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Supplemental Material

for “Inverse acoustic scattering by solid obstacles: topological sensitivity and its preliminary application”

Parameter analysis with synthetic data

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In the main text, the topological sensitivity method for inverse acoustic scattering by solid obstacles is established and applied to preliminary experimental data. This supplemental material presents the results of a parameter analysis that investigates the influence of various parameters of the input signal and testing configuration on the performance of this method using carefully designed numerical experiments.

Keywords: topological sensitivity; inverse acoustic scattering; solid obstacles; parameter analysis

1. Testing configuration

The general configuration, the same as that in the main text, is repeated here for the ease of reading. As shown in Figure 1, a solid obstacle B^{true} is embedded in a homogeneous fluid layer $\Omega = \{\boldsymbol{\xi} \mid -20d < \xi_3 < 0\}$ with Dirichlet top surface $\Sigma_T = \{\boldsymbol{\xi} \mid \xi_3 = 0\}$ and Neumann bottom surface $\Sigma_B = \{\boldsymbol{\xi} \mid \xi_3 = -20d\}$, where d is the reference length. The obstacle is a ball of radius $0.5d$ centered at $\boldsymbol{x}^c = (0.5d, -1.0d, -9.0d)$. The background acoustic medium is characterized by mass density ρ and bulk modulus κ . The solid obstacle B^{true} is elastic, homogeneous and isotropic with mass density $\rho^{\text{true}} = \rho/2$, bulk modulus $\kappa^{\text{true}} = 2\kappa$, and Poisson's ratio $\nu^{\text{true}} = 0.25$.

The problem domain is sequentially illuminated by $M = 25$ point sources equally spaced on a source grid $\Gamma^{\text{src}} = \{\boldsymbol{\xi} \mid \xi_1 = -10.0d, \xi_2 = -8d:4d:8d, \xi_3 = -18d:4d:-2d\}$. For each source location $\boldsymbol{x}^j$, $j = 1, 2, \dots, M$, the induced pressure p^{obs} is monitored by all the receivers on a measurement grid $\Gamma^{\text{obs}} = \{\boldsymbol{\xi} \mid \xi_1 = 10.0d, \xi_2 = -10d:2d:10d, \xi_3 = -18d:2d:-2d\}$. Topological sensitivity values $\mathcal{T}_\Sigma(\boldsymbol{x}^o, \boldsymbol{m}^{\text{true}}; E)$ are calculated over the three cross sections passing through the center of the obstacle. The spacing between sampling points is $0.4d$.

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Figure 1. Testing configuration.

2. Parameter analysis for time-harmonic excitation

The sectional distributions of $\mathcal{T}_\Sigma(\mathbf{x}^\circ, \mathbf{m}^{\text{true}}; E)$ as $\lambda = \pi d$ (d is the diameter of the ball) is shown in Figure 5 in the main text. The location and approximate geometry of the obstacle are indicated by pronounced negative TS values. To examine the influence of the illuminating frequency, the topological sensitivity at two other wavelengths $\lambda = 2\pi d, \pi d/2$ is computed. Figure 2 compares the sectional distributions of TS at different frequencies. Although the magnitude of topological sensitivity varies with frequency and a higher frequency generally gives more localized result with more artifacts, the distribution of TS at each frequency in the investigated frequency range is capable of providing preliminary information about the location and shape of the solid obstacle in the acoustic medium.

Figure 2. Sectional distributions ($\xi_1 = 0.5d$) of $\mathcal{T}_\Sigma(\mathbf{x}^\circ, \mathbf{m}^{\text{true}}; E)$ at different frequencies: one obstacle ($\mathbf{m}^{\text{true}} = (\rho/2, 2\kappa)$ and $\nu^{\text{true}} = 0.25$, indicated by white line), time-harmonic excitation ($\lambda = 2\pi d, \pi d, \pi d/2$).

3. Parameter analysis for transient excitation

As in the main text, a *raised cosine* wavelet [1] given by

$$R_c(t) = \text{sinc}(f_0 \bar{t}) \frac{\cos(\pi \alpha f_0 \bar{t})}{1 - (2\alpha f_0 \bar{t})^2}, \quad \bar{t} = \frac{c}{d}(t - t_0) \quad (1)$$

is used as the input signal. In the following, carefully designed numerical experiments are conducted to investigate the influence of various parameters, such as the input signal

parameters α , f_0 and t_0 , the coverage and spatial steps of the source and receiver grids, the size of the object and the distance between objects, as well as their relations.

3.1 Input signal parameters

For an input signal in the form of raised cosine (1), t_0 only affects the starting time of the pulse and thus it does not affect the reconstruction as long as the scattered wave field is observed. f_0 affects the center frequency and α affects the range of frequency. To examine their influence on the TS reconstruction, a number of numerical experiments for the single obstacle testing configuration (Figure 1) are performed using input signals with different f_0 , α parameters. The parameter sets are listed in Table 1; the time-domain input signals and their frequency spectra are displayed in Figure 3.

Table 1. Input signal parameters.

Parameter set	f_0	α	Frequency range ($\omega d/c$)	Wavelength range (λ)
0 (Original)	1.0	0.85	0-5.8	$1.08d-\infty$
1	1.0	0.40	0-4.4	$1.43d-\infty$
2	0.5	0.85	0-2.9	$2.16d-\infty$
3	2.0	0.85	0-11.6	$0.54d-\infty$

Figure 3. Input signals with different parameters: raised cosine wavelets and their frequency spectra.

The resulting TS reconstructions are shown in Figure 4. To save space, the distributions of $\kappa^{\text{est}}/\kappa$ are not displayed since the relative bulk modulus is correctly identified in all four cases. The TS results shown in Figure 4(b) are almost identical to their companion images in Figure 4(a), which indicates that the influence of α is small. Comparison of Figure 4(a), (c) and (d) shows that an input signal with higher center frequency and larger frequency range better localizes the obstacle. However, despite of the apparent differences, all the TS results are capable of approximately reconstructing and identifying the obstacle, which demonstrates the robustness of this method with frequency components in the resonance region [2, 3]. The parameter values used in the original setup are chosen referring to a previous study [4].

3.2 Coverage and spatial steps of source and receiver grids

To investigate the influence of source and receiver grids, their coverage and spatial steps are varied while the input signal takes the original form (Figure 3 with $\alpha = 0.85$, $f_0 = 1.0$). Note that with fixed distance from the obstacle, changing the range of the source/receiver grid also means changing the aperture. With reference to the testing configuration shown in Figure 1, the aperture of the source/receiver grid in

Figure 4. Sectional distributions of $\mathcal{T}_{\Sigma}(\mathbf{x}^o, \mathbf{m}^{\text{opt}}; E)$ with different input signals: one obstacle ($\mathbf{m}^{\text{true}} = (\rho/2, 2\kappa)$ and $\nu^{\text{true}} = 0.25$, indicated by red line), transient excitation (a) $f_0 = 1.0, \alpha = 0.4$, (b) $f_0 = 0.5, \alpha = 0.85$, and (c) $f_0 = 2.0, \alpha = 0.85$.

each dimension can be calculated by

$$\begin{aligned}\theta_2 &= \arctan\left(\frac{\xi_2^{\max} - x_2^c}{|\xi_1 - x_1^c|}\right) + \arctan\left(\frac{x_2^c - \xi_2^{\min}}{|\xi_1 - x_1^c|}\right) \\ \theta_3 &= \arctan\left(\frac{\xi_3^{\max} - x_3^c}{|\xi_1 - x_1^c|}\right) + \arctan\left(\frac{x_3^c - \xi_3^{\min}}{|\xi_1 - x_1^c|}\right)\end{aligned}\quad (2)$$

where x_1^c, x_2^c, x_3^c are the coordinates of the center of the obstacle, ξ_1 is the ξ_1 coordinate of the source/receiver grid, and $\xi_{2/3}^{\max}, \xi_{2/3}^{\min}$ are the maximum and minimum $\xi_{2/3}$ coordinates of the source/receiver grid. In this setting, the obstacle is centered at $\mathbf{x}^c = (0.5d, -1.0d, -9.0d)$; the source and receiver grids are located at $\xi_1 = -10d$ and $\xi_1 = 10d$ respectively. The source and receiver distributions shown in Table 2 and Table 3 are considered. The tested source and receiver combinations are listed in Table 4, where T1-T3 are to analyze the influence of the source grid, T4-T6 are for the influence of the receiver grid, and T7-T12 are to investigate their joint influence.

Table 2. Source grids.

Source grid	ξ_2	ξ_3	θ_2	θ_3	Number
0 (Original)	$-8d:4d:8d$	$-18d:4d:-2d$	74.29°	74.29°	5×5
1	$-8d:8d:8d$	$-18d:8d:-2d$	74.29°	74.29°	3×3
2	$-4d:4d:4d$	$-14d:4d:-6d$	41.41°	41.41°	3×3
3	$-4d:4d:0$	$-10d:4d:-6d$	21.39°	21.39°	2×2
4	0	$-10d$	0°	0°	1

Table 3. Receiver grids.

Receiver grid	ξ_2	ξ_3	θ_2	θ_3	Number
0 (Original)	$-10d:2d:10d$	$-18d:2d:-2d$	92.64°	79.84°	11×9
1	$-10d:4d:10d$	$-18d:4d:-2d$	92.64°	79.84°	6×5
2	$-4d:2d:4d$	$-14d:2d:-6d$	45.28°	45.28°	5×5
3	$-4d:2d:0$	$-10d:2d:-6d$	23.53°	23.53°	3×3
4	0	$-10d$	0°	0°	1

Table 4. Combinations of source and receiver grids.

Receiver grid	Source grid				
	0	1	2	3	4
0	T0	T1	T2	-	T3
1	T4	T7	-	-	-
2	T5	-	T8	-	-
3	-	-	-	T9	T10
4	T6	-	-	T11	T12

Figure 5 shows the sectional distributions of $\mathcal{T}_\Sigma(\mathbf{x}^o, \mathbf{m}^{\text{opt}}; E)$ with different source grids (T1-T3) and different receiver grids (T4-T6). The cross section ($\xi_2 = -1.0d$) is selected to better represent the differences. Together with Figure 4(a)(T0), it can be seen that the quality of TS reconstruction decreases with the increase of spacing and the decrease of range of the source/receiver grid. And it seems that the spacing has a smaller influence than the range. Figure 5(T3) and (T6) also show that the TS results are still acceptable even if there is only one source/receiver as long as the receiver/source grid covers a sufficiently large region. Comparison of Figure 5(T6) and (T3) indicates that the receiver grid has a larger influence than the source grid. The reason might be that the reflections from the top and bottom surfaces increase the aperture of the source grid.

Figure 5. Sectional distributions ($\xi_2 = -1.0d$) of $\mathcal{T}_\Sigma(\mathbf{x}^o, \mathbf{m}^{\text{opt}}; E)$ with different source grids (T1-T3) and different receiver grids (T4-T6): one obstacle ($\mathbf{m}^{\text{true}} = (\rho/2, 2\kappa)$ and $\nu^{\text{true}} = 0.25$, indicated by red line), transient excitation (raised cosine with $\alpha = 0.85$ and $f_0 = 1.0$).

Figure 6. Sectional distributions ($\xi_2 = -1.0d$) of $\mathcal{T}_\Sigma(\mathbf{x}^o, \mathbf{m}^{\text{opt}}; E)$ with different combinations of source and receiver grids (T7-T12): one obstacle ($\mathbf{m}^{\text{true}} = (\rho/2, 2\kappa)$ and $\nu^{\text{true}} = 0.25$, indicated by red line), transient excitation (raised cosine with $\alpha = 0.85$ and $f_0 = 1.0$).

Figure 6 displays the sectional distributions of $\mathcal{T}_\Sigma(\mathbf{x}^o, \mathbf{m}^{\text{opt}}; E)$ with different combinations of source and receiver grids (T7-T12), where the quality of TS reconstruction decreases with decreasing density and range of source and receiver grids. As shown in Figure 6(T9), the TS reconstruction is still acceptable when both the source and receiver grids cover a range of $4d$ (the diameter of the obstacle is d). After that, the TS reconstruction gradually deviates from the true configuration. Figure 6(T12) clearly shows

that only one source and one receiver are insufficient to yield a good obstacle reconstruction. Comparison of Figure 6(T10) and (T11) also indicates that the receiver grid has a larger influence than the source grid.

3.3 Size of the obstacle and distance between obstacles

For inverse scattering problems, the characteristics of input signals (e.g., wavelength) and source/receiver grids (e.g., spacing and range) are usually discussed with respect to the size of the obstacle (e.g., diameter) as in the above analyses, and thus the size of the obstacle often remains unchanged. In this study, the size of the obstacle and the distance between obstacles are also investigated for completeness.

With the testing configuration shown in Figure 1, the diameter of the obstacle is doubled (i.e., $2d$). The corresponding sectional distributions of $\mathcal{T}_\Sigma(\mathbf{x}^\circ, \mathbf{m}^{\text{opt}}; E)$ are shown in Figure 7. The quality of TS reconstruction is not as good as that of the results for the original smaller obstacle (Figure 4(a)). A possible reason might be that with a larger obstacle size, the coverage of the source/receiver grid becomes relatively small.

Figure 7. Sectional distributions of $\mathcal{T}_\Sigma(\mathbf{x}^\circ, \mathbf{m}^{\text{opt}}; E)$: one bigger obstacle ($\mathbf{m}^{\text{true}} = (\rho/2, 2\kappa)$ and $\nu^{\text{true}} = 0.25$, indicated by red line), transient excitation (raised cosine with $\alpha = 0.85$ and $f_0 = 1.0$).

To examine the performance of topological sensitivity for obstacles with smaller distance, the two obstacles in the example in section 4.5 in the main text are moved closer. The “stiffer” solid ball with radius $0.5d$ and material properties $(0.5\rho, 2\kappa, \nu = 0.25)$ is centered at $(-0.2d, 0, -8.8d)$, and the “softer” ellipsoidal solid object with semi-axes $(0.5d, 0.6d, 0.4d)$ and material properties $(2\rho, 0.5\kappa, \nu = 0.25)$ is centered at $(-0.2d, 0, -11.4)$. The shortest distance between these two obstacles is $0.7d$, smaller than the smallest diameter of the obstacles. Figure 8 shows the sectional distributions of $\mathcal{T}_\Sigma(\mathbf{x}^\circ, \mathbf{m}^{\text{opt}}; E)$ and $\kappa^{\text{est}}/\kappa$, where the two obstacles are clearly separated. They are also correctly identified as a stiffer obstacle and a softer one from the distribution of $\kappa^{\text{est}}/\kappa$. This example demonstrates that the topological sensitivity is capable of distinguishing obstacles with size and distance comparable to the wavelength (range) of the input signal.

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Figure 8. Sectional distributions of $\mathcal{T}_\Sigma(\mathbf{x}^o, \mathbf{m}^{\text{opt}}; E)$ and $\kappa^{\text{est}}/\kappa$: two closer obstacles (upper obstacle: $\mathbf{m}^{\text{true}} = (\rho/2, 2\kappa)$ and $\nu^{\text{true}} = 0.25$; lower obstacle: $\mathbf{m}^{\text{true}} = (2\rho, 0.5\kappa)$ and $\nu^{\text{true}} = 0.25$, both indicated by red line), transient excitation (raised cosine with $\alpha = 0.85$ and $f_0 = 1.0$).

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