]. Note that the estimate does not follow from Strichartz estimates. However, for the harmonic oscillator and the Schrödinger operator with constant magnetic field, (B.4) is known to be true [21, 19]; see also [23, 30] for corresponding results on Schrödinger operators with a Riemannian metric, but without magnetic field. We do not whether (B.4) is true under the general assumptions (2.2), (2.4) on P_0 . In any case, we could do without (B.4) at the expense of an ϵ -loss in the exponent of the L^q -part of the $X(z)$ -norm.

⁵Recall that $X(z)$ is defined as the closure of $\mathcal{D}(\mathbb{R}^n)$ in the norm $\|\cdot\|_{X(z)}$.

⁶The symbol $\lambda(x, \xi)$ in [28] depends only on the principal part of $P_0(x, \xi)$.

Theorem B.3. *Assume that $V_0 \geq 0$ and that A_0, V_0 satisfy Assumptions (2.2), (2.4). If $n = 2$, assume also that (B.4) holds. Fix $a > 0$. Then for any $z \in \mathbb{C}$ with $|\operatorname{Im} z| \geq a$, the resolvent $(\tilde{P}_0 - z)^{-1} : L^2(\mathbb{R}^n) \rightarrow \operatorname{dom}(P_0)$ extends to a bounded operator in $\mathcal{B}(X(z)', X(z))$, and we have the estimate*

$$(B.5) \quad \|(\tilde{P}_0 - z)^{-1}u\|_{\mathcal{B}(X(z)', X(z))} \leq C_0(1 + \|W\|_{L^\infty}).$$

The constant C_0 depends on n, q, μ, a , and on finitely many seminorms C_α in (2.2) and (2.4).

Proof. We assume that $W = 0$, i.e. that $\tilde{P}_0 = P_0$. The general case requires the same argument as in the proof of Theorem 2.2. We first prove the estimate

$$(B.6) \quad \|u\|_{X(z)} \lesssim \|(P_0 - z)u\|_{X(z)', \quad u \in \mathcal{D}(\mathbb{R}^n).$$

The proof differs only slightly from that of Theorem 2.2 for P_0 . For the convenience of the reader we give a sketch of the proof, highlighting the steps where the z -dependence plays a role.

$L^{q'} \rightarrow L^q$ estimate: For $n \geq 3$, interpolation between (3.12) with $q = 2n/(n-2)$ and (3.2) produces the estimate

$$(B.7) \quad \|(P_0 - z)^{-1}\|_{\mathcal{B}(L^{q'}, L^2)} \lesssim |\operatorname{Im} z|^{\frac{n}{2}(\frac{1}{2} - \frac{1}{q}) - 1}.$$

For $n = 2$, the dual of (B.4) implies (as in the proof of (3.12)) that

$$(B.8) \quad \|(P_0 - z)^{-1}\|_{\mathcal{B}(L^1, L^2)} \lesssim |\operatorname{Im} z|^{-1/2}.$$

Interpolating (B.8) with (3.2) yields (B.7) in the case $n = 2$. Inequality (3.13) can be modified to

$$(B.9) \quad \|u\|_{L^q} \lesssim |\operatorname{Im} z|^{\frac{n}{2}(\frac{1}{2} - \frac{1}{q}) - \frac{1}{2}} \|u\|_{L^2} + |\operatorname{Im} z|^{n(\frac{1}{2} - \frac{1}{q}) - 1} \|(P_0 - z)u\|_{L^{q'}}.$$

To see this, one applies the Strichartz estimates (3.7) to the function $v(x, t) = e^{-itz}u(x)$ with t localized to an interval of size $\mathcal{O}(|\operatorname{Im} z|^{-1})$. Combining (B.7) and (B.9), we obtain

$$(B.10) \quad \|u\|_{L^q} \lesssim |\operatorname{Im} z|^{n(\frac{1}{2} - \frac{1}{q}) - 1} \|(P_0 - z)u\|_{L^{q'}}.$$

$Y' \rightarrow Y$ estimate: Denote by $Y(z)$ the space defined by the part of the $X(z)$ -norm without the L^q -norm. From (3.19)–(3.22), we get

$$(B.11) \quad \|u\|_{Y(z)} \lesssim \|(P_0 - z)u\|_{Y(z)'}$$

$L^{q'} \rightarrow Y$ estimate: From (3.23), (B.7), (B.10), (3.2) and since $|\operatorname{Im} z| \geq a$, it follows that

$$(B.12) \quad \begin{aligned} \|u\|_{Y(z)}^2 &\lesssim \langle -i[P_0 - z, Op^w(\lambda)]u, u \rangle + |\operatorname{Im} z| \|u\|_{L^2}^2 \\ &\leq 2\|Op^w(\lambda)u\|_{L^q} \|(P_0 - z)u\|_{L^{q'}} + 2|\operatorname{Im} z| \|Op^w(\lambda)u\|_{L^2} \|u\|_{L^2} + |\operatorname{Im} z| \|u\|_{L^2}^2 \\ &\leq C_{\lambda, q}^2 |\operatorname{Im} z|^{-n(\frac{1}{2} - \frac{1}{q}) + 1} \|u\|_{L^q}^2 + |\operatorname{Im} z|^{n(\frac{1}{2} - \frac{1}{q}) - 1} \|(P_0 - z)u\|_{L^{q'}}^2 \\ &\quad + |\operatorname{Im} z| (1 + 2C_\lambda) \|u\|_{L^2}^2 \\ &\lesssim |\operatorname{Im} z|^{n(\frac{1}{2} - \frac{1}{q}) - 1} \|(P_0 - z)u\|_{L^{q'}}^2. \end{aligned}$$

In the third line we also used the Peter Paul inequality

$$2ab \leq \epsilon a^2 + \frac{b^2}{\epsilon} \quad \text{with } \epsilon = |\operatorname{Im} z|^{-n(\frac{1}{2} - \frac{1}{q}) + 1}.$$

The previous estimates (B.7), (B.10), (B.11), (B.12) in conjunction with the trivial L^2 -estimate (3.2) prove (B.6).

Next we prove (B.5). Consider the norm

$$|||u||| := \|u\|_{X(z)} + \|(P_0 - z)u\|_{X(z)', \quad u \in \mathcal{D}(\mathbb{R}^n).$$

By (B.6), we have

$$(B.13) \quad |||u||| \lesssim \|(P_0 - z)u\|_{X(z)'} \leq \|(P_0 - z)u\|_{L^2} \leq (1 + |z|)\|u\|_{P_0},$$

where $\|u\|_{P_0} = \|u\|_{L^2} + \|P_0 u\|_{L^2}$ is the graph norm of P_0 . Since $\mathcal{D}(\mathbb{R}^n)$ is a core for P_0 , it follows from (B.13) that

$$\overline{\mathcal{D}(\mathbb{R}^n)}^{|||\cdot|||} \supset \overline{\mathcal{D}(\mathbb{R}^n)}^{\|\cdot\|_{P_0}} = \text{dom}(P_0).$$

This implies, in particular, that (B.6) holds for all $u \in \text{dom}(P)$. Then, since $(P_0 - z) : \text{dom}(P) \rightarrow L^2$ is bijective, it follows that

$$(B.14) \quad \|(P_0 - z)^{-1}f\|_{X(z)} \lesssim \|f\|_{X(z)', \quad \text{for all } f \in L^2(\mathbb{R}^n).$$

Because $X(z) \subset L^2(\mathbb{R}^n)$ is an embedding (and therefore injective), it follows that $L^2(\mathbb{R}^n)$ is dense in $X(z)'$. Therefore, (B.5) follows from (B.14) by the BLT theorem. $\square$

APPENDIX C. PROOF OF (4.3)

We begin with the following lemma.

Lemma C.1. *Let $z \in \mathbb{C} \setminus \sigma(P_0)$, and let $u \in Y$ be such that $(P_0 - z)u \in L^{q'}(\mathbb{R}^n)$. Then there exist $(u_\epsilon)_{0 \leq \epsilon \leq 1} \subset \mathcal{D}(\mathbb{R}^n)$ such that*

- i) $u_\epsilon \rightarrow u$ in Y , and
- ii) $(P_0 - z)u_\epsilon \rightarrow (P_0 - z)u$ in $L^{q'}(\mathbb{R}^n)$,

as $\epsilon \rightarrow 0$.

Proof. The method of proof is similar as that of [34, C.2.12]. We first show that there exist $(u_\epsilon)_{0 \leq \epsilon \leq 1} \subset \mathcal{S}(\mathbb{R}^n)$ satisfying i)–ii). To this end, pick $\chi \in \mathcal{D}(\mathbb{R}^{2n})$ with $\chi \equiv 1$ in $B(0, 1)$, and set $\chi_\epsilon(x, \xi) = \chi(\epsilon x, \epsilon \xi)$. Since $\chi_\epsilon^w(x, D) : \mathcal{S}' \rightarrow \mathcal{S}$ [34, Theorem 4.1], we have that $u_\epsilon = \chi_\epsilon^w(x, D)u \in \mathcal{S}$. We claim that for every $p \in [1, \infty)$,

$$(C.1) \quad u_\epsilon \rightarrow u \quad \text{in } L^p(\mathbb{R}^n)$$

as $\epsilon \rightarrow 0$. It is sufficient to show that (C.1) holds for all $u \in \mathcal{S}(\mathbb{R}^n)$ and that

$$(C.2) \quad \|\chi_\epsilon^w(x, D)u\|_{L^p} \leq C\|u\|_{L^p}, \quad \text{for all } u \in L^p,$$

with a constant C independent of $\epsilon \in [0, 1]$. The proof of (C.1) for $u \in \mathcal{S}(\mathbb{R}^n)$ follows from a dominated convergence argument since $\chi_\epsilon(x, \xi) \rightarrow 1$ pointwise as $\epsilon \rightarrow 0$. We omit the details. We next observe that for all $\alpha \in \mathbb{N}^{2n}$, $z = (x, \xi) \in \mathbb{R}^{2n}$ and $m \in [0, |\alpha|]$, we have

$$|\partial_z^\alpha \chi_\epsilon(z)| \leq C_\alpha \epsilon^m \langle z \rangle^{m-|\alpha|},$$

where $C_\alpha = \sup_{z \in \mathbb{R}^{2n}} \langle z \rangle^{|\alpha|} |\partial_z^\alpha \chi(z)|$. In particular, since $\langle z \rangle \geq \max\{\langle \xi \rangle, \langle x \rangle\}$, we have

$$(C.3) \quad \partial_z^\alpha \chi_\epsilon \in \epsilon^m S(\langle z \rangle^m, g) \quad \text{for all } m \in [0, |\alpha|].$$

Choosing $\alpha = m = 0$, we deduce (C.2) from [31, Proposition VI.5.4].

For convenience, we set

$$(C.4) \quad T = \langle x \rangle^{-\frac{1+\mu}{2}} E_{1/2} \in Op^w(S(\langle \xi \rangle^{1/2}, g)).$$

To finish the proof of i), it remains to show that

$$\lim_{\epsilon \rightarrow 0} \|Tu - Tu_\epsilon\|_{L^2} = 0.$$

By (C.1) for $p = 2$ and with u replaced by Tu , it is sufficient to show that

$$(C.5) \quad \lim_{\epsilon \rightarrow 0} \|[T, \chi_\epsilon^w(x, D)]u\|_{L^2} = 0.$$

Using (C.3), with $m = 1/2$, $|\alpha| = 1$, and (C.4), it follows from (3.15) that

$$[T, \chi_\epsilon^w(x, D)] \in \epsilon^{1/2} Op^w(S(1, g)).$$

Here, we also used that $\langle \xi \rangle^{1/2} \leq \langle z \rangle^{1/2}$ and $\langle x \rangle^{-1} \langle \xi \rangle^{-1} \leq \langle z \rangle^{-1}$. This proves (C.5).

To prove ii), by the same argument as before, it is sufficient to show that

$$(C.6) \quad \lim_{\epsilon \rightarrow 0} \|[P_0, \chi_\epsilon^w(x, D)]u\|_{L^2} = 0.$$

Assumptions (2.2) and (2.4) imply that for z in the support of χ_ϵ ,

$$|\partial_z^\alpha P_0(x, \xi)| \leq \begin{cases} C_\alpha \epsilon^{-1} & |\alpha| = 1, \\ C_\alpha & |\alpha| > 1. \end{cases}$$

Together with (C.3), with $m = |\alpha| = 1$, this implies that

$$[P_0, \chi_\epsilon^w(x, D)] \in Op^w(S(1, g)) \subset Op^w(S_{1,0}^0).$$

Thus, [31, Proposition VI.5.4] yields

$$(C.7) \quad \|[P_0, \chi_\epsilon^w(x, D)]\|_{L^p \rightarrow L^p} \leq C,$$

with a constant C independent of ϵ . Now pick $\psi \in \mathcal{D}(\mathbb{R}^{2n})$ supported in $B(0, 1/2)$ and such that $\psi \equiv 1$ in $B(0, 1/4)$. Set $\psi_\epsilon(x, \xi) = \psi(\epsilon x, \epsilon \xi)$. Note that

$$\text{supp}(\psi_\epsilon) \cap \text{supp}(\partial_z^\alpha \chi_\epsilon) = \emptyset, \quad \text{for } |\alpha| \geq 1.$$

In particular, the support of the Weyl symbol corresponding to $[P_0, \chi_\epsilon^w(x, D)]$ has empty intersection with $\text{supp}(\psi_\epsilon)$, modulo the error term in (3.14) with $|\alpha + \beta| \geq N$ and N arbitrarily large. By (C.3) and the composition formula⁷ (3.14), we thus have that

$$[P_0, \chi_\epsilon^w(x, D)]\psi_\epsilon^w(x, D) \in \epsilon^N Op^w(S(\langle x \rangle^{-N} \langle \xi \rangle^{-N}, g)) \subset \epsilon^N Op^w(S_{1,0}^0)$$

for all $N > 0$. It follows from [31, Proposition VI.5.4] that

$$(C.8) \quad \|[P_0, \chi_\epsilon^w(x, D)]\psi_\epsilon^w(x, D)\|_{L^p \rightarrow L^p} = \mathcal{O}(\epsilon^N).$$

Combining (C.1) for $u_\epsilon = \psi_\epsilon^w(x, D)u$, (C.7) and (C.8), we get

$$\begin{aligned} \|[P_0, \chi_\epsilon^w(x, D)]u\|_{L^p} &\leq \|[P_0, \chi_\epsilon^w(x, D)]\psi_\epsilon^w(x, D)u\|_{L^p} + \|[P_0, \chi_\epsilon^w(x, D)](u - u_\epsilon)\|_{L^p} \\ &\lesssim \epsilon^N \|u\|_{L^p} + \|u - u_\epsilon\|_{L^p} \rightarrow 0 \end{aligned}$$

as $\epsilon \rightarrow 0$. This finishes the proof of (C.6). $\square$

⁷We actually use that every seminorm of the error term in (3.14) is estimated by a product of seminorms of $a(x, \xi)$ and $b(x, \xi)$ involving at least N derivatives, see e.g. the proof of [26, Theorem 1.2.16].

Proof of (4.3). From (4.2) and Lemma C.1 it follows that

$$\|u\|_Y \lesssim \|(P_0 - z)u\|_{L^{q'}}, \quad \text{for all } u \in Y \cap (P_0 - z)^{-1}(L^2 \cap L^{q'}).$$

This is equivalent to

$$\|(P_0 - z)^{-1}f\|_Y \lesssim \|f\|_{L^{q'}}, \quad \text{for all } f \in (P_0 - z)Y \cap L^2 \cap L^{q'}.$$

We can omit the intersection with $(P_0 - z)Y$ since $\text{dom}(P_0) \subset Y$ (proved below) and $(P_0 - z)\text{dom}(P_0) = L^2$ (since $z \in \mathbb{C} \setminus \sigma(P_0)$). Since $L^2 \cap L^{q'}$ is dense in $L^{q'}$, the BLT theorem provides a continuous extension $(P_0 - z)^{-1} \in \mathcal{B}(L^{q'}, Y)$.

Since $\text{dom}(P_0)$ and Y are the closures of $\mathcal{D}(\mathbb{R}^n)$ with respect to the graph norm of P_0 and $T = \langle x \rangle^{-\frac{1+\mu}{2}} E_{1/2}$, respectively, the inclusion $\text{dom}(P_0) \subset Y$ would follow if we showed that

$$\|Tu\|_{L^2} \leq a\|P_0u\|_{L^2} + b\|u\|_{L^2}, \quad \text{for all } u \in \mathcal{D}(\mathbb{R}^n),$$

for some $a, b > 0$. Such an estimate follows from the commutator bound (3.23). $\square$

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