SUPPLEMENT TO “INTEGRATED-QUANTILE-BASED ESTIMATION FOR FIRST-PRICE AUCTION MODELS”

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ABSTRACT. This supplement provides the details of the proofs which were omitted from the main text. Section 1 provides auxiliary lemmas for the proof of Theorem 1. Section 2 provides auxiliary lemmas for the proof of Theorem 2. The proofs to corollaries are collected in Section 3.

We first re-introduce the assumptions made in the main text.

Assumption 1. *There are $L \rightarrow \infty$ identical auctions, and for each auction, there are I symmetric and risk neutral bidders. Their private values are i.i.d. draws from a common distribution $F(\cdot)$.*

Assumption 2. *$F(\cdot)$ is continuously differentiable over its compact support $[\underline{v}, \bar{v}]$. There exists $\lambda > 0$ such that $\inf_{v \in [\underline{v}, \bar{v}]} f(v) \geq \lambda > 0$.*

Assumption 3. *The valuation density f is continuously differentiable.*

Assumption 4. *Let K' be the first order derivative of K . Then K satisfies (1) K has compact support and take value zero on the boundary, (2) $\int K'(u)du = \int u^2 K'(u)du = 0$, (3) $\int u K'(u)du = -1$, (4) $\int u^3 K'(u)du \neq 0$.*

1. AUXILIARY LEMMAS FOR THEOREM 1

Lemma 1. *Suppose that Assumptions 1 and 2 hold, then for any $\alpha_0 \in (0, 1)$ and uniformly over $t \in \mathcal{T}$, where $\mathcal{T}$ is a compact subset of $\mathbb{R}$,*

$$n^{2/3} \left\{ \int_{\alpha_0}^{\alpha_0 + t/n^{1/3}} Q_{b,n}(\tau) d\tau - \int_{\alpha_0}^{\alpha_0 + t/n^{1/3}} Q_b(\tau) d\tau \right\} \xrightarrow{p} 0.$$

Proof. Assumption 2 implies that Q_b is twice continuously differentiable (see [Guerre, Perrigne, and Vuong, 2000](#), Proposition 1-(iv)). By the Bahadur representation for quantile functions (see, e.g. [Bahadur, 1966](#); [Kiefer, 1967](#)), we know that uniform in $\tau \in [\delta, 1 - \delta]$,

$$Q_{b,n}(\tau) - Q_b(\tau) = \frac{\tau - \frac{1}{n} \sum_i \mathbf{1}[b_i \leq Q_b(\tau)]}{g(Q_b(\tau))} + O_{a.s.} \left(n^{-3/4} (\log n)^{1/2} (\log \log n)^{1/4} \right).$$

Since $\alpha_0 \in (0, 1)$, we have

$$\begin{aligned} n^{2/3} \int_{\alpha_0}^{\alpha_0 + t/n^{1/3}} (Q_{b,n}(\tau) - Q_b(\tau)) d\tau &= n^{2/3} \int_{\alpha_0}^{\alpha_0 + t/n^{1/3}} \left(\frac{\tau - \frac{1}{n} \sum_i \mathbf{1}[b_i \leq Q_b(\tau)]}{g(Q_b(\tau))} \right) d\tau + o_p(1) \\ &= n^{2/3} \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0 + t/n^{1/3})} \left(F(u) - \frac{1}{n} \sum_i \mathbf{1}[b_i \leq u] \right) du + o_p(1) \\ &= \frac{1}{\sqrt{n}} \sum_i n^{1/6} \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0 + t/n^{1/3})} (F(u) - \mathbf{1}[b_i \leq u]) du + o_p(1) \\ &= \frac{1}{\sqrt{n}} \sum_i \xi_n(b_i, t) + o_p(1), \end{aligned}$$

where $\xi_n(b_i, t) = n^{1/6} \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0 + t/n^{1/3})} (F(u) - \mathbf{1}[b_i \leq u]) du$. It is sufficient to show that $\frac{1}{\sqrt{n}} \sum_i \xi_n(b_i, t)$ converges uniformly to zero in probability.

Note that $\mathbb{E}[\xi_n(b_i, t)] = 0$ and the summand are i.i.d.. For each n , define a class of functions indexed by t : $\Xi_n \equiv \{\xi_n(\cdot, t) : t \in \mathcal{T}\}$. Then we can have the following observations.

(i) Let $t^* = \arg\max_{t \in \mathcal{T}} |Q_b(\alpha_0 + t/n^{1/3}) - Q_b(\alpha_0)|$ and let $\bar{\xi}(b) \equiv n^{1/6} |Q_b(\alpha_0 + t^*/n^{1/3}) - Q_b(\alpha_0)| (F(u) - \mathbf{1}[b \leq u])$. Then $\bar{\xi}(b)$ is an envelope function for Ξ_n . We also have $\mathbb{E}[\bar{\xi}^2(b)] = O(1)$ since $|Q_b(\alpha_0 + t^*/n^{1/3}) - Q_b(\alpha_0)| = O(n^{-1/3})$.

(ii) For any $\epsilon > 0$, we have $\mathbb{E}[\bar{\xi}^2(b) \mathbf{1}[\bar{\xi}(b) > \epsilon \sqrt{n}]] = o(1)$. This is because $\bar{\xi}(b) > \epsilon \sqrt{n}$ if and only if $|Q_b(\alpha_0 + t^*/n^{1/3}) - Q_b(\alpha_0)| (F(u) - \mathbf{1}[b \leq u]) > \epsilon n^{1/3}$ and the latter is a probability event with probability approaches zero, whereas $\mathbb{E}[\bar{\xi}^2]$ is bounded.

(iii) For any $\epsilon_n \downarrow 0$, there is $\sup_{(t,s) \in \mathcal{T}^2: |t-s| \leq \epsilon_n} \mathbb{E}\{\xi_n(b, t) - \xi_n(b, s)\}^2 = o(1)$. To verify this claim, assume without loss of generality that $t > 0$ and $s < 0$. Then $\xi_n(b, t) -$

$\xi_n(b, s) = n^{1/6} \int_{Q_b(\alpha_0 + s/n^{1/3})}^{Q_b(\alpha_0 + t/n^{1/3})} (F(u) - \mathbf{1}[b_i \leq u]) du$ and almost surely

$$\begin{aligned} \{\xi_n(b, t) - \xi_n(b, s)\}^2 &= n^{1/3} \left\{ \int_{Q_b(\alpha_0 + s/n^{1/3})}^{Q_b(\alpha_0 + t/n^{1/3})} (F(u) - \mathbf{1}[b_i \leq u]) du \right\}^2 \\ &\leq n^{1/3} \left\{ \int_{Q_b(\alpha_0 + s/n^{1/3})}^{Q_b(\alpha_0 + t/n^{1/3})} |F(u) - \mathbf{1}[b_i \leq u]| du \right\}^2 \\ &\leq 4n^{1/3} \left\{ Q_b(\alpha_0 + t/n^{1/3}) - Q_b(\alpha_0 + s/n^{1/3}) \right\}^2 = n^{-1/3} O(|t - s|), \end{aligned}$$

where the second inequity holds because $|F(u) - \mathbf{1}[b_i \leq u]| \leq \sup_u F(u) + 1 \leq 2$. The claim is therefore verified.

(iv) Let $\mathcal{N}(\epsilon, \Xi_n, \mathbb{L}^2(\mathcal{P}))$ be the $\mathbb{L}^2$ -covering number for Ξ_n with respect to probability measure $\mathcal{P}$, then for every $\epsilon_n \downarrow 0$, we have $\sup_{\mathcal{P}^*} \int_0^{\epsilon_n} \sqrt{\log \mathcal{N}(\epsilon \|\tilde{\xi}(b)\|_{\mathcal{P}^*, 2}, \Xi_n, \mathbb{L}^2(\mathcal{P}^*))} d\epsilon = o(1)$. This claim holds by observing that $\xi_{b,t}$ is continuously differentiable with respect to t and hence Ξ_n belongs to the parametric class (see [Van der Vaart, 2000](#), Example 19.7), which implies the convergences of the integral.

(v) We derive the limit of the covariance function. Take $t, s \in \mathcal{T}$,

$$\begin{aligned} \mathbb{E}[\xi_n(b_i, t) \xi_n(b_i, s)] &= \mathbb{E} \left[n^{1/3} \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0 + t/n^{1/3})} \mathbf{1}[b_i \leq u] du \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0 + s/n^{1/3})} \mathbf{1}[b_i \leq u] du \right] + o(1) \\ &= n^{1/3} \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0 + t/n^{1/3})} \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0 + s/n^{1/3})} \mathbb{E} \{ \mathbf{1}[\min\{u, v\} \geq b_i] \} dudv + o(1) \\ &= n^{1/3} \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0 + t/n^{1/3})} \int_{Q_b(\alpha_0)}^{Q_b(\alpha_0 + s/n^{1/3})} G(\min\{u, v\}) dudv \rightarrow 0, \end{aligned}$$

where G is the c.d.f. of the bid distribution. Therefore, $H(t, s) \equiv \lim_{n \rightarrow \infty} \mathbb{E}[\xi_n(b_i, t) \xi_n(b_i, s)] = 0$ for any $t, s \in \mathcal{T}$.

Based on (i)-(v) and [Van Der Vaart and Wellner \(1996, Theorem 2.11.22\)](#), $\frac{1}{\sqrt{n}} \sum_i \xi_n(b_i, t)$ converges weakly to a zero mean Gaussian process $\mathbb{G}$ with sample path define on $\mathcal{T}$ and with covariance function $H(t, s)$. By the property of Gaussian process, $H(t, s) = 0$ implies that the limit process $\mathbb{G}(t) = 0$ for all t almost surely. Because the mapping

$\sup_{t \in \mathcal{T}} f(\cdot) : \mathcal{C} \rightarrow \mathbb{R}$ (from the set of continuous functions defined on compact set to $\mathbb{R}$) is continuous with respect to the sup-norm, we can further apply the continuous mapping theorem and have

$$\frac{1}{\sqrt{n}} \sum_i \xi_n(b_i, \cdot) \xrightarrow{w} \mathbf{G} \Rightarrow \sup_{t \in \mathcal{T}} \frac{1}{\sqrt{n}} \sum_i \xi_n(b_i, t) \xrightarrow{d} \sup_{t \in \mathcal{T}} \mathbf{G}(t) = 0 \Rightarrow \sup_{t \in \mathcal{T}} \frac{1}{\sqrt{n}} \sum_i \xi_n(b_i, t) \xrightarrow{p} 0.$$

The conclusion of the Lemma holds.

Lemma 2. Suppose that Assumptions 1 and 2 hold, then

$$n^{2/3} \alpha_0 \left\{ Q_{b,n}(\alpha_0 + tn^{-1/3}) - Q_{b,n}(\alpha_0) - Q_b(\alpha_0 + tn^{-1/3}) + Q_b(\alpha_0) \right\} \xrightarrow{w} \frac{\alpha_0}{g(Q_b(\alpha_0))} \mathbb{B}(t),$$

where $\mathbb{B}$ is a two-sided Brownian motion.

Proof. By Van Der Vaart and Wellner (1996, Theorem 1.6.1), it is sufficient to show the result holds for a sequence of compact sets $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \dots \subseteq \mathcal{T}_k \subseteq \dots$ such that $0 \in \mathcal{T}_1$ and $\cup_{k=1}^{\infty} \mathcal{T}_k = \mathbb{R}$. Denote $\mathcal{T}_k^+ = \mathcal{T}_k \cap \mathbb{R}^+$ and $\mathcal{T}_k^- = \mathcal{T}_k \cap \mathbb{R}^-$. Given Assumption 2, we can apply Bahadur representation again (see Lemma 1) and know that uniform in τ ,

$$Q_{b,n}(\tau) - Q_b(\tau) = \frac{\tau - \frac{1}{n} \sum_i \mathbf{1}[b_i \leq Q_b(\tau)]}{g(Q_b(\tau))} + O_{a.s.}(n^{-3/4}(\log n)^{1/2}(\log \log n)^{1/4}).$$

We consider $t \geq 0$ first. Let $r_{1n} = O_{a.s.}(n^{-1/12}(\log n)^{1/2}(\log \log n)^{1/4})$, we have uniformly in $t \in \mathcal{T}_k^+$,

$$\begin{aligned} & n^{2/3} \left\{ Q_{b,n}(\alpha_0 + tn^{-1/3}) - Q_{b,n}(\alpha_0) - Q_b(\alpha_0 + tn^{-1/3}) + Q_b(\alpha_0) \right\} \\ &= \frac{n^{1/6}}{\sqrt{n}} \sum_i \left(\frac{\alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \leq Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0 + tn^{-1/3}))} - \frac{\alpha_0 - \mathbf{1}[b_i \leq Q_b(\alpha_0)]}{g(Q_b(\alpha_0))} \right) + r_{1n} \\ &= \frac{n^{1/6}}{\sqrt{n}} \sum_i \left(\frac{tn^{-1/3} - \mathbf{1}[Q_b(\alpha_0) < b_i \leq Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0))} \right) + r_{1n} + r_{2n}, \end{aligned}$$

where

$$\begin{aligned} r_{2n} &= \frac{n^{1/6}}{\sqrt{n}} \sum_i \left(\frac{\alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \leq Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0 + tn^{-1/3}))} - \frac{\alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \leq Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0))} \right) \\ &= n^{1/6} \left(\frac{1}{g(Q_b(\alpha_0 + tn^{-1/3}))} - \frac{1}{g(Q_b(\alpha_0))} \right) \frac{1}{\sqrt{n}} \sum_i \xi_i = n^{1/6} O(n^{-1/3}) O_p(1) = o_p(1), \end{aligned}$$

where $\xi_i = \alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \leq Q_b(\alpha_0 + tn^{-1/3})]$. For the leading term, it is can be shown by standard method (e.g. [Kim and Pollard, 1990](#)) that

$$\frac{n^{1/6}}{\sqrt{n}} \sum_i \left(\frac{tn^{-1/3} - \mathbf{1}[Q_b(\alpha_0) < b_i \leq Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0))} \right) \xrightarrow{w} \frac{1}{g(Q_b(\alpha_0))} \mathbb{B}(t),$$

where $\mathbb{B}$ is a Brownian motion over a sequence of compact sets $\mathcal{T}_1^+ \subseteq \mathcal{T}_2^+ \subseteq \dots \subseteq \mathcal{T}_k^+ \subseteq \dots$.

When $t < 0$, we have uniformly in $t \in \mathcal{T}_k^-$,

$$\begin{aligned} & n^{2/3} \left\{ Q_{b,n}(\alpha_0 + tn^{-1/3}) - Q_{b,n}(\alpha_0) - Q_b(\alpha_0 + tn^{-1/3}) + Q_b(\alpha_0) \right\} \\ &= \frac{n^{1/6}}{\sqrt{n}} \sum_i \left(\frac{\alpha_0 + tn^{-1/3} - \mathbf{1}[b_i \leq Q_b(\alpha_0 + tn^{-1/3})]}{g(Q_b(\alpha_0 + tn^{-1/3}))} - \frac{\alpha_0 - \mathbf{1}[b_i \leq Q_b(\alpha_0)]}{g(Q_b(\alpha_0))} \right) + \tilde{r}_{1n} \\ &= \frac{n^{1/6}}{\sqrt{n}} \sum_i \left(\frac{tn^{-1/3} + \mathbf{1}[Q_b(\alpha_0 + tn^{-1/3}) < b_i \leq Q_b(\alpha_0)]}{g(Q_b(\alpha_0))} \right) + \tilde{r}_{1n} + \tilde{r}_{2n}, \end{aligned}$$

where the two asymptotically negligible terms $\tilde{r}_{1n}$ and $\tilde{r}_{2n}$ are analogously defined as r_{1n} and r_{2n} in the proof of the case $t \geq 0$, respectively. The convergence result holds analogously over a sequence of compact sets $\mathcal{T}_1^- \subseteq \mathcal{T}_2^- \subseteq \dots \subseteq \mathcal{T}_k^- \subseteq \dots$.

The conclusion follows by combining the results for both $t \geq 0$ and $t < 0$.

Lemma 3. Suppose that Assumptions 1 and 2 hold, then

$$n^{\frac{2}{3}} \left[V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0) \right] - n^{\frac{2}{3}} \left[V(\alpha_0 + tn^{-\frac{1}{3}}) - V(\alpha_0) \right] \xrightarrow{w} \frac{\alpha_0}{(I-1)g(Q_b(\alpha_0))} \mathbb{B}(t)$$

where $\mathbb{B}$ is a two-sided Brownian motion.

Proof. Recall that for any $\tau \in (0, 1)$,

$$\begin{aligned} V_n(\tau) &= \frac{1}{n} \frac{I-2}{I-1} \sum_i b_i \mathbf{1}[b_i \leq Q_{b,n}(\tau)] + \frac{1}{I-1} \tau Q_{b,n}(\tau) + O_p(1/n) \\ &\equiv \frac{I-2}{I-1} V_{1n}(\tau) + \frac{1}{I-1} V_{2n}(\tau) + O_p(1/n). \end{aligned}$$

Likewise,

$$V(\tau) = \frac{I-2}{I-1} \int_0^\tau Q_v(t) dt + \frac{1}{I-1} \tau Q_b(\tau) \equiv \frac{I-2}{I-1} V_1(\tau) + \frac{1}{I-1} V_2(\tau).$$

The part associates with V_{1n} , that is, $n^{\frac{2}{3}} [V_{1n}(\alpha_0 + tn^{-\frac{1}{3}}) - V_{1n}(\alpha_0)] - n^{\frac{2}{3}} [V_1(\alpha_0 + tn^{-\frac{1}{3}}) - V_1(\alpha_0)]$ converges in probability to zero by Lemma 1. For the part associated with V_{2n} , note that

$$\begin{aligned} &n^{\frac{2}{3}}(I-1) [V_{2n}(\alpha_0 + tn^{-\frac{1}{3}}) - V_{2n}(\alpha_0)] - n^{\frac{2}{3}} [V_2(\alpha_0 + tn^{-\frac{1}{3}}) - V_2(\alpha_0)] \\ &= n^{2/3} Q_{b,n}(\alpha_0 + tn^{-1/3})(\alpha_0 + tn^{-1/3}) - n^{2/3} Q_{b,n}(\alpha_0)\alpha_0 - n^{2/3} Q_b(\alpha_0) \\ &\quad + tn^{-1/3})(\alpha_0 + tn^{-1/3}) + n^{2/3} Q_b(\alpha_0)\alpha_0 \\ &= n^{2/3} \alpha_0 \left\{ Q_{b,n}(\alpha_0 + tn^{-1/3}) - Q_{b,n}(\alpha_0) - Q_b(\alpha_0 + tn^{-1/3}) + Q_b(\alpha_0) \right\} \\ &\quad + n^{1/3} t \left\{ Q_{b,n}(\alpha_0 + tn^{-1/3}) - Q_b(\alpha_0 + tn^{-1/3}) \right\} \end{aligned}$$

The second right hand side term, for $|t| < K$, is uniformly bounded by order $n^{1/3} \times n^{-1/2} \times O_p(1) \xrightarrow{p} 0$. The first right hand side term is dealt with by Lemma 2.

2. AUXILIARY LEMMAS FOR THEOREM 2

Lemmas 4 to 8 shows that the sup distance between V_n and $\widehat{V}$ is small, which we adapted from Pal and Woodroffe (2006). Lemma 9 is an intermediate step for establishing the limiting distribution of the smoothed quantile estimator.

We introduce some notation. Let k_n be a sequence of integers such that $k_n \rightarrow \infty$ and $n/k_n \rightarrow \infty$. Without loss of generality we assume k_n divides n and let $\ell_n = n/k_n$. We therefore can divide $[0, n]$ into k_n equal size intervals with each interval contains ℓ_n

consecutive integers. Let $\{s_i, i = 1, 2, \dots, k_n\}$ be the set of upper boundary of those intervals such that $s_i = i\ell_n$.

For $(i-1)\ell_n \leq s < i\ell_n, i = 1, 2, \dots, k_n$, define

$$L(s) = \frac{s - (i-1)\ell_n}{\ell_n} V\left(\frac{i}{n}\right) + \frac{i\ell_n - s}{\ell_n} V\left(\frac{i-1}{n}\right),$$

and

$$L_n(s) = \frac{s - (i-1)\ell_n}{\ell_n} V_n\left(\frac{i}{n}\right) + \frac{i\ell_n - s}{\ell_n} V_n\left(\frac{i-1}{n}\right),$$

That is, L and L_n are the linear interpolation of V and V_n on k_n knots $\{s_1/n, s_2/n, \dots, s_{k_n}/n\}$, respectively. Note that since V is convex, L is necessarily convex. However L_n may not be convex since V_n is not necessarily convex. Let A_n be the event such that L_n is convex. Since L_n is convex if and only if each segment is convex, the complement of A_n can be written as

$$\begin{aligned} A_n^c &= \bigcup_{i=2}^{k_n-1} \left\{ V_n\left(\frac{(i-1)\ell_n}{n}\right) + V_n\left(\frac{(i+1)\ell_n}{n}\right) < 2V_n\left(\frac{i\ell_n}{n}\right) \right\} \\ &= \bigcup_{i=2}^{k_n} \left\{ V\left(\frac{(i-1)\ell_n}{n}\right) + V\left(\frac{(i+1)\ell_n}{n}\right) - 2V\left(\frac{i\ell_n}{n}\right) \right. \\ &\quad \left. + \Delta_n\left(\frac{(i-1)\ell_n}{n}\right) + \Delta_n\left(\frac{(i+1)\ell_n}{n}\right) - 2\Delta_n\left(\frac{i\ell_n}{n}\right) < 0 \right\}, \end{aligned}$$

where $\Delta_n \equiv V_n - V$. [Pal and Woodroffe \(2006, Proposition 2\)](#) shows that A_n^c has probability approaching zero, thus it is sufficient to consider derive the bounds of the distance conditional on A_n (see also [Kiefer and Wolfowitz, 1976, Lemma 4](#)).

Lemma 4. *Suppose that Assumption 3 is satisfied, then there exists a positive c_1 such that*

$$\min_{i=2, \dots, k_n-1} \left| V\left(\frac{(i-1)\ell_n}{n}\right) + V\left(\frac{(i+1)\ell_n}{n}\right) - 2V\left(\frac{i\ell_n}{n}\right) \right| \geq \frac{c_1}{k_n^2}.$$

Proof. By Assumption 3, there exists $c_1 > 0$ such that $Q'_v(\alpha) \geq c_1 > 0$ for all $\alpha \in [0, 1]$.

Then we have

$$\begin{aligned}
& V\left(\frac{(i-1)\ell_n}{n}\right) + V\left(\frac{(i+1)\ell_n}{n}\right) - 2V\left(\frac{i\ell_n}{n}\right) \\
&= \int_{\frac{i\ell_n}{n}}^{\frac{(i+1)\ell_n}{n}} Q_v(\alpha) d\alpha - \int_{\frac{(i-1)\ell_n}{n}}^{\frac{i\ell_n}{n}} Q_v(\alpha) d\alpha \geq \int_{\frac{i\ell_n}{n}}^{\frac{(i+1)\ell_n}{n}} \left[Q_v(\alpha) - Q_v\left(\frac{i\ell_n}{n}\right) \right] d\alpha \\
&= \frac{\ell_n}{n} \left[Q_v(\alpha_n^*) - Q_v\left(\frac{i\ell_n}{n}\right) \right] \geq c_1 \frac{\ell_n^2}{n^2} = \frac{c_1}{k_n^2}. \quad \square
\end{aligned}$$

Lemma 5. Let $\|\cdot\|$ denote the sup norm. Conditional on A_n , there is

$$\|V_n - \widehat{V}\| \leq 2\|(V_n - L_n) - (V - L)\| + 2\|V - L\|.$$

Proof. By Marshall's Lemma (see Kiefer and Wolfowitz, 1976, Lemma 3), for any convex function m , $\|\widehat{V} - m\| \leq \|V_n - m\|$. Therefore,

$$\|V_n - \widehat{V}\| \leq \|V_n - L_n\| + \|L_n - \widehat{V}\| \leq 2\|V_n - L_n\| \leq 2\|(V_n - L_n) - (V - L)\| + 2\|V - L\|,$$

where the first and third inequalities holds by triangular inequality, the second one holds by Marshall's Lemma. $\square$

Lemma 6. Suppose that Assumption 3 is satisfied, then there exists $c_3 > 0$ such that for all $s \in [0, n]$,

$$0 \leq L(s) - V(s) \leq \frac{c_3}{k_n^2}.$$

Proof. $L(s) > V(s)$ follows immediately by the convexity of V . The other inequality holds follows from a similar argument as in Lemma 4 and the fact that $Q'_v(\alpha)$ is bounded from above uniformly.

Lemma 7. Suppose that Assumptions 1 and 3 is satisfied, then

$$\|V_n - L_n - V + L\| = O_p\left(\sqrt{\frac{\log k_n}{nk_n}}\right) + O_p\left(\frac{\log n}{n}\right).$$

Proof. Define function V_P such that $V_P(j/n) = V(j/n)$ for each j/n and otherwise equals to its own interpolation. It is obvious that $\|V_P - V\| = O(1/n)$. It is then sufficient to focus on $V_n - L_n - V_P + L$. Note that all four functions are piece-wise linear, and so does there linear combinations. Therefore, the sup must be achieved at some knot(s). Based on this observations, we can write

$$\begin{aligned} & \|V_n - L_n - V_P + L\| \\ &= \max_{i=1, \dots, K_n} \max_{(i-1)\ell_n \leq j \leq i\ell_n} \left| \Delta_n(j/n) - \frac{j - (i-1)\ell_n}{\ell_n} \Delta_n(i/n) - \frac{i\ell_n - j}{\ell_n} \Delta_n((i-1)/n) \right|, \end{aligned}$$

where for $t \in [0, 1]$,

$$\begin{aligned} \Delta_n(t) &= V_n(t) - V_P(t) = V_n(t) - V(t) + O(1/n) \\ &= \frac{I-2}{I-1} \underbrace{\left\{ \sum_{i=1}^{[tn]} \frac{b(i)}{n} - \int_0^t Q_b(\alpha) d\alpha \right\}}_{\Delta_A(t)} + \frac{1}{I-1} \underbrace{\left\{ \frac{[tn]}{n} b_{(j)} - t Q_b(t) \right\}}_{\Delta_B(t)} + O(1/n) \end{aligned}$$

where $[x]$ denotes the integer part of x . Note that Δ_A is an integrated quantile process. By [Tse \(2009, Theorem 2.1\)](#), there exists a Gaussian process $\mathbb{G}_n$ and Brownian bridge $\mathbb{B}_n^A$ defined on proper measurable space such that for any $\tau < 1/6$,

$$\|\sqrt{n}\Delta_A - \psi_n\| \stackrel{a.s.}{=} O(n^{-\tau}),$$

where $\psi_n(t) = \mathbb{G}_n(t) + \int_0^t \mathbb{B}_n^A(u) dQ_b(u)$. On the other hand, by [Csorgo and Revesz \(1978,](#)

Theorem 6), there exists a sequence of Brownian bridge B_n such that $\sup_{\delta_n \leq t \leq 1-\delta_n} |g(Q_b(t))\sqrt{n}\Delta_B(t) -$

$B_n(t) \stackrel{a.s.}{=} O_p(n^{-1/2} \log n)$. We can then conclude

$$\begin{aligned}
& \|V_n - L_n - V_P + L\| \\
& \leq \max_{i=1, \dots, K_n} \max_{(i-1)\ell_n \leq j \leq i\ell_n} \left| \Delta_A(j/n) - \frac{j - (i-1)\ell_n}{\ell_n} \Delta_A(i/n) - \frac{i\ell_n - j}{\ell_n} \Delta_A((i-1)/n) \right| \\
& + \max_{i=1, \dots, K_n} \max_{(i-1)\ell_n \leq j \leq i\ell_n} \left| \Delta_B(j/n) - \frac{j - (i-1)\ell_n}{\ell_n} \Delta_B(i/n) - \frac{i\ell_n - j}{\ell_n} \Delta_B((i-1)/n) \right| + O_p(1/n) \\
& \stackrel{d}{=} \frac{1}{\sqrt{n}} \max_{i=1, \dots, K_n} \max_{(i-1)\ell_n \leq j \leq i\ell_n} \left| \psi_n(j/n) - \frac{j - (i-1)\ell_n}{\ell_n} \psi_n(i/n) - \frac{i\ell_n - j}{\ell_n} \psi_n((i-1)/n) \right| + O_p(n^{-\tau-1/2}) \\
& + \frac{1}{\sqrt{n}} \max_{i=1, \dots, K_n} \max_{(i-1)\ell_n \leq j \leq i\ell_n} \left| B_n(j/n) - \frac{j - (i-1)\ell_n}{\ell_n} B_n(i/n) - \frac{i\ell_n - j}{\ell_n} B_n((i-1)/n) \right| + O_p(\log n/n) \\
& \leq \frac{1}{\sqrt{n}} \sup_{0 \leq t-s \leq \frac{1}{k_n}} |\psi_n(t) - \psi_n(s)| + \frac{1}{\sqrt{n}} \sup_{0 \leq t-s \leq \frac{1}{k_n}} |B_n(t) - B_n(s)| + O_p(\log n/n) + O_p(n^{-\tau-1/2}) \\
& \leq \frac{\sqrt{2 \log \log n}}{\sqrt{n}} \frac{1}{\sqrt{k_n}} + \frac{1}{\sqrt{n}} \frac{\sqrt{\log \log K_n}}{\sqrt{k_n}} + O_p(\log n/n) + O_p(n^{-\tau-1/2})
\end{aligned}$$

where the last two inequalities result from the continuity module of Gaussian processes and the fact that $g(b) \geq \underline{b} > 0$ for all b (GPV Proposition 1). Recall that $k_n \propto \frac{n}{\log n}$, we can conclude that the right hand side is of order $O_p((n/\log n)^{-2/3})$.

Lemma 8. Suppose Assumptions 3 and 4 are satisfied, the $\|\hat{V} - V_n\| = O_p((n/\log n)^{-2/3})$.

Proof. The conclusion holds by Lemmas 5 to 7. □

Lemma 9. Let $z_{(i)} = n(b_{(i)} - b_{(i-1)})$ and $w_i = ((i-1)/n - \alpha) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(u - \alpha) du$. Suppose Assumption 3 is satisfied, then $\sum_i z_{(i)} w_i = o_p(1/\sqrt{nh})$.

Proof. Since b_i has bounded support, it is without loss of generality to prove the case when b_i follows the uniform distribution. Pyke (1965, Section 2.1) shows that $z_{(i)}$ are identically distributed across i . Furthermore, $\mathbb{E}[z_{(i)}] = n(n+1)^{-1}$, $V(z_{(i)}) = n^3(n+1)^{-2}(n+2)^{-1}$ and $\text{Cov}(z_{(i)} z_{(j)}) = -n^2(n+1)^{-2}(n+2)^{-1}$. Let ρ_{ij} be the correlation coefficient, so $\rho_{ij} = 1$ if $i = j$, and $\rho_{ij} = -1/n$ otherwise.

Note first that $\mathbb{E}[\sum_i z_{(i)} w_i] = n(n+1)^{-1} \sum_i w_i = (1/h) \left(\int_0^1 (u - \alpha) K(u - \alpha/h) du + O(1/n) \right) = O(1/nh) = o_p(1/\sqrt{nh})$ since $\int u K(u) = 0$ by assumption. Next consider

$$V(\sum_i z_{(i)} w_i) = \sum_i w_i^2 V(z_{(i)}) + 2 \sum_{i \neq j} w_i w_j \text{Cov}(z_{(i)}, z_{(j)}) = V(z_{(i)}) \left(\sum_i w_i^2 + 2 \sum_{i \neq j} w_i w_j \rho_{ij} \right).$$

Consider w_i , there exists a $u_i^* \in ((i-1)/n, i/n)$ such that

$$w_i = \left(\frac{i-1}{n} - \alpha \right) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(u - \alpha) du = \frac{1}{nh} \left(\frac{i-1}{n} - \alpha \right) K \left(\frac{u_i^* - \alpha}{h} \right),$$

Since the kernel function has bounded support, that is, $K(u) = 0$ if $|u| > \bar{K}$. Then $w_i \neq 0$ only if $|u_i^* - \alpha| \leq \bar{K}h$. Therefore the quantity $i - 1/n$ for nonzero w_i is around h neighborhood of α , which implies that each of the nonzero $|w_i|$ is of order $\frac{1}{nh} \times h = \frac{1}{n}$. Let i_α be the nearest integer to $n\alpha$, then we know $w_i \neq 0$ only if $|i - i_\alpha| \leq Cnh$ for some constant C , which implies that in the expression of $V(\sum_i z_{(i)} w_i)$, there are of order nh nonzero summands. Since each w_i is of order $1/n$, $\rho_{ij} = -1/n$ when $i \neq j$, $V(z_{(i)}) = O(1)$, the order of $V(\sum_i z_{(i)} w_i)$ is $O(nh \times (1/n)^2 + (nh)^2 \times (1/n)^3) = O(h/n)$, which is of smaller order than $1/nh$.

The above argument shows that $\mathbb{E}[\sum_i z_{(i)} w_i] = o_p(1/\sqrt{nh})$ and $V(\sum_i z_{(i)} w_i) = o_p(1/nh)$, therefore we can conclude that $\sum_i z_{(i)} w_i = o_p(1/\sqrt{nh})$.

3. PROOFS TO COROLLARIES

3.1. Proof of Corollary 1. Consider a $J \times 1$ vector of mutually different quantile levels $(\alpha_1, \alpha_2, \dots, \alpha_J)$, following the arguments in the proof of Theorem 1, the following events are equivalent:

$$\cap_{j=1,2,\dots,J} \{n^{\frac{1}{3}}(\hat{Q}_v(\alpha_j) - Q_v(\alpha_j)) \leq z_j\} \Leftrightarrow \cap_{j=1,2,\dots,J} \left\{ \underset{t \in [-\alpha_j n^{\frac{1}{3}}, (1-\alpha_j) n^{\frac{1}{3}}]}{\text{argmin}} \{W_{jn}(t) - z_j t\} \geq 0 \right\},$$

where for $j = 1, 2, \dots, J$

$$W_{jn}(t) = n^{\frac{2}{3}} \left[V_n(\alpha_j + tn^{-\frac{1}{3}}) - V_n(\alpha_j) \right] - n^{\frac{2}{3}} \left[V(\alpha_j + tn^{-\frac{1}{3}}) - V(\alpha_j) \right] \\ + n^{\frac{2}{3}} \left[V(\alpha_j + tn^{-\frac{1}{3}}) - V(\alpha_j) - Q_v(\alpha_j)tn^{-\frac{1}{3}} \right].$$

Following the same arguments, we have the convergence of each single component:

$$W_{jn}(t) \xrightarrow{w} \frac{\alpha_j}{(I-1)g(Q_b(\alpha_j))} \mathbb{B}_j(t) + \frac{1}{2} Q'_v(\alpha_j) t^2.$$

where $\mathbb{B}_j$ is a two sided Brownian motion. Since $\mathbb{B}_j$ is Gaussian, it remains to find their covariance. Following the arguments in Lemmas 1 to 3 and ignoring the small order terms, we know that for each given t , the joint limiting distribution of W_{jn} , $j = 1, \dots, J$, is determined by the joint limiting distribution of

$$n^{2/3} \left\{ Q_{b,n}(\alpha_j + tn^{-1/3}) - Q_{b,n}(\alpha_j) - Q_b(\alpha_j + tn^{-1/3}) + Q_b(\alpha_j) \right\}, \quad j = 1, \dots, J,$$

or alternatively, the joint limiting distribution of (for $t > 0$, the case of $t < 0$ is similar)

$$\frac{n^{1/6}}{\sqrt{n}} \sum_i \left(\frac{tn^{-1/3} - \mathbf{1}[Q_b(\alpha_j) < b_i \leq Q_b(\alpha_j + tn^{-1/3})]}{g(Q_b(\alpha_j))} \right), \quad j = 1, \dots, J,$$

To calculate the limit of covariance of above expression at different quantile levels, it is sufficient to focus on same observation index i since bids are i.i.d.. Since all the α_j are mutually different, the we have for $j \neq j'$, there is $Q_b(\alpha_j) \neq Q_b(\alpha_{j'})$ by the strict monotonicity of Q_b . Therefore,

$$\lim_{n \rightarrow \infty} n^{1/3} \mathbb{E} \left[\left(tn^{-1/3} - \mathbf{1}[Q_b(\alpha_j) < b_i \leq Q_b(\alpha_j + tn^{-1/3})] \right) \right. \\ \left. \times \left(tn^{-1/3} - \mathbf{1}[Q_b(\alpha_{j'}) < b_i \leq Q_b(\alpha_{j'} + tn^{-1/3})] \right) \right] \\ = \lim_{n \rightarrow \infty} \mathbb{E} \left[-t^2 n^{-1/3} + n^{1/3} \mathbf{1}[Q_b(\alpha_j) < b_i \leq Q_b(\alpha_j + tn^{-1/3})] \right. \\ \left. \times \mathbf{1}[Q_b(\alpha_{j'}) < b_i \leq Q_b(\alpha_{j'} + tn^{-1/3})] \right] = 0.$$

Therefore, we can conclude that $\mathbb{B}_j$ are asymptotically uncorrelated and hence independent.

Let constants (a_j, b_j) be defined as

$$a_j = \frac{\alpha_j}{(I-1)g(Q_b(\alpha_j))}, \quad b_j = \frac{1}{2}Q'_v(\alpha_j).$$

Then we have the joint limiting distribution be given by

$$\begin{aligned} & \mathbb{P} \left(\cap_{j=1,2,\dots,J} \{n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_j) - Q_v(\alpha_j)) \leq z_j\} \right) \\ & \rightarrow \mathbb{P} \left(\cap_{j=1,2,\dots,J} \left\{ \operatorname{argmax}_{t \in \mathbb{R}} \{\mathbb{B}_j(t) - t^2\} \leq \frac{z_j}{2b_j} \left(\frac{b_j}{a_j} \right)^{2/3} \right\} \right) \\ & = \prod_{j=1,\dots,J} \mathbb{P} \left(\operatorname{argmax}_{t \in \mathbb{R}} \{\mathbb{B}_j(t) - t^2\} \leq \frac{z_j}{2b_j} \left(\frac{b_j}{a_j} \right)^{2/3} \right). \end{aligned}$$

3.2. Proof to Corollary 2. Consider inverting $\widehat{Q}(\cdot)$ first. Recall that $\widehat{F}(v_0) = \sup\{\alpha : \widehat{Q}_v(\alpha) \leq v_0\}$. Consistency of $\widehat{F}(v_0)$ holds by the consistency of $\widehat{Q}_v$ and the continuity of the sup operator. It remains to work out the convergence rate and limiting distribution. Let $Z = \operatorname{argmax}_{t \in \mathbb{R}} \{\mathbb{B}(t) - t^2\}$. Now,

$$\mathbb{P} \left(n^{1/3}(\widehat{F}(v_0) - F(v_0)) < x \right) = \mathbb{P} \left(\widehat{F}(v_0) < n^{-1/3}x + F(v_0) \right)$$

Note that the event $\{\widehat{F}(v_0) < n^{-1/3}x + F(v_0)\}$ is equivalent to $\{v_0 < \widehat{Q}_v(n^{-1/3}x + F(v_0))\}$. Using the fact that $F(v_0) = \alpha_0$, $Q_v(\alpha_0) = v_0$, and $(Q'_v(\alpha_0))^{-1} = f(v_0)$, we have

$$\begin{aligned} & \mathbb{P} \left(\widehat{F}(v_0) < n^{-1/3}x + F(v_0) \right) = \mathbb{P} \left(\widehat{Q}_v(n^{-1/3}x + F(v_0)) > v_0 \right) \\ & = \mathbb{P} \left(\widehat{Q}_v(n^{-1/3}x + \alpha_0) - Q_v(n^{-1/3}x + \alpha_0) > v_0 - Q_v(n^{-1/3}x + \alpha_0) \right) \\ & = \mathbb{P} \left(\widehat{Q}_v(n^{-1/3}x + \alpha_0) - Q_v(n^{-1/3}x + \alpha_0) > -n^{-1/3}Q'_v(\alpha_0)x + O(n^{-2/3}) \right) \\ & = \mathbb{P} \left(f(v_0)n^{1/3}(\widehat{Q}_v(n^{-1/3}x + \alpha_0) - Q_v(n^{-1/3}x + \alpha_0)) < x + O(n^{-1/3}) \right) \end{aligned}$$

Repeat the proof of Theorem 1 shows that for each x , $n^{1/3}(\widehat{Q}_v(n^{-1/3}x + \alpha_0) - Q_v(n^{-1/3}x + \alpha_0))$ has the same limiting distribution as $n^{1/3}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0))$. Therefore, we have

$$\mathbb{P}\left(n^{1/3}(\widehat{F}(v_0) - F(v_0)) < x\right) \rightarrow \mathbb{P}(f(v_0)C(\alpha_0)Z < x).$$

Next, by the definition of $\widehat{F}$, we have for any positive $\eta_n \downarrow 0$,

$$\mathbb{P}\left(n^{1/3}(\widehat{F}(v_0) - F(v_0)) = x\right) \leq \mathbb{P}\left(\widehat{Q}_v(n^{-1/3}x + F(v_0)) \leq v_0 < \widehat{Q}_v(n^{-1/3}x + F(v_0) + \eta_n)\right),$$

the right hand side coverages to zero. Observe that Z is continuous, we have

$$\mathbb{P}\left(n^{1/3}(\widehat{F}(v_0) - F(v_0)) \leq x\right) \rightarrow \mathbb{P}(f(v_0)C(\alpha_0)Z \leq x).$$

Lastly, because $\hat{q}(\cdot)$ is continuous and strictly increasing, the result for $\widehat{F}^S$ follows essentially the same (but simpler) argument as above and therefore omitted.

3.3. Proof to Corollary 3. We give the sketch of the proof for brevity. Let γ_n be a deterministic diverging sequence whose rate will be determined later. For a given x , define

$$W_n(t|x) = \gamma_n^2 \left[V_n(\alpha_0 + t\gamma_n^{-1}|x) - V_n(\alpha_0|x) - Q_v(\alpha_0|x)t\gamma_n^{-1} \right].$$

Following the same argument as in Theorem 1, we have

$$\gamma_n^{-1}(\widehat{Q}_v(\alpha_0|x) - Q_v(\alpha_0|x)) \leq z \Leftrightarrow \underset{t \in [-\alpha_0\gamma_n, (1-\alpha_0)\gamma_n]}{\operatorname{argmin}} \{W_n(t|x) - zt\} \geq 0$$

Then we conduct the same decomposition:

$$\begin{aligned} W_n(t|x) &= \underbrace{\gamma_n^2 \left[V_n(\alpha_0 + t\gamma_n^{-1}|x) - V_n(\alpha_0|x) \right]}_{\equiv W_n^A(t)} - \gamma_n^2 \left[V(\alpha_0 + t\gamma_n^{-1}|x) - V(\alpha_0|x) \right] \\ &\quad + \underbrace{\gamma_n^2 \left[V(\alpha_0 + t\gamma_n^{-1}|x) - V(\alpha_0|x) - Q_v(\alpha_0|x)t\gamma_n^{-1} \right]}_{= \frac{1}{2}Q_v'(\alpha_0|x)t^2 + o(1)}. \end{aligned}$$

It remains to analyze the asymptotic behavior of W_n^A . It can be observed from the definition of $V_n(\cdot|x)$ that for any $\tau \in (0, 1)$,

$$V_n(\tau|x) = \frac{I-2}{I-1} \int_0^\tau Q_{n,b}(t|x)dt + \frac{1}{I-1} \tau Q_{n,b}(\tau|x) + O(1/n),$$

where $Q_{n,b}(\tau|x)$ is chosen to be the local polynomial estimator of [Guerre and Sabbah \(2012\)](#), whose Assumptions X, F and K can be verified to hold in our context. In particular, Assumption F is satisfied since the continuous differentiability of $F(\cdot|x)$ implies that $Q_b(\cdot|x)$ is twice continuously differentiable, as shown in [Guerre, Perrigne, and Vuong \(2000, Proposition 1-iv\)](#).

Since we only need to estimate the quantile function, we choose the order ν of the polynomial as $\nu = 0$. Using [Guerre and Sabbah \(2012, page 98\)](#)'s uniform Bahadur representation, we have for any $\tau \in (0, 1)$,

$$Q_{n,b}(\tau|x) - Q_b(\tau|x) = \frac{\beta_n(\tau)}{(nh^d)^{1/2}} + O(h^2) + O_p\left(\frac{\log n}{nh^d}\right)^{3/4},$$

where the first right hand side (RHS) is the first order approximation, the second RHS term is the bias and its order is determined by the twice continuous differentiability of Q_b , and the third RHS term is the Bahadur representation error, and β_n is defined as

$$\beta_n(\tau) = J_n^{-1} \frac{2}{(nh^d)^{1/2}} \sum_i^n \{\mathbf{1}[b_i \leq Q_b^*(\tau|x)] - \tau\} K\left(\frac{X_i - x}{h}\right),$$

where $J_n \xrightarrow{p} J$ for some constant, $K(\cdot)$ is the kernel function and Q_b^* is the argmin of the population criterion function of the local polynomial regression.

Following similar argument as in Lemmas 1 to 3, we need to make sure that both the bias term and the Bahadur representation error term converges (in probability) to zero faster than γ_n^2 . Take $\gamma_n = (nh^d)^{1/3}$, then $O(\gamma_n^2 h^2) = o(1)$ since h is chosen such that $nh^{d+3} \rightarrow 0$; $\gamma_n^2 \left(\frac{\log n}{nh^d}\right)^{3/4} = o_p(1)$ since $nh^d \rightarrow \infty$. As the consequence, the limiting behavior of $W_n^A(t)$ when $t \geq 0$ (the case of $t < 0$ similar) depends on the following dominant term (up to additive asymptotically negligible and some multiplicative constant terms):

$$\frac{1}{\sqrt{nh^d}} \sum_i^n \underbrace{(nh^d)^{1/6} \left\{ t(nh^d)^{-1/3} - \mathbf{1}[Q_b^*(\alpha_0|x) < b_i \leq Q_b^*(\alpha_0 + t(nh^d)^{-1/3}|x)] \right\}}_{\equiv \tilde{\xi}_i(t)} K\left(\frac{X_i - x}{h}\right).$$

Guerre and Sabbah (2012, Lemma A.1) shows that $\tilde{\xi}_i(t)$ has zero mean. Furthermore, for arbitrary $t, s > 0$,

$$\begin{aligned} \lim_{n \rightarrow \infty} h^{-d} \mathbb{E}[\tilde{\xi}_i(t) \tilde{\xi}_i(s)] &= \lim_{n \rightarrow \infty} h^{-d} \mathbb{E}\{\mathbb{E}[\tilde{\xi}_i(t) \tilde{\xi}_i(s) | X_i]\} \\ &= \lim_{n \rightarrow \infty} h^{-d} \mathbb{E} \left\{ K^2 \left(\frac{X_i - x}{h} \right) [\min\{t, s\} + O(\gamma_n^{-1})] \right\} \\ &= f_X(x) \min\{t, s\} \int K^2(u) du. \end{aligned}$$

It follows that $\frac{1}{\sqrt{nh^d}} \sum_i \tilde{\xi}_i(t)$ converges in distribution to normal for each t and given $\tilde{\xi}_i(t)$ is sum of indicator functions, $\frac{1}{\sqrt{nh^d}} \sum_i \tilde{\xi}_i(\cdot)$ weakly converge to a constant multiplied by a Brownian motion process. The rest of the proof follows similarly from Theorem 1.

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APPENDIX A. PROOF OF MAIN RESULTS

A.1. Proof of Theorem 1. For a generic $c > 0$, let $Z_n(c) = \operatorname{argmin}_{t \in [0,1]} \{V_n(t) - ct\}$. If the argmin is a set, then we take the sup of the set. For any $\alpha_0 \in (0, 1)$, by [van Es, Jongbloed, and Zuijlen \(1998, Theorem 2\)](#), the two following events are equivalent

$$Z_n(c) \geq \alpha_0 \Leftrightarrow \widehat{Q}_v(\alpha_0) \leq c.$$

Therefore, we have for a fixed $\alpha_0 \in [0, 1)$

$$\begin{aligned} n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \leq z &\Leftrightarrow \widehat{Q}_v(\alpha_0) \leq zn^{-\frac{1}{3}} + Q_v(\alpha_0) \Leftrightarrow Z_n(zn^{-\frac{1}{3}} + Q_v(\alpha_0)) \geq \alpha_0 \\ &\stackrel{i}{\Leftrightarrow} \operatorname{argmin}_{s \in [0,1]} \{V_n(s) - (zn^{-\frac{1}{3}} + Q_v(\alpha_0))s\} \geq \alpha_0 \\ &\stackrel{ii}{\Leftrightarrow} \operatorname{argmin}_{\{t: \alpha_0 + tn^{-\frac{1}{3}} \in [0,1]\}} \{V_n(\alpha_0 + tn^{-\frac{1}{3}}) - (zn^{-\frac{1}{3}} + Q_v(\alpha_0))(\alpha_0 + tn^{-\frac{1}{3}})\} \geq 0 \\ &\stackrel{iii}{\Leftrightarrow} \operatorname{argmin}_{t \in [-\alpha_0 n^{\frac{1}{3}}, (1-\alpha_0)n^{\frac{1}{3}}]} \{V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0) - Q_v(\alpha_0)tn^{-\frac{1}{3}} - ztn^{-\frac{2}{3}}\} \geq 0 \\ &\stackrel{iv}{\Leftrightarrow} \operatorname{argmin}_{t \in [-\alpha_0 n^{\frac{1}{3}}, (1-\alpha_0)n^{\frac{1}{3}}]} \{n^{\frac{2}{3}}V_n(\alpha_0 + tn^{-\frac{1}{3}}) - n^{\frac{2}{3}}V_n(\alpha_0) - Q_v(\alpha_0)tn^{\frac{1}{3}} - zt\} \geq 0, \end{aligned}$$

where (i) holds by definition of Z_n , (ii) holds by changing variable $s = \alpha_0 + tn^{-\frac{1}{3}}$, and (iii) and (iv) hold because the argmin stays unchanged when constants are multiplied or added to, or subtracted from the objective function.

Let $W_n(t) = n^{\frac{2}{3}} \left[V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0) - Q_v(\alpha_0)tn^{-\frac{1}{3}} \right]$, then the above displayed equation reduces to

$$n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \leq z \Leftrightarrow \operatorname{argmin}_{t \in [-\alpha_0 n^{\frac{1}{3}}, (1-\alpha_0)n^{\frac{1}{3}}]} \{W_n(t) - zt\} \geq 0$$

It remains to analyze the asymptotic behavior of $W_n(t)$. Decompose W_n as following

$$\begin{aligned} W_n(t) &= n^{\frac{2}{3}} \left[V_n(\alpha_0 + tn^{-\frac{1}{3}}) - V_n(\alpha_0) \right] - n^{\frac{2}{3}} \left[V(\alpha_0 + tn^{-\frac{1}{3}}) - V(\alpha_0) \right] \\ &\quad + n^{\frac{2}{3}} \left[V(\alpha_0 + tn^{-\frac{1}{3}}) - V(\alpha_0) - Q_v(\alpha_0)tn^{-\frac{1}{3}} \right]. \end{aligned}$$

The second component equals to $\frac{1}{2}Q'_v(\alpha_0)t^2 + o(1)$ by Assumption 2. As shown by Lemma 3 in the Supplement Material (Luo and Wan, 2016), the first right hand side term converges weakly to $\frac{\alpha_0}{(I-1)\sqrt{g(Q_b(\alpha_0))}}\mathbb{B}$, where $\mathbb{B}$ is a two sided Brownian Motion. Therefore, we have

$$W_n(t) \xrightarrow{w} \frac{\alpha_0}{(I-1)g(Q_b(\alpha_0))}\mathbb{B}(t) + \frac{1}{2}Q'_v(\alpha_0)t^2.$$

To simplify the notation, denote the constants in front of $\mathbb{B}$ and t^2 with a and b , respectively. Note that $a > 0$ and $b > 0$. By Van Der Vaart and Wellner (1996, Theorem 3.2.2) and the property of Brownian motion,

$$\begin{aligned} \operatorname{argmin}_{t \in [-\alpha_0 n^{\frac{1}{3}}, (1-\alpha_0)n^{\frac{1}{3}}]} \{W_n(t) - zt\} &\xrightarrow{d} \operatorname{argmin}_{t \in \mathbb{R}} \{a\mathbb{B}(t) + bt^2 - zt\} \stackrel{d}{\sim} \operatorname{argmin}_{t \in \mathbb{R}} \{a\mathbb{B}(t) + b(t - \frac{z}{2b})^2\} \\ &\stackrel{d}{\sim} \operatorname{argmin}_{t \in \mathbb{R}} \{\frac{a}{b}\mathbb{B}(t) + (t - \frac{z}{2b})^2\} \stackrel{d}{\sim} \left(\frac{a}{b}\right)^{2/3} \operatorname{argmin}_{t \in \mathbb{R}} \{\mathbb{B}(t) + t^2\} + \frac{z}{2b}, \end{aligned}$$

where $\stackrel{d}{\sim}$ denote equivalence in distribution. Therefore,

$$\begin{aligned} \mathbb{P}\left(n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \leq z\right) &\rightarrow \mathbb{P}\left(\left(\frac{a}{b}\right)^{2/3} \operatorname{argmin}_{t \in \mathbb{R}} \{\mathbb{B}(t) + t^2\} + \frac{z}{2b} \geq 0\right) \\ &= \mathbb{P}\left(\operatorname{argmin}_{t \in \mathbb{R}} \{\mathbb{B}(t) + t^2\} \geq -\frac{z}{2b} \left(\frac{b}{a}\right)^{2/3}\right) = \mathbb{P}\left(\operatorname{argmax}_{t \in \mathbb{R}} \{\mathbb{B}(t) - t^2\} \leq \frac{z}{2b} \left(\frac{b}{a}\right)^{2/3}\right) \end{aligned}$$

Thus we can conclude that for $C(\alpha_0) = 2a^{2/3}b^{1/3}$,

$$n^{\frac{1}{3}}(\widehat{Q}_v(\alpha_0) - Q_v(\alpha_0)) \xrightarrow{d} C(\alpha_0) \operatorname{argmax}_{t \in \mathbb{R}} \{\mathbb{B}(t) - t^2\}.$$

A.2. Proof of Theorem 2. For notation simplicity, let $K_h(\cdot) = (1/h)K(\cdot/h)$. Then

$$\begin{aligned} \hat{q}_v(\alpha) &= \int K_h(\alpha - u) d\widehat{V}(u) = \int K_h(\alpha - u) dV_n(u) + \int K_h(\alpha - u) d(\widehat{V} - V_n)(u) \\ &= \int K_h(\alpha - u) dV_n(u) + \frac{1}{h} \int K'_h(\alpha - u)(\widehat{V}(u) - V_n(u)) du \\ &= \int K_h(\alpha - u) dV_n(u) + \frac{1}{h} \int K'_h(t)(\widehat{V}(\alpha + ht) - V_n(\alpha + ht)) dt \\ &= \int K_h(\alpha - u) dV_n(u) + O_p((n/\log n)^{-2/3}/h) \quad (6) \end{aligned}$$

where the third inequality holds by integration by parts, and the last equality holds because the sup distance from V_n to its g.c.m. $\hat{V}$ is of order $O_p((n/\log n)^{-2/3})$ by Lemma 8 in the Supplement Material (Luo and Wan, 2016). Therefore it is then sufficient to focus on the first right hand side term. Since $Q_{v,n}$ is piecewise flat and is left-continuous, we have

$$\begin{aligned} \int K_h(\alpha - u) dV_n(u) - Q_v(\alpha) &= \int K_h(\alpha - u) Q_{v,n}(u) du - Q_v(\alpha) \\ &= \underbrace{\sum_{i=1}^n b_{(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u) du}_{A_n(\alpha)} - Q_b(\alpha) + \underbrace{\frac{1}{I-1} \left(\sum_{i=1}^n (i-1)(b_{(i)} - b_{(i-1)}) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u) du - \frac{\alpha}{g(Q_b(\alpha))} \right)}_{B_n(\alpha)}. \end{aligned}$$

$A_n(\alpha)$ is the standard smooth quantile estimator. Yang (1985, Theorem 1) shows that when $nh^5 \rightarrow c$, $\sqrt{nh}A_n(\alpha) \xrightarrow{P} \sqrt{c}Q_b''(\alpha) \int u^2 K(u) du = -\frac{\sqrt{c}}{3} Q_b''(\alpha) \int u^3 K'(u) du$, and when $nh^5 \rightarrow 0$, $\sqrt{nh}A_n(\alpha) \xrightarrow{P} 0$.

It remains to consider the B_n part. Define $\tilde{B}_n(\alpha)$ as

$$\tilde{B}_n(\alpha) = \sum_{i=1}^n \alpha n(b_{(i)} - b_{(i-1)}) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u) du - \frac{\alpha}{g(Q_b(\alpha))}.$$

Note first when n is large,

$$\begin{aligned} n \sum_{i=1}^n (b_{(i)} - b_{(i-1)}) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u) du &= n \sum_{i=1}^{n-1} b_{(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u) du \\ &\quad - n \sum_{i=1}^{n-1} b_{(i)} \int_{\frac{i}{n}}^{\frac{i+1}{n}} K_h(\alpha - u) du + nb_{(n)} \int_{\frac{n-1}{n}}^1 K_h(\alpha - u) du - nb_{(0)} \int_0^{1/n} K_h(\alpha - u) du \\ &\approx n \sum_{i=1}^{n-1} b_{(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u) du - n \sum_{i=1}^{n-1} b_{(i)} \int_{\frac{i}{n}}^{\frac{i+1}{n}} K_h(\alpha - u) du. \end{aligned}$$

The last equality holds because under Assumption 4, when n is large, $K_h(t) = 0$ for any $t \neq 0$.

Recall that $K_h(\cdot) = (1/h)K(\cdot/h)$, we know that

$$\begin{aligned} \tilde{B}_n(\alpha) &= \alpha n \sum_{i=1}^{n-1} b_{(i)} \left\{ \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(\alpha - u) du - \int_{\frac{i}{n}}^{\frac{i+1}{n}} K_h(\alpha - u) du \right\} \\ &= \frac{\alpha}{h^2} \sum_{i=1}^{n-1} b_{(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} K' \left(\frac{u - \alpha}{h} \right) du + O_p(1/n). \end{aligned}$$

By [Welsh \(1988, main theorem, part \(ii\)\)](#), under Assumptions [3](#) and [4](#) and $nh^5 \rightarrow c$, $\sqrt{nh}(\tilde{B}_n(\alpha) - \frac{\alpha}{g(Q_b(\alpha))}) \xrightarrow{d} N(\mathcal{B}, \mathcal{V})$, where

$$\mathcal{B} = -\frac{c^2\alpha}{6(I-1)}Q_b'''(\alpha) \int u^3 K'(u) du \quad \mathcal{V} = \frac{\alpha^2}{c(I-1)^2} (Q_b'(\alpha))^2 \int K^2(u) du.$$

When $h = cn^{-r}$ for some $\frac{1}{5} < r < \frac{1}{2}$, $\sqrt{nh^{1-r}}(\tilde{B}_n(\alpha) - Q_v(\alpha)) \xrightarrow{d} N(0, \mathcal{V})$.

Lastly, let $z_{(i)} = n(b_{(i)} - b_{(i-1)})$ and $w_i = ((i-1)/n - \alpha) \int_{\frac{i-1}{n}}^{\frac{i}{n}} K_h(u - \alpha) du$. Observe that $B_n - \tilde{B}_n = \sum_i z_{(i)} w_i$. By Lemma 9 in the Supplement Material ([Luo and Wan, 2016](#)), $B_n - \tilde{B}_n$ is of order $o_p(1/\sqrt{nh})$. Therefore we can conclude that $\sqrt{nh}B_n(\alpha) \xrightarrow{d} N(\mathcal{B}, \mathcal{V})$.

APPENDIX B. COMPUTE THE GREATEST CONVEX MINORANT OF $V_n(\cdot)$

We now describe how to compute the greatest convex minorant of $V_n(\cdot)$.

First, consider the coordinate vectors of the piecewise linear function $V_n(\cdot)$: $\{(0,0), (1/n, V_n(1/n)), \dots, (1, V_n(1))\}$. We find the smallest slope of each $(j/n, V_n(j/n))$ with respect to the origin, which defines the first partition on the g.c.m.. Let $j_1 = \operatorname{argmin}_{j \in \{1, \dots, n\}} \frac{V_n(j/n) - 0}{j/n - 0}$. The first partition is the line segment connecting $(0,0)$ and $(j_1, V_n(j_1/n))$.

Second, we find the next smallest slope, after removing the first partition from further consideration. In particular, consider the coordinate vectors $\{(j_1, V_n(j_1/n)), \dots, (1, V_n(1))\}$. Let $j_2 = \operatorname{argmin}_{j \in \{j_1+1, \dots, n\}} \frac{V_n(j/n) - V_n(j_1/n)}{j/n - j_1/n}$. The second partition is the line segment connecting $(j_1, V_n(j_1/n))$ and $(j_2, V_n(j_2/n))$.

We continue in this manner until we reach the end of the points $(1, V_n(1))$. The resulting coordinate vectors $\{(0,0), (j_1, V_n(j_1/n)), (j_2, V_n(j_2/n)), \dots, (1, V_n(1))\}$ define the greatest convex minorant of $V_n(\cdot)$, which is also piecewise linear.