Online Supplement for “Guaranteeing Performance based on
Time-stability for Energy-efficient Data Centers”
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Theoretical Foundations for Constraint (11) of the Proposed MIP Formulation

Our main claim is that through speed scaling controlled to be proportional to arrival rates, we can guarantee time-stability of queue lengths distribution based on time-homogeneous aggregate workload distribution. To validate our claim, we first consider the following problem, state a theorem, prove it and then finally return to this situation. Let us consider an $M_t/G_t/1$ queue setting has the following conditions: (i) entities arrive according to a piecewise-constant deterministic non-homogeneous Poisson process with cycles such that at time t the arrival rate is Λ_t per unit time, (ii) the amount of work an arriving entity brings is IID (with mean $1/\theta$ and CDF $H(\cdot)$) which can be modeled as a Phase-type service distribution, (iii) a single server uses a processing rate ϕ_t (in appropriate units) at time t , and (iv) thus if a service begins at time t and ϕ_t remains a constant throughout, the mean service time is $1/(\theta\phi_t)$. Based on this, we present the following theorem which enables us to obtain time-stable performance.

Theorem 1 *For some $\rho < 1$, say we choose $\phi_t = \Lambda_t/(\theta\rho)$ for all t , then in a steady state the number of entities in the system for such an $M_t/G_t/1$ queue is stochastically identical to that of a time-homogeneous $M/G/1$ queue with traffic intensity ρ , mean workload $1/\theta$ and workload distribution $H(\cdot)$.*

Proof 1 *Since we assume $\{\Lambda_t, t \geq 0\}$ is a piecewise constant deterministic process with cycles, we use the notation in Section 1.2. Each cycle has T arrival-phases (such that $T < \infty$) represented by the set $\mathcal{T} = \{1, 2, \dots, T\}$. During each arrival-phase, Λ_t remains a constant. For all $\ell \in \mathcal{T}$, the duration of arrival-phase ℓ is τ_ℓ and the entity arrival rate in arrival-phase ℓ is λ_ℓ . In addition,*

we represent $Z(t)$ as the arrival-phase in progress at time t so that the process $\{Z(t), t \geq 0\}$ is a deterministic cyclical process on state space $\mathcal{T}$. Notice that $\Lambda_t = \lambda_{Z(t)}$ for all $t \geq 0$.

Let $A_{Z(t)}$ be the age of the phase $Z(t)$ at time t , i.e. the time since the current phase started. Using $A_{Z(t)}$, we state a lemma that allows us to model a cyclic non-homogeneous Poisson process as a limit to a sequence of Markov-modulated Poisson processes (MMPP). Define for $\nu = 1, 2, 3, \dots$, a sequence of bivariate continuous time Markov chains (CTMC) $\{Z_\nu(t), t \geq 0\}$ on state space given by the Cartesian product $\mathcal{T} \times \{1, \dots, \nu\}$ such that for $\ell = 1, \dots, T-1$, the time to go from state (ℓ, j) to $(\ell, j+1)$ for $0 \leq j < \nu$ and from (ℓ, j) to $(\ell+1, 1)$ for $j = \nu$ are all according to $\exp(\nu/\tau_\ell)$. The remainder of the proof of this theorem follows from Lemma 1 and 2 given below.

Lemma 1 As $\nu \rightarrow \infty$, the CTMC $\{Z_\nu(t), t \geq 0\}$ converges to the bivariate deterministic process $\{(Z(t), A_{Z(t)}), t \geq 0\}$.

Proof 2 For any $\ell \in \mathcal{T}$, we can write down

$$\tau_\ell = \lim_{\nu \rightarrow \infty} \sum_{i=1}^{\nu} Y_i(\nu)$$

where $\{Y_i(\nu), i \geq 1\}$ is a sequence of IID random variables exponentially distributed with mean τ_ℓ/ν . Thus we can model each phase ℓ as a set of ν exponential times back to back with mean τ_ℓ/ν . This gives rise to the CTMC $\{Z_\nu(t), t \geq 0\}$ where the state (ℓ, j) denotes the phase ℓ and the j^{th} $\exp(\nu/\tau_\ell)$ time. Thus taking the limit $\nu \rightarrow \infty$, the lemma is proved.

The CTMC $\{Z_\nu(t), t \geq 0\}$ can be used to modulate an MMPP with parameter λ_ℓ at time t if $Z_\nu(t) = (\ell, j)$ for any $j \in \{1, \dots, \nu\}$. Next we state a lemma that holds for any CTMC that modulates an MMPP which drives the arrival of entities into a queue. For that consider an $MMPP/PH_t/1$ queue where the arrivals are according to an MMPP modulated by a CTMC $\{Y(t), -\infty \leq t \leq \infty\}$ with state space S such that when $Y(t) = j$, the arrival rate is $\lambda_j > 0$. Note that we start at $t = -\infty$, thus, at time 0 the system would be stationary. Recall that we assumed the amount of work can be represented as a phase-type distribution. Thus the time-varying phase-type distribution PH_t corresponds to the amount of work with CDF $H(\cdot)$ which is written as a phase-type distribution and service rate ϕ_t at time t . Let $X(t)$ be the number in the system and $U(t)$ be the phase (of the phase-type) service process is at time t in this $MMPP/PH_t/1$ queue.

Lemma 2 *The bivariate process $\{(X(t), U(t)), t \geq 0\}$ is stationary and independent of $\{Y(t), t \geq 0\}$.*

Proof 3 *Let π_j be the stationary probability for the CTMC $\{Y(t), t \geq 0\}$ and $p_{i,k}$ be the steady-state probability that a homogeneous $M/PH/1$ queue with traffic intensity ρ , mean workload $1/\theta$ and workload distribution $H(\cdot)$, would have i in the system and the service process is in phase k . By writing down the balance equations for the trivariate CTMC $\{(X(t), Y(t), U(t)), t \geq 0\}$, we can show that*

$$\hat{p}_{i,j,k} = p_{i,k}\pi_j$$

satisfies the balance equations, and the norming equation $\sum_i \sum_k p_{i,k} = 1$. Since the solution to the balance equations and norming equations is unique, the stationary probabilities of $\{(X(t), Y(t), U(t)), t \geq 0\}$ are $\hat{p}_{i,j,k}$, i.e.

$$\hat{p}_{i,j,k} = P\{X(t) = i, Y(t) = j, U(t) = k\} = p_{i,k}\pi_j$$

for all $t \geq 0$. Then, summing over all j we get $P\{X(t) = i, U(t) = k\} = p_{i,k}$ and also that $(X(t), U(t))$ is independent of $Y(t)$. Hence the lemma is proved.

In other words, by choosing an appropriate processing rate ϕ_t proportional to the arrival rate Λ_t , we have mapped a non-homogeneous $M_t/G_t/1$ queue-length process into a homogeneous $M/G/1$ queue-length process. To explain this intuitively, one can imagine recording a stationary queueing system. Then, play back the recording but in fast forward at different speeds and slow motion at different speeds for different periods of times. The latter would correspond to a our $M_t/G_t/1$ queue whereas the former would be $M/G/1$. If one were to observe at an arbitrary time in steady state, the probability of seeing i (for some $i \geq 0$) in the system must be the same in both queues.