Generalized Coles' law and outer layer conformal mapping

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ABSTRACT

The time-averaged velocity profile of a turbulent boundary layer can be predicted combining its different trends in the inner and outer regions in a single law of the wake. A new non-dimensional coordinate system that projects the time-averaged velocity profiles of the inner and the outer regions on the same non-dimensional plane is introduced, leading to a unified treatment for the mixing region. In this coordinate system, various laws of the wake are shown to be the same but a constant. The non-dimensionalization is tested on a specific law of the wake, in which the closure coefficients are regressed from wind tunnel measurements and direct numerical simulations of turbulent boundary layers under zero-pressure gradient, over a good range of boundary layer thickness based Reynolds numbers. This data fit produced profiles within 2% of the reference values. This is of practical use to numerical modellers for generating boundary layer inflow profiles.

Keywords: boundary layer model, law of the wake, scaling law, turbulent boundary layer, wake parameter

1 Introduction

The zero-pressure gradient boundary layer is an established test case for improved near-wall velocity profile formulations for turbulent wall-bounded flows. Buschmann and Gad-el-Hak (2007) review the use of scaling laws for turbulent boundary layers, demonstrating that a universally applicable functional is lacking, with neither power law nor logarithmic behaviours being ruled out. Under certain conditions, the boundary layer inner region is described by a logarithmic profile by von Kármán (1954). Its outer region is described by a defect velocity approach. Coles (1956) combined these two analytical descriptions into a single formulation, the law of the wake. Following this approach, several variants of this law have been published by Moses (1964), Finley *et al.* (1966), Granville (1976), Dean (1976), Lewkowicz (1982), and Guo *et al.* (2005).

The state-of-art in that of a law of the wake raises two issues. First, how do the above mentioned formulations compare with each other, and second, why does the law of the wake not remove the presence of two concurrent length scales in the velocity profile, namely the boundary layer thickness δ in the outer region, with its corresponding free-stream velocity u_∞ , and the viscous length scale μ/u_τ in the inner region, with its corresponding friction velocity u_τ , with μ = laminar viscosity. This research addresses both issues by introducing a novel scaling for the outer layer (y^*, u^*), where the defect velocity region and the logarithmic portion of the profile are projected on an unified non-dimensional reference system. The scaling depends on the value of the von Kármán parameter k and the smooth wall parameter B . Krogstad *et al.* (1992) determined k and B empirically by regressing measured velocity profiles. Following their approach, this work presents regressed values of k , B , and of the wake parameter Π over the boundary layer thickness-based Reynolds number range of $145 \leq R \leq 13030$, where $R = \delta u_\tau / \nu$. These give velocity profiles by the law of the wake of Finley *et al.* (1966) within $\pm 2\%$ of the measurements.

2 Unifying laws of the wake

2.1 Classical formulations

Based on the existence of a region of overlap between the inner region of the turbulent boundary layer, described by the law of the wall

$$u^+ = f(y^+), \quad (1)$$

and its outer region, described by the velocity defect law

$$1 - u/u_\infty = f(\eta), \quad (2)$$

Coles (1956) proposed the additive function

$$u^+ = k^{-1} \ln y^+ + B + \Pi k^{-1} f(\eta), \quad (3)$$

where $u^+ = u/u_\tau$, $y^+ = yu_\tau/\nu$, $\eta = y/\delta$, where y = wall-normal distance, and Π = wake parameter determined by

$$\Pi = 0.5 k(u_\infty^+ - k^{-1} \ln R - B). \quad (4)$$

where $R = \delta u_\tau/\nu$ = non-dimensional boundary layer thickness and $u_\infty^+ = u_\infty/u_\tau$ = normalized free-stream velocity. Commonly used formulations for the velocity defect law are reported in Table 1.

Table 1 Classical velocity defect law

Reference	$f(\eta)$	Eq.
Coles (1956)	$1 - \cos(\pi\eta)$	(5)
Coles's polynomial (White 1991)	$2\eta^2(3-2\eta)$	(6)
Lewkowicz (1982)	$2(3\eta^2 - 2\eta^3) - \eta\Pi^{-1}(1-\eta)(1-2\eta)$	(7)
Moses (1964)	$2(3\eta^2 - 2\eta^3) - \eta^3(3\Pi)^{-1}$	(8)
Finley <i>et al.</i> (1966)	$(\Pi^{-1} + 6)\eta^2 - (\Pi^{-1} + 4)\eta^3$	(9)

The defect laws by Coles (1956), Coles' polynomial defect law reported by White (1991), and the defect laws by Lewkowicz (1982) and Finley *et al.* (1966) satisfy

$$u = u_\infty \text{ at } y = \delta \quad (10)$$

The defect laws by Lewkowicz (1982), Moses (1964), and Finley *et al.* (1966) satisfy

$$\partial u / \partial y (y = \delta) = 0 \quad (11)$$

In an actual turbulent boundary layer, the mean velocity profile approaches Eqs. (10) and (11) asymptotically, with a rate that is strongly influenced by the free-stream turbulence level (Gostelow 2009). It is established practice to define δ in experiments at about $0.99u_\infty$ and to neglect the small velocity defect above it. For instance, Granville (1976) used $0.9969u_\infty$ and

$0.998u_\infty$ for fitting his law of the wake to the experimental data. The same simplification is adopted in the data fitting of section 3, by which the velocity defect above a reference δ is set to zero and δ is taken to match a desired finite input value.

2.2 Conformal mapping approach

Let the solution of

$$(\partial u^+ / \partial y^+) = (ky^+)^{-1}, \quad (12)$$

with u^+ defined by Eqs. (3) and (9), be y_{crit}^+ . Substituting Eq. (12) into Eq. (3) and using $f(\eta)$ from Eq. (9), then

$$y_{crit}^+ = R(I\Gamma + 1/6) / (I\Gamma + 1/4), \quad (13)$$

with y_{crit}^+ = point where the distance of Eq. (3) from the logarithmic law of the wall $u^+ = k^{-1} \ln y^+ + B$ is maximum (Jones *et al.* 2001). Its physical meaning may be expressed as the location where the effect of the dominant outer layer mixing over that of the inner layer-driven viscous dissipation is a maximum. In other words, the mean velocity profile at y_{crit}^+ is essentially determined by the convective processes in the outer layer and the influence of viscosity is weak.

Consider the transformation

$$(y^* = y^+ / y_{crit}^+, u^* = u^+ - k^{-1} \ln y_{crit}^+), \quad (14)$$

For u^* to be real, $y_{crit}^+ > 0$. Imposing $y_{crit}^+ > 0$ in Eq. (13) gives $I\Gamma > -1/6$. Equation (3), expressed in terms of the new variables (y^*, u^*) from Eq. (14), with $f(\eta)$ from Eq. (9), is

$$u^* = k^{-1} \ln y^* + B + k^{-1} (I\Gamma + 1/6)^2 (I\Gamma + 1/4)^{-2} (y^*)^2 (3 - 2y^*). \quad (15)$$

Using Eqs. (3) and (6) in Eq. (12), the root of the latter is $y_{crit}^+ = R$. Substituting this in Eq. (14) and then in Eq. (3) with Eq. (6) results in

$$u^* = k^{-1} \ln y^* + B + 2I\Gamma k^{-1} (y^*)^2 (3 - 2y^*). \quad (16)$$

Similarly, substituting Eqs. (3) and (8) in Eq. (12) gives $y_{crit}^+ = R I \Gamma / (I\Gamma + 1/12)$. Substituting this in Eq. (14) and then in Eq. (3) with Eq. (8), this becomes

$$u^* = k^{-1} \ln y^* + B + k^{-1} I \Gamma^2 (I\Gamma + 1/12)^{-2} (y^*)^2 (3 - 2y^*). \quad (17)$$

Comparing Eqs. (15), (16) and (17), the transformation of Eq. (14) is seen to project the laws of the wake of Coles, Moses and Finley *et al.* so that, in (y^*, u^*) coordinates, the three laws are identical but a constant. The form of the three laws is that of Coles equation (3), with (y^*, u^*) in place of y^+, u^+ and y/δ .

Having determined a common functional in (y^*, u^*) coordinates enables to see the three laws of the wake as belonging to the same class. It also enables to study what key elements make them different from each other. The three laws use different values of y_{crit}^+ . This suggests that in Coles' law re-stated as Eq. (16), where $y_{crit}^+ = R$, the viscous effects on the mean velocity profile decrease monotonically to the boundary layer edge, at $y^+ = R$, while

momentum transport effects have the opposite trend. In Eq. (15), the re-statement of Finley *et al.*'s law, transport effects over viscous effects reach a maximum below the boundary layer edge R , for instance, $y_{crit}^+ = 0.89R$ for $\Pi=0.5$. In the re-stated Moses' law of Eq. (16), this maximum occurs further down the boundary layer, for instance, at $y_{crit}^+ = 0.86R$ for $\Pi=0.5$.

3 Validation

3.1 Predictions of outer layer velocity profile at various R

Equation (3), with $f(\eta)$ from Eq. (9), was fitted to a range of streamwise velocity data (Spalart 1988, Erm and Joubert 1991, Graaff and Eaton 2000, Österlund 1999) from turbulent boundary layers under zero pressure gradient over the range $145 \leq R \leq 13,030$. The values of u_∞^+ and R are given by the above authors and are reported in Table 2. Note that k , Π , and B are unknown parameters to be determined by regression, as by Krogstad *et al.* (1992).

Table 2 Velocity profiles from experiment and direct numerical simulation

R_θ	R	u_∞^+	Π	k	Symbol
300	145	18.25	0.253	0.390	◦
697	335	20.25	0.227	0.411	*
1003	460	21.5	0.321	0.415	Δ
1430	640	22.4	0.308	0.393	.
2900	1192	24.33	0.357	0.390	◀
3654	1365	25.38	0.575	0.402	×
5200	2000	26	0.490	0.406	▶
12633	4436	28.62	0.635	0.404	□
13000	4770	28	0.484	0.412	◇
22845	8000	30.15	0.615	0.407	+
31000	13030	30	0.383	0.419	☆

Figure 1a shows the non-dimensional mean streamwise velocity defect $(1-u/u_\infty)$ versus non-dimensional wall distance η . The symbols used are given in Table 2 and denote values from experiment and direct numerical simulation. An incremental vertical shift of $(1-u/u_\infty) = 0.50$ was applied to all curves for clarity and the 0 label on the ordinate of Fig. 1a refers to the $R = 145$ curve. The experimental data almost collapse on each fitted curve, suggesting that the curve fit has captured most of the u^+ dependence on δ , u_∞ , u_τ , and R .

The quality of the data fit was further investigated as by Buschmann and Gad-el-Hak (2007) by evaluating the Fractional Difference (FD), defined as

$$FD = 100k(u_a^+ - u_e^+)/(\ln y_i^+) \quad (18)$$

where u_a^+ = value given by Eq. (3), with $f(\eta)$ from Eq. (9), and u_e^+ = corresponding measured value for a given y_i^+ in a discretized velocity profile of N points. Buschmann and Gad-el-Hak (2007) computed FD for a range of exponential and logarithmic laws. Figure 1b shows FD from Eq. (3), with $f(\eta)$ from Eq. (9), following Buschmann and Gad-el-Hak (2007).

Over the range $30 \leq y^+ \leq R$, over which k and Π were optimized, Fig. 1b shows that the fitted velocity data are within $\pm 2\%$ of the measured values. Multiple crossings of the $FD = 0$ line indicate the absence of a systematic u^+ under-prediction or over-prediction over this y^+ range, for all Reynolds numbers tested R . If $y^+ < 30$, the law of the wake (3), with $f(\eta)$ from Eq. (9), does not follow the experimental profile through the laminar sub-layer, where $u^+ =$

y^+ , which is a known feature of the von Kármán law of the wall. This gives a rising FD as $y^+ < 30$, as in most of the logarithmic laws reviewed by Buschmann and Gad-el-Hak (2007). In Fig. 1b, the lighter curves denote the lower reference Reynolds numbers, as in Buschmann and Gad-el-Hak (2007). Over the range $30 \leq y^+ \leq R$, there appears not to be a clear Reynolds number trend in the FD error from the even placement of all curves about the FD = 0 axis.

Figure 1c shows the fitted velocity profiles of Fig. 1a without the vertical shift. In this figure, the symbols of Table 2 label each curve by its reference Reynolds number. Plotting the velocity profiles in defect velocity form shows a good scaling of the outer region, where all curves collapse as $\eta \rightarrow 1$. The curves are shown to satisfy the boundary conditions of Eqs. (10) and (11) as $\eta \rightarrow 1$. At $\eta < 1$, the curves depart among each other, with the higher Reynolds number profiles having a more pronounced concavity and a correspondingly lower velocity defect.

Figure 2a shows the fitted wake parameter Π versus the free-stream Reynolds number, from the regression method by Krogstad *et al.* (1992). A least-squares linear regression is applied to Π , from which its standard deviation is obtained. Assuming the differences in Π are normal, independent, with constant variance, the 50% confidence interval band for Π is plotted in Fig. 2a. A positive correlation between Π and the free-stream Reynolds number is shown in Fig. 2a. The width of the 50% confidence interval band is narrower than the regressed variation of Π over the range $145 \leq R \leq 13030$, indicating that the regressed Π trend is significant.

Using the method by Krogstad *et al.* (1992), a similar linear regression was performed for the fitted von Kármán constant k versus the free-stream Reynolds number, indicating a positive correlation between k and the freestream Reynolds number. This inference is at odds with the inverse correlation for k of Barenblatt *et al.* (2000) and is further mitigated by the relatively large standard deviation for k , namely $0.0095 \leq \sigma^{0.5}(k) \leq 0.011$, as obtained by regression. The significant standard deviation in k motivates further work, as suggested by Barenblatt *et al.* (2000), for a more complex, Reynolds number-dependent scaling law for the logarithmic portion of the velocity profile.

Figure 1 Fitting velocity profiles to measurements over the range $134 \leq R \leq 13,030$ (a) Profiles from experiment and direct numerical simulation in Table 2, lines by Eq. (9), (b) Normalized percentage difference FD between fitted and measured velocity profiles. Low R , light gray lines, high R , black lines, with line patterns as in (a), (c) Predicted velocity defect profiles. Symbols see Table 2

Figure 2 (a) Regressed wake parameter Π versus Reynolds number R , with $\circ$ Π from regression by Krogstad *et al.* (1992), (—) least squares fit, (---) $\pm 50\%$ confidence interval band, (b) Velocity profiles from experiment and direct numerical simulation in coordinates of Eq. (14). Symbols denote time-averaged velocity data at different R , see Table 2

3.2 Velocity profiles in modified logarithmic coordinates

George and Castillo (1997) defined the limit of the overlap region and the outer boundary layer in terms of (y^+, u^+) and $(y/\delta, u/u_\infty)$. Rather than using two different scaling coordinates, time-averaged velocity measurements are here presented using the unified coordinate system (y^*, u^*) from Eq. (14). Just like with (y^+, u^+) and $(y/\delta, u/u_\infty)$, the normalization (y^*, u^*) aims to collapse the experimental velocity data on a single profile, without any zone restriction. y_{crit}^+ in Eq. (14) is univocally defined and a function of the wake parameter and the normalized boundary layer thickness. It is computed from Eq. (13) using R from experiment and direct numerical simulation and using Π regressed following Krogstad *et al.* (1992).

Figure 2b shows the experimental velocity profiles in the logarithmic coordinates of Eq. (14). The projection shows an appreciable data collapse in the logarithmic profile region, over the range $0.2 \leq y^* \leq 1.0$, above the viscous sub-layer that is not modelled by the law of

the wake. Above $y^*=1.0$, the velocity profiles asymptote to different values of the normalized free-stream velocity u_∞^* , as in the conventional (y^+, u^+) scaling (Buschmann and Gad-el-Hak 2007). However, the profiles about $y^*=1.0$ are stacked parallel to one another, following essentially the same trend. This positive result derives from the new scaling (y^*, u^*) that uses R and u_∞^+ as scaling parameters. Since $R = f(u_\tau/\nu, \delta)$ and $u_\infty^+ = u_\infty/u_\tau$, these parameters embed both the inner and the outer scaling of Eqs. (1) and (2), upholding the fundamental notion of the classical inner, outer, and the overlap regions, where the inner scaling of Eq. (1) and the outer scaling of Eq. (2) apply.

Barenblatt *et al.* (2000), Chauhan *et al.* (2005) and Nagib and Chauhan (2008) proved that the value of k and B are Reynolds number dependent. As both are used to define y_{crit}^+ in Eq. (13), their Reynolds number-dependence affects the data collapse. Still, the data fit shown in Figs. 1a and 2b by an appropriate regression of k and Π is useful for modelling boundary layers at Reynolds numbers commonly encountered in engineering practice.

4 Conclusions

A new non-dimensional coordinate system (y^*, u^*) has been defined, which projects the time-averaged velocity profiles of the inner and outer regions of a turbulent boundary layer on the same non-dimensional plane. This enables a unified treatment of these two regions. The three laws of the wake considered herein are identical but a constant. This transformation highlights the commonality between these laws, offering a unified coordinate system in which their dependence on the common scaling parameter y_{crit}^+ was studied. A preliminary interpretation of the physical meaning of y_{crit}^+ is given, based on the known flow physics in the turbulent boundary layer outer region. Through conformal mapping, the variety of different analytical approximations for a turbulent velocity profile in the literature may be ordered, albeit this is currently limited to laws of the wake of logarithmic form. This may help the on-going search for a physics-based unified analytical description of the time-averaged turbulent boundary layer velocity profile.

The law of the wake of Finley *et al.* was used to regress experimental mean velocity profiles over a relatively wide Reynolds number range of $145 \leq R \leq 13,030$. By optimizing the law of the wake closure coefficients k and Π using an appropriate norm, the formulation shows to be a good fit to the measurements if $y^+ \geq 30$, with a fractional difference error of $\pm 2\%$ that is comparable to previous work.

Tabulated values of k and Π enable the practical use of the fitted laws of the wake as input to computational fluid dynamic simulations at the test Reynolds numbers. The benefits of using this approach is that R and u_∞^+ , which are typically used in computational fluid dynamics to specify a desired wall-bounded inflow state, can be matched explicitly rather than by iteration. The correlations of the von Kármán constant and the wake parameter with the free-stream Reynolds number display a significant standard deviation for k . This requires attention if attempting to interpolate k and Π to obtain time-averaged velocity profiles at Reynolds number different than these tested.

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Notations

B Logarithmic law constant

k	von Kármán constant
N	Number of points in experimental velocity profiles
R	Reynolds number
u	Streamwise velocity component
u_τ	Wall friction velocity
y	Wall-normal distance
δ	Boundary layer thickness
θ	Momentum thickness
ν	Kinematic viscosity
Π	Wake parameter

Subscripts

a	Analytical prediction
$crit$	Maximum difference value between log-law and law of the wake
e	Measured value
∞	Free-stream value

Superscripts

$+$	Normalized value in wall units
$*$	Normalized value by Eq. (14)

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