1 Appendix A: To consider or not consider soil quality dynamics

When farmers do not consider soil quality dynamics, considering both cases, the first-order conditions of our problem can be rewritten as:

$$H_m = pf_m - c_1 = 0 \tag{1}$$
$$H_u = -c_2 = 0 \tag{2}$$

Since the farmer does not consider soil quality dynamics or the detrimental impact of productive inputs on soil quality, he does not internalize the additional cost of using productive inputs in terms of the marginal value of soil quality.
However, according to condition (2), the optimal use of conservation practices is such that the investment is equal to zero at any point in time. That is, when the dynamics of soil quality are not considered, no soil conservation investments are made. Hence, we are always in a situation of underinvestment in soil quality.

One can still expected that soil quality will attain a long-term equilibrium (Smith et al, 2000) such that:

$$\dot{s} = -\delta(m)s + g(u) = 0 \Leftrightarrow s^S = \frac{g(0)}{\delta(m^S)} \tag{3}$$

When comparing the long-term soil quality equilibrium (s^S) and not considering soil quality dynamics and the optimum soil quality level (s^*) when considering soil quality dynamics, one obtains:

$$s^S = \frac{g(0)}{\delta(m^S)} \quad \text{and} \quad s^* = \frac{g(u^*)}{\delta(m^*)} \tag{4}$$

In addition, from conditions (**??**) and (1) of the two optimization problems, at any point of time, and in particular, for the bundles (m^*, s^*) and (m^S, s^S), we have:

$$pf_m(m^*, s^*) - c_1 - \mu\delta_m(m^*)s^* = 0 \quad \text{and} \quad pf_m(m^S, s^S) - c_1 = 0 \tag{5}$$
$$\Leftrightarrow pf_m(m^*, s^*) - c_1 - \mu\delta_m(m^*)s^* = pf_m(m^S, s^S) - c_1 \tag{6}$$
$$\Leftrightarrow f_m(m^*, s^*) - \frac{\mu}{p}\delta_m(m^*)s^* = f_m(m^S, s^S) \tag{7}$$

due to the more complex relationship between productive inputs m and soil quality s. In addition to the cooperation effect in terms of production, the detrimental impact of productive inputs on soil quality dynamics is considered. In this second case, several situations are plausible depending on the initial soil quality.

- $m^* < m^S$ and $s^* > s^S$
  From the assumptions of our model:

$$m^* < m^S \Rightarrow \delta(m^*) < \delta(m^S) \tag{8}$$
$$g(u^*) > g(0) \Rightarrow \frac{g(u^*)}{\delta(m^*)} > \frac{g(0)}{\delta(s^S)} \tag{9}$$
$$s^* > s^S \tag{10}$$

- $m^* = m^S$ and $s^* > s^S$
  From the assumptions of our model:

$$m^* = m^S \Rightarrow \delta(m^*) = \delta(m^S) \tag{11}$$
$$g(u^*) > g(0) \Rightarrow \frac{g(u^*)}{\delta(m^*)} > \frac{g(0)}{\delta(s^S)} \tag{12}$$
$$s^* > s^S \tag{13}$$

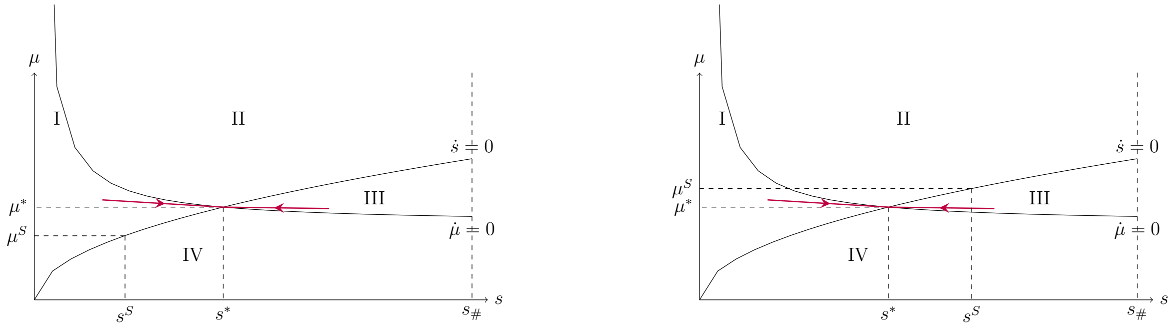
These two cases are consistent with (7). Their interpretation is fairly intuitive: these are situations do when not consider soil quality dynamics; thus, the farmer uses productive inputs without compensating for the reduction in soil quality, which degrades his soil quality below the optimum. There is overuse of productive inputs and underuse of soil quality investments.

- $m^* > m^S$

$$m^* > m^S \Rightarrow \delta(m^*) > \delta(m^S) \tag{14}$$
$$g(u^*) > g(0) \Rightarrow \frac{g(u^*)}{\delta(m^*)} \gtreqless \frac{g(0)}{\delta(s^S)} \tag{15}$$
$$s^* \gtreqless s^S \tag{16}$$

This can correspond to different situations. One is where initial soil quality is above the optimum and sufficiently high for long-term soil quality to stabilize above the optimum, even when the farmer does not compensate for the impact of productive inputs on his soil. This is possible if the farmer also use fewer productive inputs than optimal, thus causing less damage. In the other situation, initial soil quality is not sufficiently high for the reduced use of productive inputs to compensate for the lack of investment in soil quality.

In most cases, not considering soil quality dynamics leads to a long-term equilibrium level of soil quality that is lower than optimal. This can be observed when the farmer



Phase diagram: Not considering soil quality dynamics: case 2

overuses or underuses productive inputs compared to the cases where the farmer considers soil quality. Indeed, in all cases, no investments are made in soil quality. The damage, whether natural or caused by the use of productive inputs, is not compensated for. In one of the situations described, a sufficiently high initial soil quality level can still lead to a long-term equilibrium of soil quality that is higher than optimal. This case corresponds to a situation where the cooperation relationship between soil quality and productive inputs is underused.

The problem is that in all cases, the long-term equilibrium of soil quality is not stable: these are situations that cross $\dot{s} = 0$, so the strategies followed by the farmer remain non-optimal, with underinvestment in soil quality leading to depletion of the resource.

2 Appendix B: Computation of the soil investment model

Phase diagram and stability properties of our problem: ambiguity due to the prevalence of the cooperating benefits and the marginal damages to soil quality

The long-run or steady state equilibrium of the optimal control problem is determined by the intersection of the ($\dot{\mu} = 0$) and ($\dot{s} = 0$) demarcation curves such that:

$$
\begin{aligned}
&A(s,\mu) = \dot{\mu} \gtreqqless 0 \\
&\text{if} \;\; \mu(r + \delta(m(s,\mu))) - pf_s(m(s,\mu), s) \gtreqqless 0 && (17) \\
&B(s,\mu) = \dot{s} \gtreqqless 0 \\
&\text{if} \;\; -\delta(m(s,\mu))s + g(u(s,\mu)) \gtreqqless 0 && (18)
\end{aligned}
$$

The slopes of the stationary loci are given by:

$$\left.\frac{d\mu}{ds}\right|_{B=\dot{s}=0} = -\frac{\partial H_\mu/\partial s}{\partial H_\mu/\partial \mu} = -\frac{-\delta_m m_s s - \delta(m) + g_u u_s}{-\delta_m m_\mu s + g_u u_\mu} = -\frac{-\delta_m m_s s - \delta(m)}{-\delta_m m_\mu s + g_u u_\mu} \tag{19}$$

$$\begin{aligned}\left.\frac{d\mu}{ds}\right|_{A=\dot{\mu}=0} &= -\frac{\partial(\mu r - H_s)/\partial s}{\partial(\mu r - H_s)/\partial \mu} \\ &= -\frac{\partial(\mu(r+\delta(m(s,\mu))) - pf_s(s,m(s,\mu)))/\partial s}{\partial(\mu(r+\delta(m(s,\mu))) - pf_s(s,m(s,\mu)))/\partial \mu} = -\frac{\delta_m m_s \mu - pf_{ss} - pf_{sm} m_s}{r + \delta_m m_\mu \mu + \delta(m) - pf_{sm} m_\mu}\end{aligned} \tag{20}$$

To determine the stability properties of our problem, i.e., whether all solutions converge toward the steady state, one can evaluate the Jacobian matrix:

$$J = \begin{bmatrix} \partial\dot{s}/\partial s & \partial\dot{s}/\partial\mu \\ \partial\dot{\mu}/\partial s & \partial\dot{\mu}/\partial\mu \end{bmatrix} = \begin{bmatrix} H_{\mu s} & H_{\mu\mu} \\ -H_{ss} & r - H_{s\mu} \end{bmatrix} = \begin{bmatrix} \underset{(?)}{-\delta_m m_s s - \delta} & \underset{(+)}{-\delta_m m_\mu s + g_u u_\mu} \\ \underset{(?)}{m_s(-H_{ms}) - pf_{ss}} & \underset{(?)}{r + m_\mu(-H_{ms}) + \delta} \end{bmatrix} \tag{21}$$

at the steady sate (s^*, μ^*). Computing the trace of the Jacobian matrix, it appears that:

$$tr[J] = -\delta_m m_s s - \delta + r + m_\mu(-H_{ms}) + \delta = -m_s(\delta_m s - \delta_m s) + r = r > 0 \tag{22}$$

Since the eigenvalues of the Jacobian matrix equal its trace, at least one eigenvalue is positive, which implies that the fixed point (here, the intersection of the ($\dot{\mu} = 0$) and ($\dot{s} = 0$) demarcation curves) is not locally asymptotically stable (Caputo, 2005). If the determinant of the Jacobian matrix is negative, then the steady state is a local saddle point (Hediger, 2003; Narain and Fisher, 2006). Otherwise, if the determinant of the Jacobian matrix is positive, the steady state is an unstable node or at the center of an unstable spiral (Caputo, 2005) such that the system is not converging toward a steady state.

With a general form of the problem, that is, without specifying the functional forms of the different functions considered, the existence of an equilibrium can be found in the case where $H_{ms} > 0$. However, no conclusion can be made in the case where $H_{ms} < 0$.

When the marginal cooperating benefits are higher than the marginal damages to soil quality: Phase diagram and stability properties of our problem

In the case where $H_{ms} > 0$, which corresponds to the case where the marginal benefits of using productive inputs in terms of revenues is higher than the damages in terms of the marginal value of soil quality, there is a steady state equilibrium, since the

Jacobian matrix is such that:

$$
\begin{aligned}
\det J &= \begin{vmatrix} H_{\mu s} & H_{\mu\mu} \\ -H_{ss} & r - H_{s\mu} \end{vmatrix} = H_{\mu s}(r - H_{s\mu}) - H_{\mu\mu}(-H_{ss}) \\
&= (-\delta_m m_s s - \delta(m) + u_s)(r + \delta_m m_\mu \mu + \delta(m) - pf_{sm} m_\mu) \\
&\quad - (-\delta_m m_\mu s + u_\mu g_u)(\delta_m m_s \mu - pf_{ss} - pf_{sm} m_s) \\
&= (-\delta_m m_s s - \delta(m))(r + m_\mu(-H_{sm}) + \delta(m)) - (-\delta_m m_\mu s + u_\mu g_u)(m_s(-H_{ms}) - pf_{ss}) \\
&= (-\delta_m m_s s - \delta(m))(r + m_\mu(-H_{sm}) + \delta(m)) - (-\delta_m m_\mu s + u_\mu g_u)\left(\left(-\frac{H_{ms}}{H_{mm}}\right)(-H_{ms}) - pf_{ss}\right) \\
&= (-\delta_m m_s s - \delta(m))(r + m_\mu(-H_{sm}) + \delta(m)) - (-\delta_m m_\mu s + u_\mu g_u)\left(\frac{H_{ms}^2 - pf_{ss} H_{mm}}{H_{mm}}\right) \\
&= (-\delta_m m_s s - \delta(m))(r + m_\mu(-H_{sm}) + \delta(m)) \\
&\quad - (-\delta_m m_\mu s + u_\mu g_u)\left(\frac{p^2(f_{ms}^2 - f_{ss} f_{mm}) + \mu\delta_m(\mu\delta_m - 2pf_{sm}) + pf_{ss}\mu\delta_{mm} s}{H_{mm}}\right) \\
&< 0 \qquad (23)
\end{aligned}
$$

From conditions (??), (??) and (??) and equations (??) to (??), given positive r and p and assuming that $H_{ms} > 0$, then $H_{\mu s} < 0$, $r - H_{s\mu} > 0$, $H_{\mu\mu} > 0$ and $H_{\mu\mu}(-H_{ss}) > 0$. From these results, the determinant of the Jacobian matrix is negative.

The slopes of the stationary loci are given by:

$$
\begin{aligned}
\left.\frac{d\mu}{ds}\right|_{B=0} &= -\frac{\partial H_\mu / \partial s}{\partial H_\mu / \partial \mu} \\
&= -\frac{H_{\mu s}}{H_{\mu\mu}} > 0 \qquad (24) \\
\left.\frac{d\mu}{ds}\right|_{A=0} &= -\frac{\partial(\mu r - H_s)/\partial s}{\partial(\mu r - H_s)/\partial \mu} \\
&= -\frac{-H_{ss}}{r - H_{s\mu}} < 0 \qquad (25)
\end{aligned}
$$

From conditions (??), (??) and (??) and equations (??) to (??), given positive r and p and assuming that $H_{ms} > 0$, the gradient of the ($\dot{s} = 0$) curve is positive. Given these conditions, the gradient of the ($\dot{\mu} = 0$) curve is negative.

In addition, the slopes of the trajectories in the (s, μ) space are such that:

$$
\frac{d\mu}{ds} = \left(\frac{d\mu}{dt}\right) \cdot \left(\frac{dt}{ds}\right) = \frac{\dot{\mu}}{\dot{s}} \qquad (26)
$$

Hence, when a trajectory goes through a locus where $\dot{\mu} = 0$, it has a slope zero, and when it goes through a locus where $\dot{s} = 0$, it has an infinite slope.

Furthermore, when $\dot{s}=0$ and $\dot{\mu}=0$ and in the case where the steady state is a local saddle point (which is the case when $H_{ms}>0$), we have:

$$\left[\underbrace{\frac{\partial\dot{s}}{\partial s}}_{-}\underbrace{\frac{\partial\dot{\mu}}{\partial\mu}}_{+}-\underbrace{\frac{\partial\dot{\mu}}{\partial s}}_{+}\underbrace{\frac{\partial\dot{s}}{\partial\mu}}_{+}\right]<0\Leftrightarrow\frac{-\partial\dot{s}/\partial s}{\partial\dot{s}/\partial\mu}>\frac{-\partial\dot{\mu}/\partial s}{\partial\dot{\mu}/\partial\mu} \tag{27}$$

from which one can conclude that the slope of the $\dot{s}=0$ isocline is greater than the slope of the $\dot{\mu}=0$ isocline in the neighborhood of the steady state. This is true if and only if the steady state is a local saddle point (Caputo, 2005).

Comparative statics of case 2, when $H_{ms}>0$

We aim to estimate the impact of a change in a given parameter; here, c_1, c_2, p and r are the costs associated with soil degrading practices m, soil quality investment (or conservation practices), crop prices and discount rates, respectively. When one parameter changes, all variables change. However, the other parameters remain fixed and have a zero differential. To study this change, we evaluate the total differentials at the original equilibrium, that is, the total differentials of the first-order conditions (FOCs) when $\dot{\mu}=\dot{s}=0$.

The FOCs at equilibrium are such that:

$$\tilde{H}_m=pf_m-c_1-\mu\delta_m s=0 \tag{28}$$
$$\tilde{H}_u=-c_2+\mu g_u=0 \tag{29}$$
$$\tilde{H}_\mu=-\delta(m)s+g(u)=0 \tag{30}$$
$$\dot{\mu}-r\mu=-\tilde{H}_s\Leftrightarrow\dot{\mu}=r\mu-pf_s+\delta(m)\mu=\mu(r+\delta(m))-pf_s=0 \tag{31}$$

The total differentials of the system are such that:

$$(pf_{mm}-\mu\delta_{mm}s)dm+0du-\delta_m s d\mu+(pf_{ms}-\mu\delta_m)ds+f_m dp-dc_1+0dc_2+0dr=0 \tag{32}$$
$$0dm+\mu g_{uu}du+g_u d\mu+0ds+0dp+0dc_1-dc_2+0dr=0 \tag{33}$$
$$-\delta_m s dm+g_u du+0d\mu-\delta(m)ds+0dp+0dc_1+0dc_2+0dr=0 \tag{34}$$
$$(\mu\delta_m-pf_{sm})dm+0du+(r+\delta(m))d\mu-pf_{ss}ds-f_s dp+0dc_1+0dc_2+\mu dr=0 \tag{35}$$

The determinant of the matrix of the system, denoted B is positive:

$$
\begin{aligned}
|B| &= \begin{vmatrix} pf_{mm}-\mu\delta_{mm}s & 0 & -\delta_m s & pf_{ms}-\mu\delta_m \\ 0 & \mu g_{uu} & g_u & 0 \\ -\delta_m s & g_u & 0 & -\delta \\ \mu\delta_m - pf_{sm} & 0 & r+\delta & -pf_{ss} \end{vmatrix} = \mu g_{uu} \begin{vmatrix} H_{mm} & -\delta_m s & H_{ms} \\ -\delta_m s & 0 & -\delta \\ -H_{ms} & r+\delta & -pf_{ss} \end{vmatrix} \\
&- g_u \begin{vmatrix} H_{mm} & -\delta_m s & H_{ms} \\ 0 & g_u & 0 \\ -H_{ms} & r+\delta & -pf_{ss} \end{vmatrix} \\
&= \mu g_{uu} \left(\delta_m s \begin{vmatrix} -\delta_m s & -\delta \\ -H_{ms} & -pf_{ss} \end{vmatrix} - (r+\delta) \begin{vmatrix} H_{mm} & H_{ms} \\ -\delta_m s & -\delta \end{vmatrix} \right) - g_u \left(g_u \begin{vmatrix} H_{mm} & H_{ms} \\ -H_{ms} & -pf_{ss} \end{vmatrix} \right) \\
&= \mu g_{uu}(\delta m_s(\delta_m s p f_{ss} - H_{ms}\delta) - (r+\delta)(-H_{mm}\delta + H_{ms}\delta_m s)) - g_u^2(-H_{mm}H_{ss} + H_{ms}^2) \\
&= \mu g_{uu}((\delta_m s)^2 H_{ss} - \delta_m s \delta H_{ms} + (r+\delta)H_{mm}\delta - (r+\delta)H_{ms}\delta_m s) - g_u^2(-H_{mm}H_{ss} + H_{ms}^2) > 0
\end{aligned} \tag{36}
$$

Applying Cramer's rule, we obtain the following comparative statics for the case where the damages caused by the use of productive inputs are more than offset by their cooperating benefits with soil quality in terms of revenue ($H_{ms} > 0$):

$$m = m(\overset{-}{c_1}, \overset{-}{c_2}, \overset{+}{p}, \overset{?}{r}) \tag{37}$$

$$u = u(\overset{-}{c_1}, \overset{-}{c_2}, \overset{+}{p}, \overset{-}{r}) \tag{38}$$

$$\mu = \mu(\overset{-}{c_1}, \overset{+}{c_2}, \overset{+}{p}, \overset{-}{r}) \tag{39}$$

$$s = s(\overset{-}{c_1}, \overset{-}{c_2}, \overset{?}{p}, \overset{?}{r}) \tag{40}$$

Using this method, some impacts are ambiguously signed. Hence, an alternative methodology is used to determine the impacts of changes in the discount rate and in the crop price on the steady state. Indeed, it is not the FOCs that are taken into account but the ($\dot{s} = 0$) and ($\dot{\mu} = 0$) equations, while using the expressions of m and u as implicit functions of soil quality s and marginal soil quality μ.

Hence, we have the following set of equations:

$$\dot{s} = H_\mu = -\delta(m^*(s,\mu)) + g(u^*(s^*,\mu^*)) = 0 \tag{41}$$

$$\dot{\mu} = r\mu - H_s = \mu^*(r + \delta(m^*(s,\mu))) - pf_s(m^*(s,\mu), s) = 0 \tag{42}$$

Differentiating the system with respect to s, μ, p and r yields:

$$(-\delta_m m_s - \delta(m) + g_u u_s)ds + (g_u u_\mu - \delta_m m_\mu s)d\mu + 0dr + 0dp = 0 \quad (43)$$

$$(\mu\delta_m m_s - pf_{sm}m_s - pf_{ss})ds + (r + \delta(m) + \mu\delta_m m_\mu - pf_{sm}m_\mu)d\mu + \mu dr - f_s dp = 0 \quad (44)$$

Only considering changes in r gives the following system:

$$\begin{bmatrix} (-\delta_m m_s - \delta(m)) & (g_u u_\mu - \delta_m m_\mu s) \\ (-H_{ms}m_s - pf_{ss}) & (r + \delta - m_\mu H_{ms}) \end{bmatrix} \begin{bmatrix} ds/dr \\ d\mu/dr \end{bmatrix} = \begin{bmatrix} 0 \\ -\mu \end{bmatrix} \quad (45)$$

Applying Cramer's rule yields the following results:

$$\frac{ds}{dr} = \frac{\begin{vmatrix} 0 & (g_u u_\mu - \delta_m m_\mu s) \\ -\mu & (r + \delta(m) - m_\mu H_{ms}) \end{vmatrix}}{|J|} = \frac{\mu(g_u u_\mu - \delta_m m_\mu s)}{|J|} < 0 \quad (46)$$

$$\frac{d\mu}{dr} = \frac{\begin{vmatrix} (-\delta_m m_s - \delta(m)) & 0 \\ (-H_{ms}m_s - pf_{ss}) & -\mu \end{vmatrix}}{|J|} = \frac{\mu(\delta_m m_s s + \delta(m))}{|J|} < 0 \quad (47)$$

Similarly, only considering changes in p yields the following system:

$$\begin{bmatrix} -\delta_m m_s - \delta(m) & g_u u_\mu - \delta_m m_\mu s \\ -H_{ms}m_s - pf_{ss} & r + \delta - m_\mu H_{ms} \end{bmatrix} \begin{bmatrix} ds/dr \\ d\mu/dr \end{bmatrix} = \begin{bmatrix} 0 \\ f_s \end{bmatrix} \quad (48)$$

Applying Cramer's rule yields the following results:

$$\frac{ds}{dp} = \frac{\begin{vmatrix} 0 & g_u u_\mu - \delta_m m_\mu s \\ f_s & r + \delta(m) - m_\mu H_{ms} \end{vmatrix}}{|J|} = \frac{-f_s(g_u u_\mu - \delta_m m_\mu s)}{|J|} > 0 \quad (49)$$

$$\frac{d\mu}{dp} = \frac{\begin{vmatrix} -\delta_m m_s - \delta(m) & 0 \\ -H_{ms}m_s - pf_{ss} & f_s \end{vmatrix}}{|J|} = \frac{f_s(\delta_m m_s s + \delta(m))}{|J|} > 0 \quad (50)$$

The comparative statics for the case where the damage caused by the use of productive inputs is more than offset by its cooperating benefits with soil quality in terms of revenue ($H_{ms} > 0$) are the following:

$$m = m(\overset{-}{c_1}, \overset{-}{c_2}, \overset{+}{p}, \overset{-}{r}) \quad (51)$$

$$u = u(\overset{-}{c_1}, \overset{-}{c_2}, \overset{+}{p}, \overset{-}{r}) \quad (52)$$

$$\mu = \mu(\overset{-}{c_1}, \overset{+}{c_2}, \overset{+}{p}, \overset{-}{r}) \quad (53)$$

$$s = s(\overset{-}{c_1}, \overset{-}{c_2}, \overset{+}{p}, \overset{-}{r}) \quad (54)$$

When the marginal cooperating benefits are higher than the marginal damages to soil quality: Comparative dynamics

To conduct our comparative dynamic analysis, we used the same methodology as in case 1, that is, the methodology proposed by Caputo (2005) *via* envelope methods. It is a general method of comparative dynamics that can be applied to any sufficiently smooth optimal control problem using a primal-dual approach (see Caputo, 2005 - chapter 11). The conditions necessary to use the theorems or corollary proposed by Caputo (2005) have to be verified for the specific set-up of our soil quality investment model when farming practices both positively and negatively affect soil quality. However, since the comparative dynamics conducted following Caputo (2005) do not consider directly

The primal form of our soil quality investment model is such that:

$$V(\alpha) \equiv \max_{m(.),u(.)} J[m(.), u(.), s(.)] \equiv \max_{m(.),u(.)} \int_0^T e^{-rt}[pf(s(t), m(t)) - c_1 m(t) - c_2 u(t)]dt \tag{55}$$

$$\text{s.t.} \quad \dot{s}(t) = k(m(t), s(t), u(t)) = -\delta(m(t))s(t) + g(u(t)), \tag{56}$$

$$s(0) = s_0, s(T) = s_T \tag{57}$$

where $\alpha \equiv (p, c_1, c_2, r)$ is the vector of time-independent parameters. We denote the optimal paths $z(t; \alpha)$, $v(t; \alpha)$ and $w(t; \alpha)$ of soil quality, soil degrading practices, and investments in soil conservation practices, respectively. The comparative dynamic analysis is conducted on the vector of parameters $\alpha \equiv (p, c_1, c_2, r)$.

We use the dynamic envelope theorem proposed in Caputo (2005). According to the theorem, when the assumptions (A.1) through (A.4) hold, the partial derivative of the optimal value function with respect to a parameter can be obtained by differentiating the Hamiltonian of the optimal control problem, evaluating it along the optimal paths (that is, for $s(t) = z(t; \alpha)$, $m(t) = v(t; \alpha)$ and $u(t) = w(t; \alpha)$) and finally integrating the result over the planning horizon.

Before doing so, let us verify that assumptions (A.1) to (A.4) hold for our soil quality investment problem. The assumptions mentioned in Caputo (2005 - page 288) and applied to our case are the following:

(A.1) $f(.) \in C^{(2)}$ and $k(.) \in C^{(2)}$ on their respective domains.

(A.2) There exists a unique optimal solution to problem (P) for each $\beta \in B(\beta^\circ; \delta)$, which we denote by the quadruplet $(z(t; \alpha), v(t; \alpha), w(t; \alpha), \lambda(t; \alpha)$, where $B(\beta^\circ; \delta)$

is an open $2+2N+A$ ball centered at the given value of the parameter β° of radius $\delta > 0$.

(A.3) The vector-valued functions $z(.), v(.), w(.), \lambda(.)$ are $C^{(1)}$ in $(t;\beta)$ for all $(t;\beta) \in [t_0^\circ, t_1^\circ] \times B(\beta^\circ;\delta)$.

(A.4) $V(.) \in C^{(2)}$ in β for all $\beta \in B(\beta^\circ;\delta)$.

Because of the assumptions made for the production function and the soil quality dynamics function, (A.1) holds. In addition, from the Mangasarian Sufficient Conditions theorem, since the Hamiltonian $\tilde{H}$ of our problem is strictly concave in m, u, and s when μ is the co-state variable, there is a unique global maximum of $J[.]$ [1]. (A.3) and (A.4) are assumed to hold.

Hence, applying Theorem 11.1 yields:

$$V_p(\alpha) \equiv \int_0^T y(t;\alpha)e^{-rt}dt > 0 \tag{58}$$

$$V_{c_1}(\alpha) \equiv -\int_0^T v(t;\alpha)e^{-rt}dt < 0 \tag{59}$$

$$V_{c_2}(\alpha) \equiv -\int_0^T w(t;\alpha)e^{-rt}dt < 0 \tag{60}$$

$$V_r(\alpha) \equiv -\int_0^T t\pi(t;\alpha)e^{-rt}dt \lesseqgtr 0 \tag{61}$$

where $y(t;\alpha) \equiv f(z(t;\alpha), v(t;\alpha))$ is the value of the production function of the farm, and $\pi(t;\alpha) \equiv pf(z(t;\alpha), v(t;\alpha)) - c_1 v(t;\alpha) - c_2 w(t;\alpha)$ is the instantaneous profits along the optimal path.

[1] The Hessian matrix $\mathcal{H}$ of the Hamiltonian $\tilde{H}$ when examining the concavity of $\tilde{H}$ is such that:

$$\mathcal{H}(m,u,s) = \begin{bmatrix} H_{mm} & H_{mu} & H_{ms} \\ H_{um} & H_{uu} & H_{us} \\ H_{sm} & H_{su} & H_{ss} \end{bmatrix} = \begin{bmatrix} H_{mm} & 0 & H_{ms} \\ 0 & H_{uu} & 0 \\ H_{sm} & & H_{ss} \end{bmatrix}$$

Note that $\mathcal{H}$ is a square symmetric matrix of order $n = 3$. If the $n = 3$ leading principal minors D_k (*i.e.*, the determinants of the $(k \times k)$ matrix obtained by eliminating the $n-k$ last rows and $n-k$ last columns of the matrix) are alternatively < 0 (k odd) and > 0 (k even), then $\mathcal{H}$ is negative definite.
Here, we have the following in the case where $H_{ms} > 0$:

$$\begin{aligned} &D_1 = H_{mm} < 0 \\ &D_2 = H_{mm}H_{uu} - (H_{mu})^2 = H_{mm}H_{uu} > 0 \\ &D_3 = H_{uu}(H_{mm}H_{ss} - (Hms)^2 = H_{uu}(-\mu s\delta_{mm} spf_{ss} + p^2(f_{mm}f_{ss} - (f_{sm})^2) \\ &\quad + \mu\delta_m(2pf_{sm} - \mu\delta_m)) < 0 \end{aligned}$$

Hence, $\mathcal{H}$ is negative definite. If the Hessian matrix of a function f is negative definite $\forall x \in \mathbb{R}^n$, then f is strictly concave and we can conclude that $\tilde{H}$ is indeed strictly concave in m, u and s when μ is the co-state variable.

The information obtained from the dynamic envelope theorem is relative to the cumulative discounted profit and production functions. Equations (58), (59) and (60) are unambiguously signed: according to the assumptions of our model, the production function cannot be negative, nor can the productive inputs or investment in soil quality conservation practices be negative. However, equation (61) is not ambiguously signed. Indeed, although $V(\alpha) > 0$ must hold for the farm to be able to thrive in the market, it may be possible for instantaneous profits along the optimal path to be positive or negative at any given point. This could be the case when an important investment in soil quality is made that does not instantaneously yield productivity gains. However, one could add a constraint whereby instantaneous profits have to be positive, in which case $V_r(\alpha) < 0$.

In our model, the integrand function of the soil quality investment model is linear in $\gamma \equiv (p, c_1, c_2)$. Thus, the model satisfies the conditions of Corollary 11.2 (Caputo, 2005). This implies that the optimal value function $V(.)$ is locally convex in γ. Hence, when differentiating equations (58) to (60), one can use the convexity of $V(.)$ to determine the signs of the second partial derivatives and infer from those signs the own-price effects:

$$V_{pp}(\alpha) \equiv \frac{\partial}{\partial p}\int_0^T y(t;\alpha)e^{-rt}dt = \int_0^T \frac{\partial y}{\partial p}(t;\alpha)e^{-rt}dt \geq 0 \tag{62}$$

$$V_{c_1c_1}(\alpha) \equiv -\frac{\partial}{\partial c_1}\int_0^T v(t;\alpha)e^{-rt}dt = -\int_0^T \frac{\partial v}{\partial c_1}(t;\alpha)e^{-rt}dt \geq 0 \tag{63}$$

$$V_{c_2c_2}(\alpha) \equiv -\frac{\partial}{\partial c_2}\int_0^T w(t;\alpha)e^{-rt}dt = -\int_0^T \frac{\partial w}{\partial c_2}(t;\alpha)e^{-rt}dt \geq 0 \tag{64}$$

Equation (**??**) shows that the cumulative discounted crop production is not decreasing in the crop price. Note that it is the discounted production function slope integrated over the entire planning horizon that is not decreasing. For a given and finite period, crop production could be decreasing while the crop price is increasing. While over the short term such a behavior could appear irrational, as long as equation (62) is verified over the entire planning horizon, such a behavior could be rational. Similar reasoning applies with respect to the impacts of increases in the cost of soil degrading practices and the cost of conservation practices. Equations (63) and (64) demonstrate that the cumulative discounted use of soil degrading practices and the cumulative discounted investment in conservation practices are non-increasing in their own prices.

The comparative dynamics of the discount rate r cannot be derived through the use of Corollary 11.2 since the integrand function $F(.)$ for our soil quality investment model:

$$F(t, m, u, s; \alpha) \equiv [pf(s, m) - c_1 m - c_2 u]e^{-rt} \tag{65}$$

is not convex in the discount rate r. Hence, to conduct the comparative dynamic analysis of the discount rate, we rely on Theorem 11.2 proposed in Caputo (2005).

From Theorem 11.2, with $\alpha \equiv (p, c_1, c_2, r)$ such that the discount rate r is the fourth element of the parameter vector α and since $L_{\alpha\alpha}(\beta)$ is a negative semi-definite matrix, we have:

$$
\begin{aligned}
L_{rr}(\beta) = &-\int_0^T [F_{rs}(t, z(t;\alpha), v(t;\alpha);\alpha)\frac{\partial z}{\partial r}(t;\alpha) + F_{rm}(t, z(t;\alpha), v(t;\alpha;\alpha)\frac{\partial v}{\partial r}(t;\alpha) \\
&+ F_{ru}(t, z(t;\alpha), v(t;\alpha);\alpha)\frac{\partial w}{\partial r}(t;\alpha)]dt \qquad (66) \\
= &-\int_0^T [pf_s(t, z(t;\alpha), v(t;\alpha))\frac{\partial z}{\partial r}(t;\alpha) + [pf_m(z(t;\alpha), v(t;\alpha))\frac{\partial v}{\partial r}(t;\alpha) - c_1] \\
&- c_2\frac{\partial w}{\partial r}(t;\alpha)]te^{-rt}dt \leq 0 \qquad (67)
\end{aligned}
$$

Similarly to the previous equations describing the comparative dynamics of our model, equation (67)describes the impact of a change in the discount rate on soil quality investment model over the entire planning horizon. However, the comparative dynamics of the discount rate are not easy to interpret, contrary to the comparative dynamics of the crop prices and the respective costs of soil degrading practices and soil conservation practices.