Technical Appendix:

Inference for Time Series Regression Models With Weakly Dependent and Heteroscedastic Errors

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Theorems 3.1 and 3.2 are proved in this technical appendix. We first introduce some notations. Denote by P^* , E^* , var^* the probability, expectation, and variance, respectively, conditional on data $\mathcal{X}_n$. Let $\stackrel{\mathcal{D}}{=}$ indicate the equality in distribution. For a vector $x = (x_1, \dots, x_p)'$, let $\|x\| = (\sum_{i=1}^p x_i^2)^{1/2}$ be the Euclidian norm. If A is a $p \times p$ symmetric nonnegative definite matrix with $A = P\Lambda P'$, $PP' = I_p$, where I_p is the $p \times p$ identity matrix, and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_p)$ is a diagonal matrix with $\lambda_1 \geq \dots \geq \lambda_p \geq 0$, we define the root of A as $A^{1/2} = P\Lambda^{1/2}P'$, where $\Lambda^{1/2} = \text{diag}(\lambda_1^{1/2}, \dots, \lambda_p^{1/2})$. We write $Y_n = o_p^*(1)$ if for any $\epsilon > 0$, $P^*\{|Y_n| > \epsilon\} \rightarrow 0$ in probability, and $Y_n = O_p^*(1)$ if there exists a constant $M > 0$ such that for all large n , $P^*\{|Y_n| > M\} < \epsilon$ with probability arbitrarily close to one, as defined in Chang and Park (2003). Define $Q_N(r) = \lfloor Nr \rfloor^{-1} \sum_{t=1}^{\lfloor Nr \rfloor + p - 1} F(t/n)F(t/n)'$ and $B_{N,F}(r) = N^{-1/2} \sum_{t=1}^{\lfloor Nr \rfloor + p - 1} F_{t,n}u_{t,n}$. Define $\bar{\omega} = \max\{\omega^*, \sup_{r \in [0,1]} \omega(r)\}$ and $\bar{\nu} = \sup_{r \in [0,1]} \nu(r)$. Notice that since $\omega(r)$ and $\nu(r)$ have only finitely many breaks on $[0,1]$ interval, $\bar{\omega} < \infty$ by (A1) (i) and $\bar{\nu} < \infty$. In the following argument, all $\sup_r$ indicates supremum over $\{r \in [\epsilon, 1]\}$. The positive constant C is generic and may vary from place to place. We use $\mathcal{I}_j$ or $\mathcal{I}_{N,j}$ for $j = 1, 2, \dots$ in different proofs to indicate different objects in a simple way.

The following lemmas contain some key results needed in the proofs of Theorems 3.1 and 3.2 and they may be of independent interest.

LEMMA **0.1**. Let $A_p = (a_{p,ij})_{i,j \leq p}$ and $B_p = (b_{p,ij})_{i,j \leq p}$ be $p \times p$ matrices with $p = 1, 2, \dots$ and let $C_p = \max_{i,j} \{a_{p,ij}^2, b_{p,ij}^2\}$. Then

$$\{\det(A_p) - \det(B_p)\}^2 \leq (2p^3 C_p)^{p-1} \|A_p - B_p\|_F^2. \quad (1)$$

Proof of Lemma 0.1. Let $A_{p,ij}$ be the $(p-1) \times (p-1)$ matrix of A_p with i th row and j th column deleted. If $p = 1$, it is trivial. If (1) holds for p , then

$$\begin{aligned} |\det(A_{p+1}) - \det(B_{p+1})|^2 &= \left| \sum_{j=1}^{p+1} (-1)^{j+1} \{a_{p+1,1j} \det(A_{p+1,1j}) - b_{p+1,1j} \det(B_{p+1,1j})\} \right|^2 \\ &\leq \sum_{j=1}^{p+1} a_{p+1,1j}^2 \{\det(A_{p+1,1j}) - \det(B_{p+1,1j})\}^2 + \sum_{j=1}^{p+1} (a_{p+1,1j} - b_{p+1,1j})^2 \{\det(B_{p+1,1j})\}^2 \\ &\leq (2p^3 C_{p+1})^{p-1} \{C_{p+1} \sum_{j=1}^{p+1} \|A_{p+1,1j} - B_{p+1,1j}\|_F^2 + \|A_{p+1} - B_{p+1}\|_F^2 \sum_{j=1}^{p+1} \|B_{p+1,1j}\|_F^2\} \\ &\leq (2p^3 C_{p+1})^{p-1} \{C_{p+1}(p+1) \|A_{p+1} - B_{p+1}\|_F^2 + \|A_{p+1} - B_{p+1}\|_F^2 (p+1) p^2 C_{p+1}\} \\ &\leq \{2(p+1)^3 C_{p+1}\}^p \|A_{p+1} - B_{p+1}\|_F^2. \end{aligned}$$

By induction, (1) holds for all p . ◇

LEMMA **0.2**. Assume (R1)-(R2).

$$(i) \sup_{r \in [\epsilon, 1]} \|Q_N(r) - Q_r\|_F = O(n^{-1})$$

$$(ii) \sup_{r \in [\epsilon, 1]} \|Q_r^{-1}\|_F < \infty \text{ and } \sup_{r \in [\epsilon, 1]} \|Q_N(r)^{-1}\|_F < \infty \text{ for all } N.$$

$$(iii) \sup_{r \in [\epsilon, 1]} \|Q_N(r)^{-1} - Q_r^{-1}\|_F = O(n^{-1})$$

Proof of Lemma 0.2. (i) Our goal is to show that, for all $i, j = 1, \dots, p$,

$$\sup_r \left| r^{-1} \int_0^{\lfloor nr \rfloor / n} f_i(s) f_j(s) ds - (nr)^{-1} \sum_{t=1}^{\lfloor nr \rfloor} f_i(t/n) f_j(t/n) \right| = O(n^{-1}). \quad (2)$$

Once (2) is shown, we have $\sup_{r \in [\epsilon, 1]} \|Q_N(r) - Q_r\|_F \leq \left\{ \sum_{i,j \leq p} (C n^{-1})^2 \right\}^{1/2} = p C n^{-1} = O(n^{-1})$ so that the part (i) of this lemma is complete.

Now we prove (2). For each $i = 1, \dots, p$ and $r \in [\epsilon, 1]$, define $\mathcal{S}_{n,i,r} = \{s : s = t/n \text{ for some } t = 1, \dots, \lfloor nr \rfloor, \text{ or } s \text{ is a break point on } s \in [0, \lfloor nr \rfloor / n]\}$. Let

$\{s_k\}_{k=1,\dots,m_{i,r}}$ denote the elements of $\mathcal{S}_{n,i,r}$ in ascending order, where $m_{i,r}$ is the number of elements in $\mathcal{S}_{n,i,r}$. Let $s_0 = 0$. For each i , define a step function $f_{n,i}(s)$ on $s \in [0, 1]$ such that $f_{n,i}(s) = f_i(s_k)$ for $s \in [s_{k-1}, s_k)$, $k = 1, \dots, m_{i,1}$, and $f_{n,i}(1) = f_i(1)$. Then this step function is very close to $f_i(s)$ in the sense that $\sup_{s \in [0,1]} |f_i(s) - f_{n,i}(s)| \leq Cn^{-1}$, due to the piecewise Lipschitz continuity of $f_i(s)$. Then, for all $i, j = 1, \dots, p$, $|f_i(s)f_j(s) - f_{n,i}(s)f_{n,j}(s)| \leq |f_i(s)||f_j(s) - f_{n,j}(s)| + |f_i(s) - f_{n,i}(s)||f_{n,j}(s)|$ and the boundedness of $f_i(s)$ and $f_{n,j}(s)$ imply that

$$\sup_{r \in [\epsilon, 1]} \left| r^{-1} \int_0^{\lfloor nr \rfloor / n} f_i(s)f_j(s)ds - r^{-1} \int_0^{\lfloor nr \rfloor / n} f_{n,i}(s)f_{n,j}(s)ds \right| = O(n^{-1}). \quad (3)$$

Also, notice that $\sup_r |\sum_{k=1}^{m_{i,r}} f_{n,i}(s_k) - \sum_{t=1}^{\lfloor nr \rfloor} f_i(t/n)| = O(1)$, since the number of breaks of $f_i(s)$ is finite. Similarly, for all $i, j = 1, \dots, p$, $\sup_r |\sum_{k=1}^{m_{i,j,r}} f_{n,i}(s_k)f_{n,j}(s_k) - \sum_{t=1}^{\lfloor nr \rfloor} f_i(t/n)f_j(t/n)| = O(1)$, where $m_{i,j,r}$ is the number of elements in $\mathcal{S}_{n,i,r} \cup \mathcal{S}_{n,j,r}$. Thus we have

$$\sup_{r \in [\epsilon, 1]} \left| (nr)^{-1} \sum_{k=1}^{m_{i,j,r}} f_{n,i}(s_k)f_{n,j}(s_k) - (nr)^{-1} \sum_{t=1}^{\lfloor nr \rfloor} f_i(t/n)f_j(t/n) \right| = O(n^{-1}). \quad (4)$$

Since $r^{-1} \int_0^{\lfloor nr \rfloor / n} f_{n,i}(s)f_{n,j}(s)ds = (nr)^{-1} \sum_{k=1}^{m_{i,j,r}} f_{n,i}(s_k)f_{n,j}(s_k)$, by the triangular inequality, (3) and (4) imply (2). This completes the proof.

(ii) Let $Q_{r,ij}$ be the $(p-1) \times (p-1)$ matrix is a submatrix of Q_r with i th row and j th column deleted. The (i, j) cofactor of Q_r is defined as $\text{cof}(Q_r, i, j) = (-1)^{i+j} \det(Q_{r,ij})$, which is a function of $(p-1)$ product of $\{r^{-1} \int_0^r f_i(s)ds\}_{1 \leq i \leq p}$. Thus we can write $\text{cof}(Q_r, i, j) = r^{-(p-1)} \int_0^r \dots \int_0^r h_{i,j} \{F(s_1), \dots, F(s_{p-1})\} ds_1 \dots ds_{p-1}$ for a piecewise continuous function $h_{i,j}$ with at most finite breaks. Let $\bar{h}_{i,j} = \sup_{s_1, \dots, s_{p-1}} h_{i,j} \{F(s_1), \dots, F(s_{p-1})\}$ and $\bar{h} = \max_{i,j} \bar{h}_{i,j}$. The (i, j) th element of Q_r^{-1} can be written as

$$\{\det(Q_r)\}^{-1} r^{-(p-1)} \int_0^r \dots \int_0^r h_{i,j} \{F(s_1), \dots, F(s_{p-1})\} ds_1 \dots ds_{p-1},$$

using the same $h_{i,j}$ above, and since $\bar{h}$ is bounded due to the piecewise Lipschitz conti-

nity of f_i 's, we have $\|Q_r^{-1}\|_F \leq \{\det(Q_r)\}^{-1}\bar{h}p$ so that $\sup_r \|Q_r^{-1}\|_F \leq \bar{h}p \sup_{r \in [\epsilon, 1]} \{\det(Q_r)\}^{-1}$, which is bounded due to (R1). Similarly, the (i, j) th element of $Q_N(r)^{-1}$ can be written as

$$[\det\{Q_N(r)\}]^{-1} [Nr]^{-(p-1)} \sum_{t_1, \dots, t_{p-1}}^{[Nr]+p-1} h_{i,j} \{F(t_1/n), \dots, F(t_{p-1}/n)\}$$

so that $\|Q_N(r)^{-1}\|_F \leq [\det\{Q_N(r)\}]^{-1}\bar{h}p$.

Now we need to show $\sup_r [\det\{Q_N(r)\}]^{-1} < \infty$. By Lemma 0.1, we have $|\det\{Q_N(r)\} - \det(Q_r)| \leq C\|Q_N(r) - Q_r\|_F$, where C is bounded if all elements of $Q_N(r)$ and Q_r are bounded, which follow from (R1) and Lemma 0.2 (i). In fact, the inequality holds uniformly over $r \in [\epsilon, 1]$, thus we have $\sup_r |\det\{Q_N(r)\} - \det(Q_r)| = o(1)$ by Lemma 0.2 (i). It follows from (R1) and Lemma 0.2 (i) that $\sup_r |[\det\{Q_N(r)\}]^{-1} - \{\det(Q_r)\}^{-1}| \leq |\det\{Q_N(r)\} - \det(Q_r)| |\det\{Q_N(r)\}|^{-1} |\det(Q_r)|^{-1} = o(1)$. Thus $\sup_r [\det\{Q_N(r)\}]^{-1} \leq \sup_r \{\det(Q_r)\}^{-1} + 1 < \infty$ for $N \geq N_0$ for some N_0 .

(iii) Note that for any $p \times p$ matrices A and B , we have $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$. Due to the sub-multiplicative property of the Frobenius norm, we have $\|Q_N(r)^{-1} - Q_r^{-1}\|_F \leq \|Q_N(r)^{-1}\|_F \|Q_N(r) - Q_r\|_F \|Q_r^{-1}\|_F$. By Lemma 0.2 (i) and (ii), $\sup_r \|Q_N(r)^{-1} - Q_r^{-1}\|_F \leq Cn^{-1}$, which completes the proof. $\diamond$

REMARK 0.1. Notice that the proof of Lemma 0.2 (i) only works for $\lfloor nr \rfloor \geq 1$, which means this proof does not go through in general if we work on $r \in [0, 1]$. For some special cases, e.g., $p = 1$ and $f_1(s) = c$ with constant c , then the uniformness can be relaxed to be over $[0, 1]$.

In the following proof of the invariance principle of the error process when the linear process (A1) is assumed, we use the BN decomposition (Beveridge and Nelson, 1981) technique introduced in Phillips and Solo (1992). The BN decomposition is

$$\mathbf{C}(L) = \mathbf{C}(1) - (1 - L)\tilde{\mathbf{C}}(L),$$

where $\tilde{\mathbf{C}}(L) = \sum_{j=0}^{\infty} \tilde{c}_j L^j$, $\tilde{c}_j = \sum_{k=j+1}^{\infty} c_k$. Then the original error process can be decomposed into

$$u_{t,n} = \mathbf{C}(L)\varepsilon_{t,n} = \mathbf{C}(1)\varepsilon_{t,n} + \tilde{\varepsilon}_{t-1,n} - \tilde{\varepsilon}_{t,n}, \quad (5)$$

where $\tilde{\varepsilon}_{t,n} = \tilde{\mathbf{C}}(L)\varepsilon_{t,n} = \sum_{j=0}^{\infty} \tilde{c}_j \varepsilon_{t-j,n}$.

LEMMA 0.3. *Recall that $B_{N,F}(r) = N^{-1/2} \sum_{t=1}^{\lfloor Nr \rfloor + p-1} F_{t,n} u_{t,n}$. Assume (R1)-(R2). Then there exist iid standard normal random variables $V_1, \dots, V_n$ and a centered Gaussian process $\bar{B}_{N,F}$ such that $\{\bar{B}_{N,F}(\lfloor N\epsilon \rfloor / N), \dots, \bar{B}_{N,F}(N/N)\}$ have the same distribution as $\{B_{N,F}(\lfloor N\epsilon \rfloor / N), \dots, B_{N,F}(N/N)\}$, and*

$$\sup_{r \in [\epsilon, 1]} \left\| \bar{B}_{N,F}(r) - \hat{B}_{N,F}(r) \right\| = o_p(1),$$

where $\hat{B}_{N,F}(r) = \mathbf{C}(1)N^{-1/2} \sum_{t=1}^{\lfloor Nr \rfloor + p-1} F_{t,n} \omega_{t,n} V_t$ if (A1) and $\hat{B}_{N,F}(r) = \Gamma N^{-1/2} \sum_{t=1}^{\lfloor Nr \rfloor + p-1} F_{t,n} \nu_{t,n} V_t$ if (A2).

Proof of Lemma 0.3. We first prove for (A1). If $p = 1$ and $F(\cdot) \equiv 1$, from the BN decomposition (5), we have $\sum_{t=1}^{\lfloor nr \rfloor} u_{t,n} = C(1) \sum_{t=1}^{\lfloor nr \rfloor} \varepsilon_{t,n} + \tilde{\varepsilon}_{0,n} - \tilde{\varepsilon}_{\lfloor nr \rfloor, n}$ so that

$$\sup_{r \in [\epsilon, 1]} \left| N^{-1/2} \sum_{t=1}^{\lfloor Nr \rfloor + p-1} u_{t,n} - C(1)N^{-1/2} \sum_{t=1}^{\lfloor Nr \rfloor + p-1} \omega_{t,n} e_t \right| \leq \frac{|\tilde{\varepsilon}_{0,n}|}{N^{1/2}} + \frac{\sup_{r \in [\epsilon, 1]} |\tilde{\varepsilon}_{\lfloor Nr \rfloor + p-1, n}|}{N^{1/2}}. \quad (6)$$

Our goal is to show that the left-hand side of (6) is $o_p(1)$, which holds if

$$\tilde{\varepsilon}_{0,n}^2 = O_p(1) \quad (7)$$

and

$$\max_{1 \leq k \leq n} n^{-1/2} |\tilde{\varepsilon}_{k,n}| \xrightarrow{P} 0. \quad (8)$$

Notice that $\sum_{j=0}^{\infty} |\tilde{c}_j|^\kappa < \infty$ for any $\kappa \geq 1$ by (A1) (ii) and Lemma 2.1 of Phillips and Solo (1992). Since $E(\tilde{\varepsilon}_{0,n}^2) = \sum_{j=0}^{\infty} \tilde{c}_j^2 \omega_{-j,n}^2 \leq \bar{\omega}^2 \sum_{j=0}^{\infty} \tilde{c}_j^2 < \infty$, (7) holds using the Markov inequality. For (8), by Minkowski's inequality, $\sup_{1 \leq k \leq n} E|\tilde{\varepsilon}_{k,n}|^4 \leq \sup_{1 \leq k \leq n} (\sum_{j=0}^{\infty} \|\tilde{c}_j \omega_{k-j,n} e_{k-j}\|_4)^4 \leq C\bar{\omega}^4 (\sum_{j=0}^{\infty} |\tilde{c}_j|)^4 < \infty$. Then for any $M > 0$,

$$\begin{aligned} P(\max_{1 \leq k \leq n} |\tilde{\varepsilon}_{k,n}| > n^{1/2} M) &= P(\bigcup_{k=1}^n \{|\tilde{\varepsilon}_{k,n}| > n^{1/2} M\}) \leq \sum_{k=1}^n P(|\tilde{\varepsilon}_{k,n}| > n^{1/2} M) \\ &\leq (n^{1/2} M)^{-4} \sum_{k=1}^n E|\tilde{\varepsilon}_{k,n}|^4 = O(n^{-1}) \end{aligned}$$

by Markov inequality, and (8) follows.

Also, since e_t are iid $(0,1)$ with finite fourth moment, there exist iid $N(0,1)$ random variables $\{V_1, \dots, V_n\}$ on a richer probability space such that for any $M > 0$,

$$P \left\{ \max_{1 \leq k \leq n} \left| \sum_{t=1}^k \omega_{t,n} e_t - \sum_{t=1}^k \omega_{t,n} V_t \right| \geq n^{1/2} M \right\} \leq C(n^{1/2} M)^{-4} \sum_{t=1}^n E|e_t|^4 = O(n^{-1}) \quad (9)$$

using, for example, Lemma 2 of Csörgő et al. (2003). Then by (6), (7), (8), and (9), the result follows for (A1) with $p = 1$ and $F(\cdot) \equiv 1$.

For a general p and $F(\cdot)$, notice that $f_j(\cdot)$ is piecewise Lipschitz continuous for each $j = 1, \dots, p$. If $\omega_{t,n}$ is replaced with $\omega_{t,n} f_j(t/n)$, the same proof goes through for all $j = 1, \dots, p$, and we have

$$\sup_r \|B_{N,F}(r) - C(1)N^{-1/2} \sum_{t=1}^{\lfloor Nr \rfloor + p-1} F(t/n) \omega_{t,n} V_t\| = o_p(1).$$

For (A2), if $p = 1$ and $F(\cdot) \equiv 1$, our model is a special case of Zhou (2013)'s setting and the result follows by Proposition 5 of Zhou (2013). For a general p and $F(\cdot)$, the same proof works for $\nu_{t,n} f_j(t/n)$, in place of $\nu_{t,n}$, for all $j = 1, \dots, p$, which completes the proof. $\diamond$

Proof of Theorem 3.1. We first prove for (A1). Define a step function $\omega_n(s) = \omega(t/n)$ for $s \in [t/n, (t+1)/n)$ for each $t = 0, \dots, n-1$ if $\omega(s)$ has no break in the interval. If $\omega(s)$ has k break points $s_1, \dots, s_k$ in an interval $(t/n, (t+1)/n)$, i.e., $s_0 := t/n < s_1 < \dots < s_k < s_{k+1} := (t+1)/n$, then define $\omega_n(s) = \omega(s_i)$ for $s \in [s_i, s_{i+1}]$ and $i = 0, \dots, k$. Let $\omega_n(1) = \omega(1)$. Similarly, for all $i = 1, \dots, p$, define step functions $f_{n,i}(s)$ from $f_i(s)$, and let $F_n(s) = \{f_{n,1}(s), \dots, f_{n,p}(s)\}'$. Let $\check{B}_{N,F,\mathbf{C},\omega}(r) = \mathbf{C}(1) \int_0^{(\lfloor Nr \rfloor + p-1)/n} F_n(s) \omega_n(s) dB(s)$, $\bar{B}_{N,F,\mathbf{C},\omega}(r) = \mathbf{C}(1) \int_0^r F_n(s) \omega_n(s) dB(s)$, and $B_{F,\mathbf{C},\omega}(r) = \mathbf{C}(1) \int_0^r F(s) \omega(s) dB(s)$.

Due to the triangular inequality of the sup norm, $\sup_r \|\check{B}_{N,F,\mathbf{C},\omega}(r) - B_{F,\mathbf{C},\omega}(r)\| \leq \sup_r \|\check{B}_{N,F,\mathbf{C},\omega}(r) - \bar{B}_{N,F,\mathbf{C},\omega}(r)\| + \sup_r \|\bar{B}_{N,F,\mathbf{C},\omega}(r) - B_{F,\mathbf{C},\omega}(r)\| =: \mathcal{I}_1 + \mathcal{I}_2$. We have

$\mathcal{I}_1 = \sup_r \|\check{B}_{N,F,\mathbf{C},\omega}(r) - \bar{B}_{N,F,\mathbf{C},\omega}(r)\| = o_p(1)$ because $\sup_r |r - (\lfloor Nr \rfloor + p - 1)/n| \leq p/n$ and $\sup_r \|\int_{(\lfloor Nr \rfloor + p - 1)/n}^r F_n(s)\omega_n(s)dB(s)\| \leq C(p+1)\sup_{t=1,\dots,n}|B(t/n) - B\{(t-1)/n\}| = o_p(1)$. Due to the way $\omega_n(s)$ is defined in the beginning of this proof, we have $\sup_r |\omega_n(r) - \omega(r)| = o(1)$ so that $\sup_r \|F_n(r)\omega_n(r) - F(r)\omega(r)\| = o(1)$ is implied by (R1). Therefore, $\mathcal{I}_2 = \sup_r \|\bar{B}_{N,F,\omega}(r) - B_{F,\omega}(r)\| = o_p(1)$ holds by Proposition 5.19 of Kurtz (2001). Then we have

$$\sup_{r \in [\epsilon, 1]} \|\check{B}_{N,F,\mathbf{C},\omega}(r) - B_{F,\mathbf{C},\omega}(r)\| = o_p(1). \quad (10)$$

Note that $\sup_{s \in [0, 1]} |f_i(s)\omega(s)| < C$ for some constant C which does not depend on i so that we have $\sup_r \|B_{F,\mathbf{C},\omega}(r)\|^2 \leq \mathbf{C}(1)^2 \sum_{i=1}^p \sup_r \left\{ \int_0^r f_i(s)\omega(s)dB(s) \right\}^2 = O_p(1)$. By the triangular inequality of the Euclidean norm and the sub-multiplicative property of the Frobenius norm, we have $\|Q_N(r)^{-1}\check{B}_{N,F,\mathbf{C},\omega}(r) - Q_r^{-1}B_{F,\mathbf{C},\omega}(r)\| \leq \|Q_N(r)^{-1} - Q_r^{-1}\|_F \|B_{F,\mathbf{C},\omega}(r)\| + \|Q_N(r)^{-1}\|_F \|\check{B}_{N,F,\mathbf{C},\omega}(r) - B_{F,\mathbf{C},\omega}(r)\|$. Thus by (10) and Lemma 0.2 (ii) and (iii), we have

$$\sup_{r \in [\epsilon, 1]} \|Q_N(r)^{-1}\check{B}_{N,F,\mathbf{C},\omega}(r) - Q_r^{-1}B_{F,\mathbf{C},\omega}(r)\| = o_p(1). \quad (11)$$

From Lemma 0.3, there exist iid standard normal random variables $V_1, \dots, V_n$ and a centered Gaussian process $\bar{B}_{N,F}$ such that $\{\bar{B}_{N,F}(\lfloor N\epsilon \rfloor/N), \dots, \bar{B}_{N,F}(N/N)\}$ have the same distribution as $\{B_{N,F}(\lfloor N\epsilon \rfloor/N), \dots, B_{N,F}(N/N)\}$, and $\sup_{r \in [\epsilon, 1]} \|\bar{B}_{N,F}(r) - \hat{B}_{N,F}(r)\| = o_p(1)$, where $\hat{B}_{N,F}(r) = \mathbf{C}(1)N^{-1/2} \sum_{t=1}^{\lfloor Nr \rfloor + p - 1} F_{t,n}\omega_{t,n}V_t$. Thus we have from Lemma 0.2 (ii)

$$\sup_{r \in [\epsilon, 1]} \|Q_N(r)^{-1}\bar{B}_{N,F}(r) - Q_N(r)^{-1}\hat{B}_{N,F}(r)\| = o_p(1). \quad (12)$$

Notice that

$$\{\hat{B}_{N,F}(r)\}_{r=\epsilon}^1 \stackrel{\mathcal{D}}{=} \left\{ \mathbf{C}(1) \sum_{t=1}^{\lfloor Nr \rfloor + p - 1} F(t/n)\omega(t/n) [B(t/n) - B\{(t-1)/n\}] \right\}_{r=\epsilon}^1 \stackrel{\mathcal{D}}{=} \{\check{B}_{N,F,\mathbf{C},\omega}(r)\}_{r=\epsilon}^1. \quad (13)$$

By (11) and (13), $Q_N(r)^{-1}\hat{B}_{N,F}(r) \Rightarrow Q_r^{-1}B_{F,\mathbf{C},\omega}(r)$. Then Theorem 3.1 (i) follows

from (12) and the fact that $\{Q_N(r)^{-1}B_{N,F}(r)\}_{r=\epsilon}^1 \stackrel{D}{=} \{Q_N(r)^{-1}\bar{B}_{N,F}(r)\}_{r=\epsilon}^1$. The proof is complete for (A1).

The same proof goes for (A2) by replacing $\omega(s)$ with $\nu(s)$ and $\mathbf{C}(1)$ with Γ . $\diamond$

Now we prove the bootstrap consistency of the SN quantity. The following lemmas present some key steps.

LEMMA 0.4. *If $\{u_{t,n}\}$ are generated from (A1),*

$$\sup_{r \in [\epsilon, 1]} \left| n^{-1} \sum_{t=1}^{\lfloor nr \rfloor} u_{t,n}^2 - \mathbf{D}(1) \int_0^r \omega^2(s) ds \right| = o_p(1),$$

where $\mathbf{D}(1) = \sum_{j=0}^{\infty} c_j^2$,

Proof of Lemma 0.4. We define some notations first. Let $\mathbf{D}_j(L) = \sum_{k=0}^{\infty} c_k c_{k+j} L^k = \sum_{k=0}^{\infty} d_{jk} L^k$ for $j = 0, 1, \dots$, where $d_{jk} = c_k c_{k+j}$. Define $\tilde{\mathbf{D}}_j(L)$ from $\mathbf{D}_j(L)$ for each j in the same way as we define $\tilde{\mathbf{C}}(L)$ from $\mathbf{C}(L)$, i.e., $\tilde{\mathbf{D}}_j(L) = \sum_{k=0}^{\infty} \tilde{d}_{jk} L^k$ and $\tilde{d}_{jk} = \sum_{s=k+1}^{\infty} d_{js} = \sum_{s=k+1}^{\infty} c_s c_{s+j}$. Denote by $\mathbf{D}(L) = \mathbf{D}_0(L)$, which is consistent with the above definition for $\mathbf{D}(1)$. Let $\varepsilon_{t-1,n}^D = \sum_{j=1}^{\infty} \mathbf{D}_j(1) \varepsilon_{t-j,n}$, $\tilde{u}_{at,n} = \tilde{\mathbf{D}}_0(L) \varepsilon_{t,n}^2$, and $\tilde{u}_{bt,n} = \sum_{j=1}^{\infty} \tilde{\mathbf{D}}_j(L) \varepsilon_{t,n} \varepsilon_{t-j,n}$. Using the BN decomposition argument of (Phillips and Solo, 1992, pp. 978–9), we can write

$$u_{t,n}^2 = \mathbf{D}(1) \varepsilon_{t,n}^2 - (1-L) \tilde{u}_{at,n} + 2 \varepsilon_{t,n} \varepsilon_{t-1,n}^D - 2(1-L) \tilde{u}_{bt,n}$$

and $n^{-1} \sum_{t=1}^{\lfloor nr \rfloor} u_{t,n}^2 = \mathbf{D}(1) n^{-1} \sum_{t=1}^{\lfloor nr \rfloor} \varepsilon_{t,n}^2 - n^{-1} (\tilde{u}_{a\lfloor nr \rfloor} - \tilde{u}_{a0}) + 2n^{-1} \sum_{t=1}^{\lfloor nr \rfloor} \varepsilon_{t,n} \varepsilon_{t-1,n}^D - 2n^{-1} (\tilde{u}_{b\lfloor nr \rfloor} - \tilde{u}_{b0}) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4$. Our goal is to show that

$$\sup_{r \in [\epsilon, 1]} |\mathcal{I}_1 - \mathbf{D}(1) \int_0^r \omega^2(s) ds| = o_p(1) \quad (14)$$

and

$$\sup_{r \in [\epsilon, 1]} |\mathcal{I}_j| = o_p(1), \quad j = 2, 3, 4. \quad (15)$$

To prove (14), we need to show

$$\sup_{r \in [\epsilon, 1]} |n^{-1} \sum_{t=1}^{\lfloor nr \rfloor} (\varepsilon_{t,n}^2 - \omega_{t,n}^2)| = o_p(1).$$

Since $\{\varepsilon_{t,n}\}_{t=1}^n$ are independent, using Kolmogorov's inequality, $P(\max_{1 \leq k \leq n} |\sum_{t=1}^k (\varepsilon_{t,n}^2 - \omega_{t,n}^2)| > nM) \leq (nM)^{-2} \sum_{k=1}^n \text{var}(\varepsilon_{k,n}^2 - \omega_{k,n}^2) = (nM)^{-2} \sum_{k=1}^n \omega_{k,n}^4 (Ee_k^4 - 1) = O(n^{-1})$. Thus (14) holds.

Now we show (15). For $j = 2$, observe that $E|\tilde{\mathbf{D}}_0(L)\varepsilon_{t,n}^2|^2 = E|\sum_{k=0}^{\infty} \tilde{d}_{0k}\varepsilon_{t-k,n}^2|^2 = \sum_{k=0}^{\infty} \tilde{d}_{0k}^2 E\varepsilon_{t-k,n}^4 + \sum_{k \neq k'} \tilde{d}_{0k} \tilde{d}_{0k'} E\varepsilon_{t-k,n}^2 E\varepsilon_{t-k',n}^2 \leq C\{\sum_{k=0}^{\infty} \tilde{d}_{0k}^2 + (\sum_{k=0}^{\infty} \tilde{d}_{0k})^2\} < \infty$, where the last inequality is due to (A1) (ii) and Lemma 2.1 of Phillips and Solo (1992). Then for any $M > 0$, $P(\sup_{r \in [\epsilon, 1]} |n^{-1} \tilde{u}_{a[\lfloor nr \rfloor, n]}| > M) \leq P(\max_{1 \leq t \leq n} |\tilde{\mathbf{D}}_0(L)\varepsilon_{t,n}^2| > nM) \leq \sum_{t=1}^n P(|\tilde{\mathbf{D}}_0(L)\varepsilon_{t,n}^2| > nM) \leq (nM)^{-2} \sum_{t=1}^n E|\tilde{\mathbf{D}}_0(L)\varepsilon_{t,n}^2|^2 = O(n^{-1})$, which proves (15) for $j = 2$.

For $j = 3$, notice that $\{\varepsilon_{t,n}\varepsilon_{t-1,n}^D\}$ is a martingale difference sequence with respect to $\{\mathcal{E}_t\}$, where $\mathcal{E}_t = \sigma(e_t, e_{t-1}, \dots)$ is the sigma field generated by $\{e_j\}_{j=-\infty}^t$, which implies that $E(\sum_{t=1}^n \varepsilon_{t,n}\varepsilon_{t-1,n}^D)^2 = \sum_{t=1}^n E(\varepsilon_{t,n}\varepsilon_{t-1,n}^D)^2 = O(n)$ because $\varepsilon_{t,n}$ and $\varepsilon_{t-1,n}^D$ are independent and have finite second moments, i.e., $E(\varepsilon_{t-1,n}^D)^2 = E(\sum_{j=1}^{\infty} D_j(1)\varepsilon_{t-j,n})^2 \leq C \sum_{j=1}^{\infty} D_j(1)^2 < \infty$ due to (A1) (ii), (iii) and Lemma 3.6 of Phillips and Solo (1992). Then by Doob's martingale inequality, for any $M > 0$, $P\{\max_k |\sum_{t=1}^k \varepsilon_{t,n}\varepsilon_{t-1,n}^D| > nM\} \leq (nM)^{-2} E(\sum_{t=1}^n \varepsilon_{t,n}\varepsilon_{t-1,n}^D)^2 = O(n^{-1})$.

For $j = 4$, $P(\sup_r |\tilde{u}_{b[\lfloor nr \rfloor, n]}| > nM) \leq (nM)^{-2} \sum_{t=1}^n E\tilde{u}_{bt,n}^2 = O(n^{-1})$ if $\sup_{1 \leq t \leq n} E\tilde{u}_{bt,n}^2 < \infty$, which holds by the same argument used in the proof of Lemma 5.9 of Phillips and Solo (1992).

Thus (15) holds and the proof is complete. $\diamond$

LEMMA 0.5. Assume (B1). Fix $r \in [\epsilon, 1]$. If $\{u_{t,n}\}$ are generated from (A1),

$$n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} W_t u_{t,n} = n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} W_t \mathbf{C}(L)\varepsilon_{t,n} \xrightarrow{\mathcal{D}} N\left(0, \mathbf{D}(1) \int_0^r \omega^2(s) ds\right)$$

in probability and if (A2),

$$n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} W_t u_{t,n} = n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} W_t \eta_t \nu_{t,n} \xrightarrow{\mathcal{D}} N \left(0, \int_0^r \nu^2(s) ds \right)$$

in probability.

Proof of Lemma 0.5. We first prove for (A1), which is complete once we show the Lindeberg condition

$$n^{-1} \sum_{t=1}^n E^* \{ W_t^2 u_{t,n}^2 \mathbf{1}(|W_t u_{t,n}| > n^{1/2} \delta) \} = o_p(1) \quad (16)$$

and the uniform consistency of the conditional variance

$$\sup_{r \in [\epsilon, 1]} \left| \text{var}^* \left(n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} W_t u_{t,n} \right) - \mathbf{D}(1) \int_0^r \omega^2(s) ds \right| = o_p(1). \quad (17)$$

Notice that $u_{t,n}^4 = O_p(1)$ since for any $t = 1, \dots, n$, by Minkowski's inequality, $E(u_{t,n}^4) = E(\sum_{j=0}^{\infty} c_j \varepsilon_{t-j,n})^4 \leq (\sum_{j=0}^{\infty} \|c_j \varepsilon_{t-j,n}\|_4)^4 \leq \bar{\omega}^4(E\varepsilon_0^4)(\sum_{j=0}^{\infty} |c_j|)^4 < \infty$, which is due to (A1) (ii) and (iii). Then $n^{-1} \sum_{t=1}^n E^* \{ W_t^2 u_{t,n}^2 \mathbf{1}(|W_t u_{t,n}| > n^{1/2} \delta) \} \leq (n\delta)^{-2} \sum_{t=1}^n E^* \{ W_t^4 u_{t,n}^4 \mathbf{1}(|W_t u_{t,n}| > n^{1/2} \delta) \} \leq (n\delta)^{-2} \sum_{t=1}^n E^*(W_t^4 u_{t,n}^4) \leq (n\delta)^{-2} C \sum_{t=1}^n u_{t,n}^4 = O_p(n^{-1})$ so that the Lindeberg condition (16) holds. The uniform consistency of the conditional variance (17) follows from Lemma 0.4 because $\text{var}^*(n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} W_t u_{t,n}) = n^{-1} \sum_{t=1}^{\lfloor nr \rfloor} u_{t,n}^2$. Thus the proof is complete for (A1).

Now we prove for (A2). Notice that $E(u_{t,n}^4) = E(\nu_{t,n}^4 \eta_t^4) \leq \bar{\nu}^4 E(\eta_t^4) = \bar{\nu}^4 E(\eta_0^4) < \infty$ holds uniformly over all t , since $\bar{\nu} = \sup_{r \in [0, 1]} \nu(s) < \infty$ and η_t is stationary with finite fourth moment. Thus by using the same argument as above, the Lindeberg condition (16) follows. Now we show the uniform consistency of the variance

$$\sup_{r \in [\epsilon, 1]} \left| \text{var}^* \left(n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} W_t \nu_{t,n} \eta_t \right) - \int_0^r \nu^2(s) ds \right| = o_p(1). \quad (18)$$

Since $\text{var}^*(n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} W_t \nu_{t,n} \eta_t) = n^{-1} \sum_{t=1}^{\lfloor nr \rfloor} \nu_{t,n}^2 \eta_t^2$, (18) follows if

$$\sup_{r \in [\epsilon, 1]} \left| n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} \nu_{t,n}^2 (\eta_t^2 - 1) \right| = o_p(1) \quad (19)$$

is shown. Notice that $\max_t \nu_{t,n}^2 \leq \bar{\nu}^2$ so we can ignore $\nu_{t,n}$ part in (19). Define $S_k = \sum_{t=1}^k (\eta_t^2 - 1)$ and $S_k^* = \max_{1 \leq t \leq k} |S_t|^2$. Let d be a smallest positive integer such that $n \leq 2^d$ and $q = 2 + \delta/2$ with $\delta > 0$ as in (A2) (ii). For some $m_1 \leq m_2$, by Minkowski's inequality, $\|\sum_{t=m_1+1}^{m_2} (\eta_t^2 - 1)\|_q \leq \sum_{t=m_1+1}^{m_2} \|(\eta_t^2 - 1)\|_q \leq C(m_2 - m_1)$ since $\|(\eta_0^2 - 1)\|_q < \infty$ due to (A2) (ii). Thus we have, by Proposition 1(i) in Wu (2007), $E(\max_{1 \leq k \leq n} |\sum_{t=1}^k (\eta_t^2 - 1)|)^q \leq \|S_{2^d}^*\|_q^q \leq \{\sum_{z=0}^d (\sum_{m=1}^{2^{d-z}} \|\sum_{t=2^z(m-1)+1}^{2^z m} (\eta_t^2 - 1)\|_q^{1/q})^q\} \leq C\{\sum_{z=0}^d (2^{d-z} 2^z)^{1/q}\}^q = (d+1)^q 2^d = O\{(\log n)^q n\}$. Thus, for any $M > 0$, $P(\sup_{r \in [\epsilon, 1]} |\sum_{t=1}^{\lfloor nr \rfloor} (\eta_t^2 - 1)| > n^{1/2} M) \leq n^{-q/2} M^{-q} E(\max_{1 \leq k \leq n} |\sum_{t=1}^k (\eta_t^2 - 1)|)^q = O\{(\log n)^q n^{1-q/2}\} = O\{(\log n)^{2+\delta/2} n^{-\delta/4}\} = o(1)$. Now (19) is proved, which completes the proof for (A2). $\diamond$

LEMMA 0.6. *Assume (B1). If $\{u_{t,n}\}$ are generated from (A1) or (A2), then for $\epsilon \leq r_1 < r_2 \leq 1$ and $n \geq n_0$ for some positive integer n_0 ,*

$$E^* \left| n^{-1/2} \sum_{t=\lfloor nr_1 \rfloor + 1}^{\lfloor nr_2 \rfloor} W_t u_{t,n} \right|^4 \leq \bar{C}(\mathcal{X}_n) \{(r_2 - r_1)^2 + n^{-1}(r_2 - r_1)\},$$

where $\bar{C}(\mathcal{X}_n) = O_p(1)$.

Proof of Lemma 0.6. Notice that W_t 's are iid $(0,1)$,

$$\begin{aligned} n^{-2} E^* \left| \sum_{t=\lfloor nr_1 \rfloor + 1}^{\lfloor nr_2 \rfloor} W_t u_{t,n} \right|^4 &\leq n^{-2} \sum_{t=\lfloor nr_1 \rfloor + 1}^{\lfloor nr_2 \rfloor} E^* |W_t u_{t,n}|^4 + n^{-2} \left\{ \sum_{t=\lfloor nr_1 \rfloor + 1}^{\lfloor nr_2 \rfloor} E^* |W_t u_{t,n}|^2 \right\}^2 \\ &\leq C n^{-2} \sum_{t=\lfloor nr_1 \rfloor + 1}^{\lfloor nr_2 \rfloor} u_{t,n}^4 + n^{-2} \left(\sum_{t=\lfloor nr_1 \rfloor + 1}^{\lfloor nr_2 \rfloor} u_{t,n}^2 \right)^2 := \mathcal{I}_1 + \mathcal{I}_2. \end{aligned}$$

Since $E(\mathcal{I}_1) = C n^{-2} \sum_{t=\lfloor nr_1 \rfloor + 1}^{\lfloor nr_2 \rfloor} E u_{t,n}^4 \leq C \bar{\omega}^4 E e_0^4 n^{-2} (\lfloor nr_2 \rfloor - \lfloor nr_1 \rfloor) = O(1) n^{-1} (r_2 - r_1)$ and $E(\mathcal{I}_2^{1/2}) = n^{-1} \sum_{t=\lfloor nr_1 \rfloor + 1}^{\lfloor nr_2 \rfloor} E u_{t,n}^2 \leq C \bar{\omega}^4 n^{-1} (\lfloor nr_2 \rfloor - \lfloor nr_1 \rfloor) = O(1) (r_2 - r_1)$, we have $\mathcal{I}_1 + \mathcal{I}_2 = O_p(1) n^{-1} (r_2 - r_1) + O_p(1) (r_2 - r_1)^2$. Thus the proof is complete. $\diamond$

LEMMA 0.7. *Assume (R1)-(R2) and (B1).*

(I) *Assume (A1). Then*

$$n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} W_t F_{t,n} u_{t,n} = n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} W_t F_{t,n} C(L) \varepsilon_t \Rightarrow \{\mathbf{D}(1)\}^{1/2} B_{F,\omega}(r) \quad \text{in probability}$$

(II) Assume (A2). Then

$$n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} W_t F_{t,n} u_{t,n} = n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} W_t F_{t,n} \nu_{t,n} \eta_t \Rightarrow B_{F,\nu}(r) \quad \text{in probability.}$$

Proof of Lemma 0.7. First we show

$$n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} W_t u_{t,n} \Rightarrow \int_0^r \varpi(s) dB(s) \quad \text{in probability,} \quad (20)$$

where $\varpi(s) = \{\mathbf{D}(1)\}^{1/2} \omega(s)$ for (A1) and $\varpi(s) = \nu(s)$ for (A2). The finite-dimensional convergence of (20),

$$\left(n^{-1/2} \sum_{t=1}^{\lfloor nr_1 \rfloor} W_t u_{t,n}, \dots, n^{-1/2} \sum_{t=1}^{\lfloor nr_k \rfloor} W_t u_{t,n} \right) \xrightarrow{\mathcal{D}} \left\{ \int_0^{r_1} \varpi(s) dB(s), \dots, \int_0^{r_k} \varpi(s) dB(s) \right\}$$

for any $k \in \mathbb{N}$ and $r_1, \dots, r_k$, follows from a similar argument presented in Lemma 0.5 and the Cramér-Wold device in $D[\epsilon, 1]$ space (see Theorem 29.16 of Davidson (1994)). The tightness follows from Lemma 0.6 and the argument of Theorem 2.1 in Shao and Yu (1996). This completes the proof of (20).

Since $f_j(s)$ are piecewise Lipschitz continuous, the above argument works if $\omega(s)$ is replaced by $\omega(s)f_j(s)$ for (A1) and $\nu(s)$ is replaced by $\nu(s)f_j(s)$ for (A2), respectively, for all $j = 1, \dots, p$. Then using the Cramér-Wold device, the proof is now complete.

◇

Proof of Theorem 3.2. Define $Q_{N,W}(r) = N^{-1/2} \sum_{t=1}^{\lfloor Nr \rfloor + p-1} F_{t,n} F'_{t,n} W_t$, $B_{F,\varpi}(r) = \int_0^r F(s) \varpi(s) dB(s)$, and $\varpi(s) = \{\mathbf{D}(1)\}^{1/2} \omega(s)$ if (A1) and $\varpi(s) = \nu(s)$ if (A2), as defined in the above proof for Lemma 0.7. Observe that

$$N^{-1/2} \lfloor Nr \rfloor (\hat{\beta}_{\lfloor Nr \rfloor}^* - \hat{\beta}_N) = \mathcal{I}_{N,1}(r) + \mathcal{I}_{N,2}(r), \quad (21)$$

where $\mathcal{I}_{N,1}(r) = \{Q_N(r)\}^{-1} (N^{-1/2} \sum_{t=1}^{\lfloor Nr \rfloor + p-1} F_t u_{t,n} W_t)$ and $\mathcal{I}_{N,2}(r) = \{Q_N(r)\}^{-1} Q_{N,W}(r)$

$(\beta - \widehat{\beta}_N)$. From Lemma 0.7 and Lemma 0.1 (i), we have

$$\mathcal{I}_{N,1}(r) \Rightarrow Q_r^{-1} B_{F,\varpi}(r) \quad \text{in probability.} \quad (22)$$

Let $Q_{N,W}^{i,j}(r) = N^{-1/2} \sum_{t=1}^{\lfloor Nr \rfloor} W_{t,ij}$ be the (i, j) th component of $Q_{N,W}(r)$, where $W_{t,ij} = f_i(t/n)f_j(t/n)W_t$. Define $S_{k,ij} = \sum_{t=1}^k W_{t,ij}$ and $S_{k,ij}^* = \max_{t \leq k} |S_{t,ij}|^2$. Let d be a smallest positive integer such that $n \leq 2^d$. By Proposition 1(i) in Wu (2007) and the fact that $W_{t,ij}$ is an iid process, we have $E \max_{k \leq n} S_{k,ij}^2 \leq \|S_{2^d}^*\|_2^2 \leq \{\sum_{z=0}^d (\sum_{m=1}^{2^{d-z}} \|\sum_{t=2^z(m-1)+1}^{2^z m} W_{t,ij}\|_2^2)^{1/2}\}^2 \leq C(d+1)^2 2^d \leq C(\log n)^2 n$, where the constant C does not depend on i or j . Thus we have $\sup_{r \in [\epsilon, 1]} \{Q_{N,W}^{i,j}(r)\}^2 = N^{-1} \max_{k \leq n} S_{k,ij}^2 = O_p^*\{(\log n)^2\}$ uniformly on i and j , which leads to

$$\sup_{r \in [\epsilon, 1]} \|Q_{N,W}(r)\|_F = O_p^*\{(\log n)^2\}. \quad (23)$$

Then by the sub-multiplicativity of the Frobenius norm, Lemma 0.2 (ii), (23), and Theorem 3.1, we have

$$\sup_{r \in [\epsilon, 1]} \|\mathcal{I}_{N,2}(r)\| \leq \sup_{r \in [\epsilon, 1]} \|Q_N(r)^{-1}\|_F \sup_{r \in [\epsilon, 1]} \|Q_{N,W}(r)\|_F \|\beta - \widehat{\beta}_N\| = O_p^*\{(\log n)^2 n^{-1/2}\} = o_p^*(1). \quad (24)$$

Then Theorem 3.2 (i) and (ii) are proved in view of (21), (22), and (24). The proof is therefore complete. $\diamond$

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