Supplementary material to the article "Improved gradient statistic in heteroskedastic generalized linear models"

1 Bartlett identities and cumulants

The joint cumulants of the logarithm of the likelihood function are given by $\kappa_{rs} = \mathbb{E}(\partial^2 \ell / \partial\theta_r \theta_s)$, $\kappa_{r,s} = \mathbb{E}(\partial\ell/\partial\theta_r \partial\ell/\partial\theta_s)$, $\kappa_{rst} = \mathbb{E}(\partial^3\ell/\partial\theta_r\theta_s\theta_t)$, $\kappa_{r,st} = \mathbb{E}(\partial\ell/\partial\theta_r \partial^2\ell/\partial\theta_s\theta_t)$, and so on. We define the derivatives of the cumulants as $\kappa_{rs}^{(t)} = \partial\kappa_{rs}/\partial\theta_t$, $\kappa_{rs}^{(tu)} = \partial\kappa_{rs}/\partial\theta_t\partial\theta_u$ etc. All κ's refer to the sample and are in general of order n.

1.1 Bartlett identities

The identities below can be found in Lawley (1956).

- $\kappa_r = 0$,
- $\kappa_{rs} + \kappa_{r,s} = 0$,
- $\kappa_{rst} + \kappa_{r,st} - \kappa_{st}^{(r)} = 0$,
- $\kappa_{r,s,t} - 2\kappa_{rst} + \Sigma_{(3)}\kappa_{rs}^{(t)} = 0$,
- $\kappa_{rstu} + \kappa_{r,stu} - \kappa_{stu}^{(r)} = 0$,
- $\kappa_{r,s,tu} = \kappa_{rstu} - \kappa_{rtu}^{(s)} - \kappa_{stu}^{(r)} + \kappa_{tu}^{(rs)} - \kappa_{rs,tu}$,
- $\kappa_{r,s,t,u} = -3\kappa_{rstu} + 2\Sigma_{(4)}\kappa_{rst}^{(u)} - \Sigma_{(6)}\kappa_{rs}^{(tu)} + \Sigma_{(3)}\kappa_{rs,tu}$,
- $\kappa_{rstu} + \Sigma_{(4)}\kappa_{r,stu} + \Sigma_{(3)}\kappa_{rs,tu} + \Sigma_{(6)}\kappa_{r,s,tu} + \kappa_{r,s,t,u} = 0$,

where $\Sigma_{(k)}$ represents the sum of all combinations of indices.

1.2 Cumulants

Consider the logarithm of the likelihood function for $\boldsymbol{\beta}$ and $\boldsymbol{\lambda}$ in HGLM. We use the indexes in lowercase $(r, s, t, \ldots)$ to denote the components of the vector $\boldsymbol{\beta}$ and the indices in uppercase $(R, S, T, \ldots)$ to represent the vector $\boldsymbol{\lambda}$'s components. We adopt the notation introduced by Lawley (1956) for the derivatives of the logarithm of the likelihood function: $U_r = \partial\ell(\boldsymbol{\beta}, \boldsymbol{\lambda})/\partial\beta_r$, $U_{rS} = \partial^2\ell(\boldsymbol{\beta}, \boldsymbol{\lambda})/\partial\beta_r\partial\lambda_S$, $U_{r,S} = \partial\ell(\boldsymbol{\beta}, \boldsymbol{\lambda})/\partial\beta_r\partial\ell(\beta, \lambda)/\partial\lambda_S$, $U_{rsT} = \partial^3\ell(\boldsymbol{\beta}, \boldsymbol{\lambda})/\partial\beta_r\partial\beta_s\lambda_T$, etc.

- $\kappa_{rs} = -\sum_{\ell=1}^{n} \phi_\ell w_\ell x_{\ell r} x_{\ell s}$,
- $\kappa_{RS} = \sum_{\ell=1}^{n} d_{2\ell}\phi_{1\ell}^2 s_{\ell R} s_{\ell S}$,
- $\kappa_{rst} = -\sum_{\ell=1}^{n} \phi_\ell (f_\ell + 2g_\ell) x_{\ell r} x_{\ell s} x_{\ell t}$,
- $\kappa_{rs,t} = \sum_{\ell=1}^{n} \phi_\ell g_\ell x_{\ell r} x_{\ell s} x_{\ell t}$,
- $\kappa_{r,s,t} = \sum_{\ell=1}^{n} \phi_\ell (f_\ell - g_\ell) x_{\ell r} x_{\ell s} x_{\ell t}$,
- $\kappa_{Rs,t} = -\kappa_{Rst} = -\kappa_{R,s,t} = -\kappa_{st}^{(R)} = \sum_{\ell=1}^{n} \phi_{1\ell} w_\ell x_{\ell s} x_{\ell t} s_{\ell R}$,
- $\kappa_{R,st} = \kappa_{RSt} = \kappa_{R,S,t} = \kappa_{R,St} = \kappa_{RS,t} = \kappa_{RS}^{(t)} = \kappa_{Rs}^{(t)} = \kappa_{Rs}^{(T)} = 0$,
- $\kappa_{RST} = \sum_{\ell=1}^{n} (d_{3\ell}\phi_{1\ell}^3 + 3d_{2\ell}\phi_{1\ell}\phi_{2\ell}) s_{\ell R} s_{\ell S} s_{\ell T}$,
- $\kappa_{RS,T} = -\sum_{\ell=1}^{n} d_{2\ell}\phi_{1\ell}\phi_{2\ell} s_{\ell R} s_{\ell S} s_{\ell T}$,
- $\kappa_{R,S,T} = -\sum_{\ell=1}^{n} d_{3\ell}\phi_{1\ell}^3 s_{\ell R} s_{\ell S} s_{\ell T}$,
- $\kappa_{RS}^{(T)} = \sum_{\ell=1}^{n} (d_{3\ell}\phi_{1\ell}^3 + 2d_{2\ell}\phi_{1\ell}\phi_{2\ell}) s_{\ell R} s_{\ell S} s_{\ell T}$,

- $$\kappa_{rstu} = \sum_{\ell=1}^{n} \phi_\ell \Bigg\{ -\frac{6}{V^3}\left(\frac{dV}{d\mu}\right)^2\left(\frac{d\mu}{d\eta}\right)^4 + \frac{3}{V^2}\frac{d^2V}{d\mu^2}\left(\frac{d\mu}{d\eta}\right)^4 + \frac{12}{V^2}\frac{dV}{d\mu}\left(\frac{d\mu}{d\eta}\right)^2\frac{d^2\mu}{d\eta^2} - \frac{3}{V}\left(\frac{d^2\mu}{d\eta^2}\right)^2 - \frac{4}{V}\frac{d\mu}{d\eta}\frac{d^3\mu}{d\eta^3}\Bigg\}_\ell x_{\ell r} x_{\ell s} x_{\ell t} x_{\ell u},$$

- $$\kappa_{rs,tu} = \sum_{\ell=1}^{n} \phi_\ell \Bigg\{ \frac{1}{V^3}\left(\frac{dV}{d\mu}\right)^2\left(\frac{d\mu}{d\eta}\right)^4 - \frac{2}{V^2}\frac{dV}{d\mu}\left(\frac{d\mu}{d\eta}\right)^2\frac{d^2\mu}{d\eta^2} + \frac{1}{V}\left(\frac{d^2\mu}{d\eta^2}\right)^2 \Bigg\}_\ell x_{\ell r} x_{\ell s} x_{\ell t} x_{\ell u},$$

- $$\kappa_{r,s,tu} = \sum_{\ell=1}^{n} \phi_\ell \left\{ \frac{1}{V^2} \frac{dV}{d\mu} \left(\frac{d\mu}{d\eta}\right)^2 \frac{d^2\mu}{d\eta^2} - \frac{1}{V^3} \left(\frac{dV}{d\mu}\right)^2 \left(\frac{d\mu}{d\eta}\right)^4 \right\}_\ell x_{\ell r} x_{\ell s} x_{\ell t} x_{\ell u},$$

- $$\kappa_{r,s,t,u} = \sum_{\ell=1}^{n} \phi_\ell \left\{ \frac{1}{V^3} \left(\frac{dV}{d\mu}\right)^2 \left(\frac{d\mu}{d\eta}\right)^4 + \frac{1}{V^2} \frac{d^2V}{d\mu^2} \left(\frac{d\mu}{d\eta}\right)^4 \right\}_\ell x_{\ell r} x_{\ell s} x_{\ell t} x_{\ell u},$$

- $$\kappa_{rst,u} = \sum_{\ell=1}^{n} \phi_\ell \left\{ \frac{2}{V^3} \left(\frac{dV}{d\mu}\right)^2 \left(\frac{d\mu}{d\eta}\right)^4 - \frac{1}{V^2} \frac{d^2V}{d\mu^2} \left(\frac{d\mu}{d\eta}\right)^4 - \frac{3}{V^2} \frac{dV}{d\mu} \left(\frac{d\mu}{d\eta}\right)^2 \frac{d^2\mu}{d\eta^2} + \frac{1}{V} \frac{d\mu}{d\eta} \frac{d^3\mu}{d\eta^3} \right\}_\ell x_{\ell r} x_{\ell s} x_{\ell t} x_{\ell u},$$

- $$\kappa^{(r)}_{stu} = \sum_{\ell=1}^{n} \phi_\ell \left\{ -\frac{4}{V^3} \left(\frac{dV}{d\mu}\right)^2 \left(\frac{d\mu}{d\eta}\right)^4 + \frac{2}{V^2} \frac{d^2V}{d\mu^2} \left(\frac{d\mu}{d\eta}\right)^4 + \frac{9}{V^2} \frac{dV}{d\mu} \left(\frac{d\mu}{d\eta}\right)^2 \frac{d^2\mu}{d\eta^2} - \frac{3}{V} \left(\frac{d^2\mu}{d\eta^2}\right)^2 - \frac{3}{V} \frac{d\mu}{d\eta} \frac{d^3\mu}{d\eta^3} \right\}_\ell x_{\ell r} x_{\ell s} x_{\ell t} x_{\ell u},$$

- $$\kappa^{(rs)}_{tu} = \sum_{\ell=1}^{n} \phi_\ell \left\{ -\frac{2}{V^3} \left(\frac{dV}{d\mu}\right)^2 \left(\frac{d\mu}{d\eta}\right)^4 + \frac{1}{V^2} \frac{d^2V}{d\mu^2} \left(\frac{d\mu}{d\eta}\right)^4 + \frac{5}{V^2} \frac{dV}{d\mu} \left(\frac{d\mu}{d\eta}\right)^2 \frac{d^2\mu}{d\eta^2} - \frac{2}{V} \left(\frac{d^2\mu}{d\eta^2}\right)^2 - \frac{2}{V} \frac{d\mu}{d\eta} \frac{d^3\mu}{d\eta^3} \right\}_\ell x_{\ell r} x_{\ell s} x_{\ell t} x_{\ell u},$$

- $\kappa_{rstU} = -\sum_{\ell=1}^{n} \phi_{1\ell}(f_\ell + 2g_\ell) x_{\ell r} x_{\ell s} x_{\ell t} s_{\ell U}$,

- $\kappa_{r,stU} = -\kappa_{rs,t,U} = \kappa_{rs,tU} = \sum_{\ell=1}^{n} \phi_{1\ell} g_\ell x_{\ell r} x_{\ell s} x_{\ell t} s_{\ell U}$,

- $\kappa_{r,s,tU} = \sum_{\ell=1}^{n} \phi_{1\ell}(f_\ell - g_\ell) x_{\ell r} x_{\ell s} x_{\ell t} s_{\ell U}$,

- $\kappa_{rst,U} = 0$,

- $\kappa_{RStu} = \kappa_{RS,t,u} = \kappa^{(R)}_{Stu} = -\kappa_{RSt,u} = -\sum_{\ell=1}^{n} \phi_{2\ell} w_\ell x_{\ell t} x_{\ell u} s_{\ell R} s_{\ell S}$,

- $\kappa_{Rt,Su} = -\kappa_{R,St,u} = \frac{1}{2}\kappa_{R,S,t,u} = \sum_{\ell=1}^{n} \phi_{1\ell}^2 \phi_\ell^{-1} w_\ell x_{\ell t} x_{\ell u} s_{\ell R} s_{\ell S}$,

- $\kappa_{R,Stu} = \kappa_{R,S,tu} = \kappa_{RS,tu} = \kappa^{(u)}_{RSt} = 0$,

- $\kappa_{RSTU} = \sum_{\ell=1}^{n}(d_{4\ell}\phi_{1\ell}^4 + 6d_{3\ell}\phi_{1\ell}^2\phi_{2\ell} + 3d_{2\ell}\phi_{2\ell}^2 + 4d_{2\ell}\phi_{1\ell}\phi_{3\ell})s_{\ell R}s_{\ell S}s_{\ell T}s_{\ell U}$,
- $\kappa_{R,S,TU} = -\sum_{\ell=1}^{n} d_{3\ell}\phi_{1\ell}^2\phi_{2\ell}s_{\ell R}s_{\ell S}s_{\ell T}s_{\ell U}$,
- $\kappa_{RS,TU} = -\sum_{\ell=1}^{n} d_{2\ell}\phi_{2\ell}^2 s_{\ell R}s_{\ell S}s_{\ell T}s_{\ell U}$,
- $\kappa_{R,S,T,U} = -\sum_{\ell=1}^{n} d_{4\ell}\phi_{1\ell}^4 s_{\ell R}s_{\ell S}s_{\ell T}s_{\ell U}$,
- $\kappa_{RS}^{(TU)} = \sum_{\ell=1}^{n}(d_{4\ell}\phi_{1\ell}^4 + 5d_{3\ell}\phi_{1\ell}^2\phi_{2\ell} + 2d_{2\ell}\phi_{2\ell}^2 + 2d_{2\ell}\phi_{1\ell}\phi_{3\ell})s_{\ell R}s_{\ell S}s_{\ell T}s_{\ell U}$,
- $\kappa_{RST}^{(U)} = \sum_{\ell=1}^{n}(d_{4\ell}\phi_{1\ell}^4 + 6d_{3\ell}\phi_{1\ell}^2\phi_{2\ell} + 3d_{2\ell}\phi_{2\ell}^2 + 3d_{2\ell}\phi_{1\ell}\phi_{3\ell})s_{\ell R}s_{\ell S}s_{\ell T}s_{\ell U}$.

2 Hypothesis about $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \boldsymbol{\lambda}^\top)^\top$ subsets

2.1 Subset of $\boldsymbol{\beta}$

Suppose that we are interested in testing $\mathcal{H}_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_1^{(0)}$ against $\mathcal{H}_1 : \boldsymbol{\beta}_1 \neq \boldsymbol{\beta}_1^{(0)}$. In this case, $\boldsymbol{Z}_\lambda = \boldsymbol{Z}_{\lambda_2}$. Thus, the coefficients of the improved gradient statistics are given by:

$$\begin{aligned} A_{11} = 3\{&\mathbf{1}^\top\boldsymbol{\Phi}(\boldsymbol{F}+2\boldsymbol{G})[\boldsymbol{Z}_{\beta d}(\boldsymbol{Z}_\beta + \boldsymbol{Z}_{\beta_2})\boldsymbol{Z}_{\beta_2 d} - 2\boldsymbol{Z}_{\beta_2 d}\boldsymbol{Z}_{\beta_2}\boldsymbol{Z}_{\beta_2 d} + 2(\boldsymbol{Z}_\beta - \boldsymbol{Z}_{\beta_2}) \odot \boldsymbol{Z}_{\beta_2}^{(2)}] \\ &\times \boldsymbol{\Phi}(\boldsymbol{F}+2\boldsymbol{G})\mathbf{1}\} \\ &+ 6\{\mathbf{1}^\top\boldsymbol{\Phi}_1\boldsymbol{W}[(\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d})\boldsymbol{Z}_\lambda\boldsymbol{Z}_{\beta_2 d} + 2(\boldsymbol{Z}_\beta - \boldsymbol{Z}_{\beta_2}) \odot \boldsymbol{Z}_{\beta_2} \odot \boldsymbol{Z}_{\lambda_2}]\boldsymbol{\Phi}_1\boldsymbol{W}\mathbf{1} \\ &- 6\{\mathbf{1}^\top\boldsymbol{\Phi}_1\boldsymbol{W}(\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d})\boldsymbol{Z}_\lambda\boldsymbol{Z}_{\lambda d}(\boldsymbol{D}_3\boldsymbol{\Phi}_1^3 + 3\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2)\mathbf{1}\}, \end{aligned}$$

$$A_{12} = 12\{\mathbf{1}^\top\boldsymbol{\Phi}(\boldsymbol{F}+\boldsymbol{G})[\boldsymbol{Z}_{\beta d}\boldsymbol{Z}_\beta\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d}\boldsymbol{Z}_{\beta_2}\boldsymbol{Z}_{\beta_2 d} + \boldsymbol{Z}_\beta^{(3)} - \boldsymbol{Z}_{\beta_2}^{(3)}]\boldsymbol{\Phi}(\boldsymbol{F}+\boldsymbol{G})\mathbf{1}\},$$

$$\begin{aligned} A_{13} = &- 6\{\mathbf{1}^\top\boldsymbol{\Phi}(\boldsymbol{F}+2\boldsymbol{G})[(\boldsymbol{Z}_\beta^{(2)} - \boldsymbol{Z}_{\beta_2}^{(2)}) \odot (\boldsymbol{Z}_\beta + \boldsymbol{Z}_{\beta_2}) + (\boldsymbol{Z}_{\beta d} - 3\boldsymbol{Z}_{\beta_2 d})\boldsymbol{Z}_{\beta_2}\boldsymbol{Z}_{\beta_2 d} \\ &\qquad + (\boldsymbol{Z}_{\beta d} + \boldsymbol{Z}_{\beta_2 d})\boldsymbol{Z}_\beta\boldsymbol{Z}_{\beta d} + 2\boldsymbol{Z}_{\beta_2}^{(2)} \odot (\boldsymbol{Z}_\beta - \boldsymbol{Z}_{\beta_2})]\boldsymbol{\Phi}(\boldsymbol{F}+\boldsymbol{G})\mathbf{1}\} \\ &- 12\{\mathbf{1}^\top\boldsymbol{\Phi}_1\boldsymbol{W}[(\boldsymbol{Z}_\beta^{(2)} - \boldsymbol{Z}_{\beta_2}^{(2)}) \odot \boldsymbol{Z}_\lambda]\boldsymbol{\Phi}_1\boldsymbol{W}\mathbf{1}\} \\ &+ 12\{\mathbf{1}^\top\boldsymbol{\Phi}_1\boldsymbol{W}[(\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d})\boldsymbol{Z}_\lambda\boldsymbol{Z}_{\lambda d}](\boldsymbol{D}_3\boldsymbol{\Phi}_1^3 + 2\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2)\mathbf{1}\}, \end{aligned}$$

$$
\begin{aligned}
A_{14} = & \, 6\{\mathbf{1}^\top \boldsymbol{\Phi} \boldsymbol{B}(\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d})\boldsymbol{Z}_{\beta_2 d}\mathbf{1}\} + 6\{\mathbf{1}^\top \boldsymbol{\Phi}_2 \boldsymbol{W}(\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d})\boldsymbol{Z}_{\lambda d}\mathbf{1}\} \\
& - 6\{\mathbf{1}^\top \boldsymbol{\Phi} \boldsymbol{C}(\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d})(3\boldsymbol{Z}_{\beta_2 d} + \boldsymbol{Z}_{\beta d})\mathbf{1}\} \\
& + 12\{\mathbf{1}^\top \boldsymbol{\Phi} \boldsymbol{E}(\boldsymbol{Z}^{(2)}_{\beta d} - \boldsymbol{Z}^{(2)}_{\beta_2 d})\mathbf{1}\},
\end{aligned}
$$

$$
\begin{aligned}
A_{21} = & -3\Bigg\{\mathbf{1}^\top \boldsymbol{\Phi}(\boldsymbol{F} + 2\boldsymbol{G})\Bigg[\frac{1}{4}(\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d})(\boldsymbol{Z}_{\beta} - \boldsymbol{Z}_{\beta_2})(3\boldsymbol{Z}_{\beta d} + \boldsymbol{Z}_{\beta_2 d}) \\
& + (\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d})\boldsymbol{Z}_{\beta_2}(\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d}) + \frac{1}{2}(\boldsymbol{Z}_{\beta} - \boldsymbol{Z}_{\beta_2})^{(2)} \odot (\boldsymbol{Z}_{\beta} + 3\boldsymbol{Z}_{\beta_2})\Bigg]\boldsymbol{\Phi}(\boldsymbol{F} + 2\boldsymbol{G})\mathbf{1}\Bigg\} \\
& - 3\{\mathbf{1}^\top \boldsymbol{\Phi}_1 \boldsymbol{W}[(\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d})\boldsymbol{Z}_{\lambda}(\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d}) + 2(\boldsymbol{Z}_{\beta} - \boldsymbol{Z}_{\beta_2})^{(2)} \odot \boldsymbol{Z}_{\lambda}]\boldsymbol{\Phi}_1 \boldsymbol{W}\mathbf{1}\},
\end{aligned}
$$

$$
\begin{aligned}
A_{22} = & \, 6\{\mathbf{1}^\top \boldsymbol{\Phi}(\boldsymbol{F} + 2\boldsymbol{G})[(\boldsymbol{Z}_{\beta} - \boldsymbol{Z}_{\beta_2}) \odot (\boldsymbol{Z}^{(2)}_{\beta} - \boldsymbol{Z}^{(2)}_{\beta_2}) + (\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d})(\boldsymbol{Z}_{\beta}\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2}\boldsymbol{Z}_{\beta_2 d})] \\
& \times \boldsymbol{\Phi}(\boldsymbol{F} + \boldsymbol{G})\mathbf{1}\},
\end{aligned}
$$

$$
A_{23} = 3\{\mathbf{1}^\top \boldsymbol{\Phi}(2\boldsymbol{C} - \boldsymbol{B})(\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d})^{(2)}\mathbf{1}\},
$$

$$
A_3 = \mathbf{1}^\top \boldsymbol{\Phi}(\boldsymbol{F} + 2\boldsymbol{G})\left[\frac{3}{4}(\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d})(\boldsymbol{Z}_{\beta} - \boldsymbol{Z}_{\beta_2})(\boldsymbol{Z}_{\beta d} - \boldsymbol{Z}_{\beta_2 d}) + \frac{1}{2}(\boldsymbol{Z}_{\beta} - \boldsymbol{Z}_{\beta_2})^{(3)}\right]\boldsymbol{\Phi}(\boldsymbol{F} + 2\boldsymbol{G})\mathbf{1}.
$$

2.2 Subset of $\boldsymbol{\lambda}$

Suppose that we are interested in testing $\mathcal{H}_0 : \boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_1^{(0)}$ against the alternative hypothesis $\mathcal{H}_0 : \boldsymbol{\lambda}_1 \neq \boldsymbol{\lambda}_1^{(0)}$. In this case, $\boldsymbol{Z}_{\beta} = \boldsymbol{Z}_{\beta_2}$. Thus, the coefficients of the improved gradient statistics are given by:

$$
\begin{aligned}
A_{11} = & \, 3\{\mathbf{1}^\top \boldsymbol{\Phi}_1 \boldsymbol{W}[\boldsymbol{Z}_{\beta d}(\boldsymbol{Z}_{\lambda} - \boldsymbol{Z}_{\lambda_2})\boldsymbol{Z}_{\beta d} + 2(\boldsymbol{Z}_{\lambda} - \boldsymbol{Z}_{\lambda_2}) \odot \boldsymbol{Z}^{(2)}_{\beta}]\boldsymbol{\Phi}_1 \boldsymbol{W}\mathbf{1}\} \\
& - 3\{\mathbf{1}^\top \boldsymbol{\Phi}_1 \boldsymbol{W}[\boldsymbol{Z}_{\beta d}(\boldsymbol{Z}_{\lambda} - \boldsymbol{Z}_{\lambda_2})\boldsymbol{Z}_{\lambda_2 d}](\boldsymbol{D}_3 \boldsymbol{\Phi}_1^3 + 3\boldsymbol{D}_2 \boldsymbol{\Phi}_1 \boldsymbol{\Phi}_2)\mathbf{1}\} \\
& - 3\{\mathbf{1}^\top (\boldsymbol{D}_3 \boldsymbol{\Phi}_1^3 + 3\boldsymbol{D}_2 \boldsymbol{\Phi}_1 \boldsymbol{\Phi}_2)[\boldsymbol{Z}_{\lambda d}(\boldsymbol{Z}_{\lambda} + \boldsymbol{Z}_{\lambda_2})\boldsymbol{Z}_{\beta d} - 2\boldsymbol{Z}_{\lambda_2 d}\boldsymbol{Z}_{\lambda_2}\boldsymbol{Z}_{\beta d}]\boldsymbol{\Phi}_1 \boldsymbol{W}\mathbf{1}\} \\
& + 3\{\mathbf{1}^\top (\boldsymbol{D}_3 \boldsymbol{\Phi}_1^3 + 3\boldsymbol{D}_2 \boldsymbol{\Phi}_1 \boldsymbol{\Phi}_2)[\boldsymbol{Z}_{\lambda d}(\boldsymbol{Z}_{\lambda} + \boldsymbol{Z}_{\lambda_2})\boldsymbol{Z}_{\lambda_2 d} - 2\boldsymbol{Z}_{\lambda_2 d}\boldsymbol{Z}_{\lambda_2}\boldsymbol{Z}_{\lambda_2 d} + 2(\boldsymbol{Z}_{\lambda} - \boldsymbol{Z}_{\lambda_2}) \odot \boldsymbol{Z}^{(2)}_{\lambda_2}] \\
& \times (\boldsymbol{D}_3 \boldsymbol{\Phi}_1^3 + 3\boldsymbol{D}_2 \boldsymbol{\Phi}_1 \boldsymbol{\Phi}_2)\mathbf{1}\},
\end{aligned}
$$

$$A_{12} = 12\{\mathbf{1}^\top(\boldsymbol{D}_3\boldsymbol{\Phi}_1^3 + 2\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2)[\boldsymbol{Z}_{\lambda d}\boldsymbol{Z}_\lambda\boldsymbol{Z}_{\lambda d} - \boldsymbol{Z}_{\lambda_2 d}\boldsymbol{Z}_{\lambda_2}\boldsymbol{Z}_{\lambda_2 d} + \boldsymbol{Z}_\lambda^{(3)} - \boldsymbol{Z}_{\lambda_2}^{(3)}] \times (\boldsymbol{D}_3\boldsymbol{\Phi}_1^3 + 2\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2)\mathbf{1}\},$$

$$\begin{aligned} A_{13} =& - 12\{\mathbf{1}^\top\boldsymbol{\Phi}_1\boldsymbol{W}[\boldsymbol{Z}_\beta^{(2)} \odot (\boldsymbol{Z}_\lambda - \boldsymbol{Z}_{\lambda_2})]\boldsymbol{\Phi}_1\boldsymbol{W}\mathbf{1}\} \\ &+ 12\{\mathbf{1}^\top\boldsymbol{\Phi}_1\boldsymbol{W}[\boldsymbol{Z}_{\beta d}(\boldsymbol{Z}_\lambda\boldsymbol{Z}_{\lambda d} - \boldsymbol{Z}_{\lambda_2}\boldsymbol{Z}_{\lambda_2 d})](\boldsymbol{D}_3\boldsymbol{\Phi}_1^3 + 2\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2)\mathbf{1}\} \\ &- 6\{\mathbf{1}^\top(\boldsymbol{D}_3\boldsymbol{\Phi}_1^3 + 3\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2)[(\boldsymbol{Z}_\lambda^{(2)} - \boldsymbol{Z}_{\lambda_2}^{(2)}) \odot (\boldsymbol{Z}_\lambda + \boldsymbol{Z}_{\lambda_2}) + (\boldsymbol{Z}_{\lambda d} - 3\boldsymbol{Z}_{\lambda_2 d})\boldsymbol{Z}_{\lambda_2}\boldsymbol{Z}_{\lambda_2 d} \\ &\quad + (\boldsymbol{Z}_{\lambda d} + \boldsymbol{Z}_{\lambda_2 d})\boldsymbol{Z}_\lambda\boldsymbol{Z}_{\lambda d} + 2\boldsymbol{Z}_{\lambda_2}^{(2)} \odot (\boldsymbol{Z}_\lambda - \boldsymbol{Z}_{\lambda_2})](\boldsymbol{D}_3\boldsymbol{\Phi}_1^3 + 2\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2)\mathbf{1}\}, \end{aligned}$$

$$\begin{aligned} A_{14} =& 6\{\mathbf{1}^\top\boldsymbol{\Phi}_2\boldsymbol{W}(\boldsymbol{Z}_{\lambda d} - \boldsymbol{Z}_{\lambda_2 d})\boldsymbol{Z}_{\beta_d}\mathbf{1}\} \\ &+ 6\{\mathbf{1}^\top(\boldsymbol{D}_4\boldsymbol{\Phi}_1^4 + 6\boldsymbol{D}_3\boldsymbol{\Phi}_1^2\boldsymbol{\Phi}_2 + 3\boldsymbol{D}_2\boldsymbol{\Phi}_2^2 + 4\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_3)(\boldsymbol{Z}_{\lambda d} - \boldsymbol{Z}_{\lambda_2 d})\boldsymbol{Z}_{\lambda_2 d}\mathbf{1}\} \\ &- 6\{\mathbf{1}^\top(\boldsymbol{D}_4\boldsymbol{\Phi}_1^4 + 6\boldsymbol{D}_3\boldsymbol{\Phi}_1^2\boldsymbol{\Phi}_2 + 3\boldsymbol{D}_2\boldsymbol{\Phi}_2^2 + 3\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_3)(\boldsymbol{Z}_{\lambda d} - \boldsymbol{Z}_{\lambda_2 d})(\boldsymbol{Z}_{\lambda d} + 3\boldsymbol{Z}_{\lambda_2 d})\mathbf{1}\} \\ &+ 12\{\mathbf{1}^\top(\boldsymbol{D}_4\boldsymbol{\Phi}_1^4 + 5\boldsymbol{D}_3\boldsymbol{\Phi}_1^2\boldsymbol{\Phi}_2 + 2\boldsymbol{D}_2\boldsymbol{\Phi}_2^2 + 2\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_3)(\boldsymbol{Z}_{\lambda d}^{(2)} - \boldsymbol{Z}_{\lambda_2 d}^{(2)})\mathbf{1}\}, \end{aligned}$$

$$\begin{aligned} A_{21} =& 3\{\mathbf{1}^\top(\boldsymbol{D}_3\boldsymbol{\Phi}_1^3 + 3\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2)(\boldsymbol{Z}_{\lambda d} - \boldsymbol{Z}_{\lambda_2 d})(\boldsymbol{Z}_\lambda - \boldsymbol{Z}_{\lambda_2})\boldsymbol{Z}_{\beta d}\boldsymbol{\Phi}_1\boldsymbol{W}\mathbf{1}\} \\ &- 3\Big\{\mathbf{1}^\top(\boldsymbol{D}_3\boldsymbol{\Phi}_1^3 + 3\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2)\Big[\frac{1}{4}(\boldsymbol{Z}_{\lambda d} - \boldsymbol{Z}_{\lambda_2 d})(\boldsymbol{Z}_\lambda - \boldsymbol{Z}_{\lambda_2})(3\boldsymbol{Z}_{\lambda d} + \boldsymbol{Z}_{\lambda_2 d}) \\ &\quad + (\boldsymbol{Z}_{\lambda d} - \boldsymbol{Z}_{\lambda_2 d})\boldsymbol{Z}_{\lambda_2}(\boldsymbol{Z}_{\lambda d} - \boldsymbol{Z}_{\lambda_2 d}) + \frac{1}{2}(\boldsymbol{Z}_\lambda - \boldsymbol{Z}_{\lambda_2})^{(2)} \odot (\boldsymbol{Z}_\lambda + 3\boldsymbol{Z}_{\lambda_2})\Big] \\ &\quad \times (\boldsymbol{D}_3\boldsymbol{\Phi}_1^3 + 3\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2)\mathbf{1}\Big\}, \end{aligned}$$

$$\begin{aligned} A_{22} =& 6\{\mathbf{1}^\top(\boldsymbol{D}_3\boldsymbol{\Phi}_1^3 + 3\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2)[(\boldsymbol{Z}_\lambda - \boldsymbol{Z}_{\lambda_2}) \odot (\boldsymbol{Z}_\lambda^{(2)} - \boldsymbol{Z}_{\lambda_2}^{(2)}) \\ &\quad + (\boldsymbol{Z}_{\lambda d} - \boldsymbol{Z}_{\lambda_2 d})(\boldsymbol{Z}_\lambda\boldsymbol{Z}_{\lambda d} - \boldsymbol{Z}_{\lambda_2}\boldsymbol{Z}_{\lambda_2 d})](\boldsymbol{D}_3\boldsymbol{\Phi}_1^3 + 2\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2)\mathbf{1}\}, \end{aligned}$$

$$A_{23} = \{\mathbf{1}^\top(\boldsymbol{D}_4\boldsymbol{\Phi}_1^4 + 6\boldsymbol{D}_3\boldsymbol{\Phi}_1^2\boldsymbol{\Phi}_2 + 3\boldsymbol{D}_2\boldsymbol{\Phi}_2^2 + 2\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_3)(\boldsymbol{Z}_{\lambda d} - \boldsymbol{Z}_{\lambda_2 d})^{(2)}\mathbf{1}\},$$

$$A_3 = \mathbf{1}^\top(\boldsymbol{D}_3\boldsymbol{\Phi}_1^3 + 3\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2)\left[\frac{3}{4}(\boldsymbol{Z}_{\lambda d} - \boldsymbol{Z}_{\lambda_2 d})(\boldsymbol{Z}_\lambda - \boldsymbol{Z}_{\lambda_2})(\boldsymbol{Z}_{\lambda d} - \boldsymbol{Z}_{\lambda_2 d}) + \frac{1}{2}(\boldsymbol{Z}_\lambda - \boldsymbol{Z}_{\lambda_2})^{(3)}\right]$$
$$\times (\boldsymbol{D}_3\boldsymbol{\Phi}_1^3 + 3\boldsymbol{D}_2\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2)\mathbf{1}.$$

3 Numerical results

3.1 Inverse Gaussian model

The simulation results presented here are based on the inverse Gaussian regression model with systematic components given by:

$$\mu_\ell = \beta_1 + \beta_2 x_{\ell 2} + \ldots + \beta_p x_{\ell p} \text{ and } \log(\phi_\ell) = \lambda_1 + \lambda_2 s_{\ell 2} + \ldots + \lambda_q s_{\ell q},$$

In Table 1, we present the rejection rates considering the hypothesis $\mathcal{H}_0 : \beta_p = 0$, $n = 30$, $q = 2$ and different values of p. We vary p to analyze the effect of the number of covariates present in the model in the various tests. Note that the usual (LR), adjusted (LR_a) and Bartlett (LR_b) likelihood ratio tests, the usual Wald test (W) and the gradient test (T) are extremely liberal, especially when the number of parameters increases. The score test (SR) is also liberal in almost all cases but has a smaller distorted size than the tests mentioned above in all situations. For example, for $p = 5$ and $\alpha = 5\%$, the test rejection rates are 17.28% (LR), 10.12% (LR_a), 9.82% (LR_b), 6.18% (SR), 28.98% (W) and 9.22% (T). Note that the influence of the number of parameters is less pronounced in the asymptotic tests SR, SR_b and T_b. For the T_b test, the Bartlett-type correction factor causes the rejection rates to be closer to the nominal levels considered and produces better rates than their usual versions. Furthermore, the tests SR and SR_b have similar performance. The bootstrap tests in general indicate a significant reduction in size distortion, especially for the likelihood ratio and Wald tests.

In Table 2, we fix the null hypothesis as $\mathcal{H}_0 : \beta_p = 0$, $p = 2$, $n = 30$ and use different values of q. The asymptotic tests SR, SR_b and T_b and the bootstrap tests are practically unaffected with the increase of q. The same does not happen with the other tests. The corrected (SR_b) and usual (SR) scores are conservative. Note also that the tests LR, W and T are liberal, and the Wald test (W) has the largest size distortion.

In Table 3, we consider the null hypothesis $\mathcal{H}_0 : \beta_p = 0$, $p = 3$, $q = 3$ and we vary the sample size by $20, 30, 40$. As the sample size increases, the rejection rates of all the tests approach the respective nominal levels. The usual Wald test (W) shows rejection rates above the nominal levels considered. For example, when $n = 20$ and $\alpha = 5\%$, the Wald test rejection rate is 19.78%, which is almost four times the nominal level set. We notice that the rejection rates of the gradient test are attenuated by the Bartlett-type correction. As an illustration, for $n = 30$ and $\alpha = 5\%$, the gradient test rate is 7.60%, while the corrected gradient test rate is 5.30%. Furthermore, the tests SR, SR_{b}, T_{b} and the bootstraps show rejection rates closer to the corresponding nominal levels for all values of n.

In Table 4, we apply the null hypothesis $\mathcal{H}_0 : \lambda_2 = \cdots = \lambda_q = 0$, $q = 3$, $p = 2$ and we diversify the sample size at $20, 30, 40$. The rejection rates of all tests approach their respective nominal levels as the sample size increases. The tests LR, W and T present high size distortions, with the distortion of W being more accentuated. Also, the tests SR and T_{b} are conservative, and the tests LR_{a} and LR_{b} have similar behavior. Bootstrap tests generally perform better than other tests.

In Table 5, we fix the null hypothesis as $\mathcal{H}_0 : \lambda_2 = \cdots = \lambda_q = 0$ and $p = 1$ and vary the sample size by $20, 30, 40$ and $q = 2, 3, 4$. As the sample size increases, the test results approach the respective nominal levels. Also, the Wald (W) test has higher size distortions as the number of parameters increases. The LR and T tests are liberal but less distorted than W. Furthermore, the test scores (SR) and corrected gradient (T_{b}) are conservative, the latter being true in almost all cases. The modified likelihood ratio tests (LR_{a} and LR_{b}) show similar results. Finally, the bootstrap and SR_{b} tests provide a considerable reduction in size distortion.

In Table 6, we present the simulation results for power. We set $n = 30$, $p = 1$, $q = 2$ and $\alpha = 10\%$ and consider the hypothesis $\mathcal{H}_1 : \lambda_2 = \epsilon$, where different values of ϵ are considered. We disregard the usual tests, namely LR, SR, W and T. As expected, the power of the tests increases as ϵ increases. The powers of the tests LR_{boot} and T_{boot} are similar, and the LR_{a} test is slightly more powerful than the others.

In Figure 1, we present the graph of the asymptotic quantiles versus the relative discrepancies of the quantiles. We consider the null hypothesis $\mathcal{H}_0 : \beta_3 = 0$, $p = q = 3$ and $n = 40$, that is, asymptotic quantiles are obtained from the χ^2_1 distribution. The figure confirms the tendency of the Wald test (W) to be too liberal. The tests of the likelihood ratio (LR) and gradient (T) are also liberal but less so than W. The other tests reject the null hypothesis less frequently, compared to the nominal level. The smallest discrepancies are given by the tests SR, SR_{b} and T_{b}.

Table 1: Rejection rates for $\mathcal{H}_0 : \beta_p = 0$; inverse Gaussian distribution with $n = 30$ and $q = 2$.

p	$\alpha(\%)$	LR	LR_{a}	LR_{b}	SR	SR_{b}	W	T	T_{b}	LR_{boot}	SR_{boot}	W_{boot}	T_{boot}
	10	12.86	10.86	10.74	10.06	10.12	16.60	11.74	10.32	10.26	10.24	10.54	10.26
2	5	7.26	5.90	5.72	5.08	5.24	10.14	5.84	5.26	5.62	5.30	5.26	5.44
	1	1.74	1.44	1.22	0.72	1.02	3.94	0.90	0.98	1.26	1.20	1.46	1.20
	10	15.68	11.12	11.22	10.74	9.96	21.14	12.92	9.90	9.86	9.90	10.72	9.98
3	5	8.68	6.12	5.78	5.56	5.26	14.50	6.32	4.90	5.20	5.12	5.58	5.22
	1	2.54	1.34	1.26	1.00	1.02	6.64	1.00	0.98	1.10	1.28	1.20	1.24
	10	20.34	13.80	13.42	10.54	9.72	29.84	14.62	9.88	10.50	9.16	12.92	9.90
4	5	12.30	7.60	7.30	5.24	4.86	22.54	7.46	4.66	5.56	4.64	6.76	5.14
	1	3.92	1.80	1.48	0.76	0.78	12.26	1.00	0.78	1.00	1.00	1.72	0.98
	10	25.12	18.28	16.84	11.68	11.04	37.28	16.64	10.34	12.04	9.48	14.32	10.32
5	5	17.28	10.12	9.82	6.18	5.88	28.98	9.22	5.26	6.20	4.98	8.22	5.44
	1	6.40	2.84	2.36	1.14	1.14	16.74	1.38	0.98	1.46	1.14	2.22	1.10
	10	29.32	21.88	19.32	12.54	11.86	39.08	20.60	11.78	12.94	9.64	15.82	11.14
6	5	20.44	13.40	11.26	6.50	6.10	30.32	11.74	6.32	6.54	4.86	8.66	5.72
	1	8.48	3.96	2.96	1.18	1.26	17.10	2.00	1.44	1.60	1.00	2.18	1.24
	10	32.40	24.48	21.28	12.16	11.04	47.28	19.76	11.62	13.54	8.86	17.00	10.90
7	5	23.22	15.06	13.48	6.74	6.20	38.82	12.00	6.58	7.14	5.02	9.36	5.98
	1	11.04	4.88	4.18	1.80	1.32	25.52	3.02	1.76	1.82	1.24	2.06	1.38

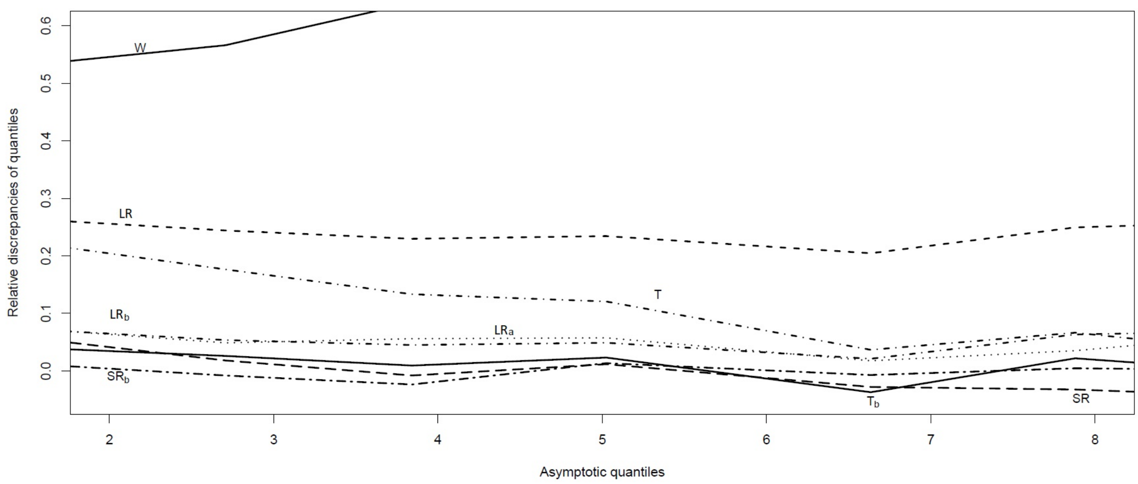

Figure 1: Relative discrepancies of quantiles - inverse Gaussian model with $n = 40$, $p = 3$ and $q = 3$.

Table 2: Rejection rates for $\mathcal{H}_0 : \beta_p = 0$; inverse Gaussian distribution with $n = 30$ and $p = 2$.

	$q = 2$			$q = 3$		
Statistic	$\alpha = 10\%$	$\alpha = 5\%$	$\alpha = 1\%$	$\alpha = 10\%$	$\alpha = 5\%$	$\alpha = 1\%$
LR	12.96	6.58	1.70	14.26	7.72	2.16
LR_{a}	10.40	5.22	1.18	11.18	5.78	1.30
LR_{b}	10.32	4.98	1.04	10.72	5.50	1.20
SR	9.62	4.36	0.58	9.64	4.44	0.68
SR_{b}	9.72	4.66	0.78	9.60	4.66	0.90
W	16.68	9.92	3.54	19.36	12.18	5.18
T	11.36	5.36	0.82	12.84	6.58	1.26
T_{b}	9.86	4.72	0.88	10.08	5.08	1.06
LR_{boot}	10.32	4.98	1.26	10.02	5.14	1.20
SR_{boot}	10.16	4.90	0.92	10.00	4.92	0.98
W_{boot}	10.16	5.22	1.24	10.02	5.04	1.30
T_{boot}	10.36	4.96	1.18	10.20	5.14	1.18
	$q = 2$			$q = 3$		
Statistic	$\alpha = 10\%$	$\alpha = 5\%$	$\alpha = 1\%$	$\alpha = 10\%$	$\alpha = 5\%$	$\alpha = 1\%$
LR	16.24	9.38	2.90	18.52	11.46	3.84
LR_{a}	12.28	6.52	1.58	14.22	8.22	2.44
LR_{b}	11.76	5.98	1.36	12.34	6.88	1.60
SR	10.10	4.68	0.76	10.04	4.58	0.60
SR_{b}	9.66	4.68	0.98	9.46	4.46	0.78
W	23.40	16.74	7.56	28.60	20.74	11.60
T	14.70	7.82	1.56	15.88	9.14	1.68
T_{b}	10.22	4.94	1.04	10.14	5.14	0.84
LR_{boot}	10.08	5.04	1.26	10.04	5.20	1.08
SR_{boot}	10.44	5.20	1.26	10.04	5.04	1.16
W_{boot}	9.84	4.94	1.26	9.88	4.90	1.16
T_{boot}	10.46	5.22	1.40	10.52	5.94	1.32

Table 3: Rejection rates for $\mathcal{H}_0 : \beta_p = 0$; inverse Gaussian distribution with $p = 3$ and $q = 3$.

n	$\alpha(\%)$	LR	LR_{a}	LR_{b}	SR	SR_{b}	W	T	T_{b}	LR_{boot}	SR_{boot}	W_{boot}	T_{boot}
	10	18.50	14.40	12.58	10.26	9.02	27.28	15.48	10.10	9.20	9.16	9.32	9.32
20	5	11.66	7.98	6.52	4.80	4.12	19.78	8.22	4.70	4.44	4.52	4.86	4.52
	1	3.84	2.30	1.56	0.80	0.92	11.00	1.40	0.92	0.92	1.34	1.14	1.10
	10	15.78	11.98	11.70	10.76	10.08	21.02	14.24	10.56	10.36	10.28	10.40	10.22
30	5	9.30	6.38	6.06	5.42	5.24	14.54	7.60	5.30	5.14	5.30	5.40	5.26
	1	2.78	1.52	1.44	0.98	1.06	6.42	1.56	1.02	1.22	1.18	1.50	1.24
	10	14.30	10.84	11.00	10.58	9.92	18.22	13.46	10.54	10.10	9.96	10.26	10.26
40	5	7.80	5.54	5.64	4.90	4.78	11.42	6.76	5.16	5.14	4.88	5.46	5.06
	1	2.02	1.18	1.10	0.90	0.98	4.58	1.24	0.94	1.10	1.30	1.28	1.22

Table 4: Rejection rates for $\mathcal{H}_0 : \lambda_2 = \cdots = \lambda_q = 0$; inverse Gaussian distribution with $p = 2$ and $q = 3$.

n	$\alpha(\%)$	LR	LR_{a}	LR_{b}	SR	SR_{b}	W	T	T_{b}	LR_{boot}	SR_{boot}	W_{boot}	T_{boot}
	10	17.72	13.04	12.96	8.60	10.52	32.52	16.16	9.92	10.18	10.08	9.82	10.18
20	5	10.60	7.20	7.02	4.28	5.30	24.16	9.42	4.08	5.02	5.20	5.02	5.28
	1	2.96	1.76	1.64	0.88	1.00	13.08	2.42	0.16	1.10	1.26	1.02	1.16
	10	14.78	11.94	11.48	9.02	10.48	25.10	13.98	9.84	9.92	10.34	9.46	10.10
30	5	7.96	5.94	5.82	4.86	5.54	17.08	7.26	4.46	5.06	5.28	4.94	4.92
	1	1.94	1.24	1.24	0.90	0.96	7.40	1.66	0.44	1.12	1.06	0.90	1.24
	10	12.52	10.20	10.36	9.50	10.30	18.78	11.90	9.44	9.74	10.34	9.22	9.60
40	5	6.78	5.44	5.60	4.40	4.96	11.64	6.26	4.52	5.04	4.96	5.32	4.92
	1	1.68	1.18	1.16	0.88	0.92	4.78	1.40	0.64	1.08	1.08	1.14	1.02

Table 5: Rejection rates for $\mathcal{H}_0 : \lambda_2 = \cdots = \lambda_q = 0$; inverse Gaussian distribution with $p = 1$.

q	n	$\alpha(\%)$	LR	LR_{a}	LR_{b}	SR	SR_{b}	W	T	T_{b}	LR_{boot}	SR_{boot}	W_{boot}	T_{boot}
		10	12.76	10.76	10.94	8.32	10.06	17.10	12.06	9.94	10.24	10.40	10.16	10.50
	20	5	6.54	5.20	5.52	3.54	4.52	10.74	5.88	4.54	5.04	4.98	4.88	4.94
		1	1.52	0.90	1.04	0.50	0.88	4.08	1.20	0.46	0.96	1.16	1.02	1.08
		10	11.56	10.40	10.28	9.22	10.18	14.84	11.38	10.06	10.10	10.44	10.00	10.08
2	30	5	6.04	5.14	5.22	4.00	4.76	8.82	5.88	4.62	5.20	4.86	5.28	5.02
		1	1.06	0.74	0.80	0.70	0.86	2.62	0.94	0.54	0.96	1.06	1.04	1.00
		10	11.34	10.14	10.48	9.52	10.42	14.08	10.98	10.14	10.34	10.46	10.26	10.28
	40	5	5.96	5.10	5.20	4.50	4.98	7.68	5.64	4.84	5.26	5.16	5.06	5.26
		1	1.34	1.02	1.02	0.94	1.10	2.10	1.14	0.78	1.10	1.42	0.86	1.10
		10	13.58	11.06	11.60	7.98	9.76	23.32	12.62	9.98	10.18	10.14	9.68	10.14
	20	5	7.96	6.18	6.36	3.86	5.16	16.16	7.04	4.26	5.70	5.22	5.24	5.78
		1	2.02	1.48	1.46	0.66	0.88	7.58	1.44	0.36	1.30	1.06	1.48	1.28
		10	12.80	10.98	10.98	8.90	10.42	19.40	12.24	9.84	10.26	10.60	9.86	10.28
3	30	5	6.60	5.34	5.28	4.58	5.26	12.56	6.04	4.26	5.06	5.38	4.56	5.12
		1	1.44	1.16	1.02	1.08	1.16	4.14	1.04	0.44	1.08	1.36	0.94	1.08
		10	11.86	10.60	10.54	8.62	9.62	17.16	11.36	9.70	10.38	9.90	10.42	10.26
	40	5	5.66	5.08	4.98	4.08	4.50	10.60	5.34	4.10	4.78	4.86	4.86	4.76
		1	1.32	1.04	0.98	0.76	0.82	3.54	1.12	0.60	1.16	0.94	1.24	1.20
		10	16.10	11.10	12.80	7.72	10.06	33.78	14.58	8.90	10.04	10.02	10.38	10.02
	20	5	8.84	5.58	6.46	3.72	4.50	25.42	7.36	2.98	4.90	4.94	5.14	4.74
		1	2.24	1.50	1.48	0.94	0.02	14.32	1.46	0.00	1.14	1.12	1.30	1.02
		10	12.84	10.82	10.68	8.36	9.86	23.32	11.82	8.86	9.72	10.24	9.84	9.90
4	30	5	6.82	5.68	5.66	4.08	4.48	15.60	6.12	3.92	4.90	4.92	4.94	4.94
		1	1.62	1.22	1.26	0.96	0.74	6.82	1.42	0.34	1.20	1.20	1.12	1.22
		10	12.70	11.30	11.14	9.28	9.98	21.58	12.06	10.12	10.66	10.52	10.98	10.66
	40	5	6.70	5.74	5.66	4.70	5.02	14.26	6.46	4.38	5.48	5.34	5.18	5.54
		1	1.44	1.00	1.06	1.14	0.94	5.12	1.22	0.44	1.14	1.20	1.10	1.24

Table 6: Power of $\mathcal{H}_1 : \lambda_2 = \epsilon$; $n = 30$, $p = 1$, $q = 2$, $\alpha = 10\%$; inverse Gaussian distribution.

ϵ	LR_{a}	LR_{b}	SR_{b}	T_{b}	LR_{boot}	SR_{boot}	W_{boot}	T_{boot}
0.5	14.36	14.14	13.72	14.06	14.20	14.36	14.20	14.12
1.5	43.82	42.86	41.20	42.78	42.86	41.60	40.78	42.94
2.5	77.40	76.74	72.94	76.58	76.42	73.22	75.76	76.38
3.5	93.94	93.78	90.56	93.66	93.64	90.82	94.28	93.64

References

Lawley, D. N. (1956). A general method for approximating to the distribution of likelihood ratio criteria. *Biometrika 43*(3/4), 295–303.