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Functional central limit theorems and $P(\phi)_1$ -processes for the relativistic and non-relativistic Nelson models

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Abstract We construct $P(\phi)_1$ -processes indexed by the full time-line, separately derived from the functional integral representations of the relativistic and non-relativistic Nelson models in quantum field theory. These two cases differ essentially by sample path regularity. Associated with these processes we define a martingale which, under an appropriate scaling, allows to obtain a central limit theorem for additive functionals of these processes. We discuss a number of examples by choosing specific functionals related to particle-field operators.

Key-words: relativistic and non-relativistic Nelson models, relativistic Schrödinger operators, ground states, Feynman-Kac representations, jump processes and diffusions, functional central limit theorems

2010 MS Classification: 47D07, 47D08, 60J75, 81Q10

1 Introduction

1.1 Definition of the non-relativistic and relativistic Nelson Hamiltonian

Nelson's model of an electrically charged spinless quantum mechanical particle coupled to a scalar boson field [30,31] has been much studied recently in rigorous quantum field theory. In this paper we are interested in constructing Markov processes associated with both the relativistic and non-relativistic Nelson models, and studying some properties of both models. These properties will be formulated in terms of central limit theorem-type behaviours of functionals of the particle-field operators. Our main contribution is a technical development to quantum field theory of Lévy or Lévy-type processes in infinite dimensions which, to the best of our knowledge, has not been attempted before.

The Nelson Hamiltonian is defined by a self-adjoint operator of the form

$$H_N = H_p \otimes \mathbb{1} + \mathbb{1} \otimes H_f + H_i, \quad (1)$$

on the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F}_b,$$

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where $\mathcal{F}_b = \oplus_{n=0}^{\infty} L_{\text{sym}}^2(\mathbb{R}^{dn})$ denotes the boson Fock space over $L^2(\mathbb{R}^d)$. Denote $\mathcal{F}_b^{(n)} = L_{\text{sym}}^2(\mathbb{R}^{dn})$. The Fock space can be identified with the space of ℓ_2 -sequences $(\psi^{(n)})_{n \in \mathbb{N}}$ such that $\psi^{(n)} \in \mathcal{F}_b^{(n)}$ and $\|\psi\|_{\mathcal{F}_b}^2 = \sum_{n=0}^{\infty} \|(\psi^{(n)})\|_{\mathcal{F}_b^{(n)}}^2 < \infty$. We denote the annihilation and creation operators by $a(f)$ and $a^*(f)$, $f \in L^2(\mathbb{R}^d)$, respectively, satisfying the canonical commutation relations

$$[a(f), a^*(g)] = (\bar{f}, g)\mathbb{1}, \quad [a(f), a(g)] = 0 = [a^*(f), a^*(g)]$$

on a dense domain of $\mathcal{F}_b$.

The components of H_N describe the Hamilton operators of the quantum mechanical particle, free field, and particle-field interaction, respectively.

Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a function describing the charge distribution of the particle, denote by $\hat{\varphi}$ its Fourier transform, and write $\tilde{\varphi}(k) = \hat{\varphi}(-k)$. For every $x \in \mathbb{R}^d$, define

$$H_i(x) = \frac{1}{\sqrt{2}} \left(a^*(\hat{\varphi}e^{-ik \cdot x}/\sqrt{\omega}) + a(\tilde{\varphi}e^{ik \cdot x}/\sqrt{\omega}) \right). \quad (2)$$

We define the interaction Hamiltonian $H_i : \mathcal{H} \rightarrow \mathcal{H}$ by the constant fiber direct integral

$$(H_i\Psi)(x) = H_i(x)\Psi(x)$$

for $\Psi \in \mathcal{H}$ such that $\Psi(x) \in D(H_i(x))$, for almost every $x \in \mathbb{R}^d$. Here we use the identification

$$\mathcal{H} \cong \int_{\mathbb{R}^d}^{\oplus} \mathcal{F}_b dx.$$

Formally, this operator is then written as

$$H_i(x) = \int_{\mathbb{R}^d} \frac{1}{\sqrt{2\omega(k)}} \left(\hat{\varphi}(k)e^{-ik \cdot x} a^*(k) + \hat{\varphi}(-k)e^{ik \cdot x} a(k) \right) dk, \quad (3)$$

where $a^*(k)$ and $a(k)$ are formal kernels of the creation and annihilation operators, respectively.

Next denote by

$$d\Gamma(\omega) : \mathcal{F}_b \rightarrow \mathcal{F}_b$$

the second quantization of the multiplication operator

$$\omega(k) = \sqrt{|k|^2 + \nu^2}, \quad \nu \geq 0$$

acting on $L^2(\mathbb{R}^d)$, defined by

$$d\Gamma(\omega)\Phi^{(n)}(k_1, \dots, k_n) = \left(\sum_{j=1}^n \omega(k_j) \right) \Phi^{(n)}(k_1, \dots, k_n).$$

Here ω denotes the dispersion relation, with ν giving the mass of a single boson. We define the free field Hamiltonian by

$$H_f = d\Gamma(\omega),$$

which formally can be written as

$$H_f = \int_{\mathbb{R}^d} \omega(k) a^*(k) a(k) dk. \quad (4)$$

The non-relativistic and relativistic Nelson Hamiltonians differ by the choice of particle Hamiltonian H_p . Consider the operator

$$H_p = (-\hbar^2 c^2 \Delta + m^2 c^4)^{1/2} - mc^2 + V,$$

on $L^2(\mathbb{R}^d)$, where V is a multiplication operator describing the potential, $m \geq 0$ is the rest mass of the particle, and $\hbar, c$ are physical constants (Planck's constant and the speed of light, respectively). We denote the kinetic term by

$$H_p^0 = (-\hbar^2 c^2 \Delta + m^2 c^4)^{1/2} - mc^2.$$

This operator has been much studied in mathematics modelling the (semi-) relativistic operator [3, 8, 17, 35] in quantum theory. We will consider the following variants of this operator in this paper:

massive relativistic case: for $m > 0$ and in units in which $c = \hbar = 1$ we have the massive relativistic Schrödinger operator

$$H_p = H_p^0 + V \quad \text{with} \quad H_p^0 = (-\Delta + m^2)^{1/2} - m \quad (5)$$

massless relativistic case: for $m = 0$ and in units in which $c = \hbar = 1$ we have the massless relativistic Schrödinger operator

$$H_p = H_p^0 + V \quad \text{with} \quad H_p^0 = (-\Delta)^{1/2} \quad (6)$$

non-relativistic case: in the limit $c \rightarrow \infty$ and in units in which $\hbar = m = 1$ we have the non-relativistic Schrödinger operator

$$H_p = H_p^0 + V \quad \text{with} \quad H_p^0 = -\frac{1}{2}\Delta. \quad (7)$$

When H_p is given by (5) or (6), we say that the operator H_N is the relativistic Nelson Hamiltonian, and for (7) we call it non-relativistic Nelson Hamiltonian.

Denote by $\text{Spec}(K)$ the spectrum of a self-adjoint operator K . We will use the following standing assumptions throughout this paper.

Assumption 1 The following conditions hold:

- (i) $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k)$ and $\hat{\varphi}/\sqrt{\omega}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^d)$.
- (ii) The potential V is of Kato-class in the sense of Definition 2 below.
- (iii) For any of the three choices (5)-(7) above, the operator H_p has a unique, strictly positive ground state $\varphi_p \in D(H_p)$, with $H_p \varphi_p = E_p \varphi_p$, $\|\varphi_p\|_{L^2(\mathbb{R}^d)} = 1$, where

$$E_p = \inf \text{Spec}(H_p).$$

- (iv) For any of the three choices (5)-(7), the operator H_N has a unique, strictly positive ground state $\varphi_g \in D(H_N)$ with $H_N \varphi_g = E \varphi_g$, $\|\varphi_g\|_{\mathcal{H}} = 1$, and $\||x|^{d+1} \varphi_g\| < \infty$, where

$$E = \inf \text{Spec}(H_N).$$

Remark 1 Using (i) of Assumption 1 it is seen that H_i is symmetric, and infinitesimally small with respect to $H_p \otimes \mathbb{1} + \mathbb{1} \otimes H_f$, thus H_N is self-adjoint on $D(H_p \otimes \mathbb{1} + \mathbb{1} \otimes H_f)$.

Remark 2 If the infrared regular condition $\int_{\mathbb{R}^d} |\hat{\varphi}(k)|^2 / \omega(k)^3 dk < \infty$ is satisfied, then a unique ground state of H_N exists. This was shown in e.g., [2, 12, 13, 19, 34].

For the non-relativistic Nelson Hamiltonian various spectral properties are known. IR divergences are studied in [1, 14, 27, 26], UV divergence in [15, 16, 22, 29–31] and a Gibbs measure associated with the ground state in [4]. So far, most papers have focused on the non-relativistic Nelson model; for a monograph-length discussion see also [25].

Remark 3 The following are known on the spatial decay of the ground state φ_g . For (5) and (7) it is known that

$$\|(e^{\beta|x|} \otimes \mathbb{1})\varphi_g\| < \infty$$

holds when (i) $\lim_{|x| \rightarrow \infty} V(x) = \infty$ or (ii) $\lim_{|x| \rightarrow \infty} V_-(x) - E > 0$, where $V_-(x) = -\min\{0, V(x)\}$; for details we refer to [25]. For (6), when $\lim_{|x| \rightarrow \infty} V(x) = \infty$, exponential decay (8) continues to hold, but when $\lim_{|x| \rightarrow \infty} V(x) - E > 0$, only

$$\||x|^{d+1} \otimes \mathbb{1}\varphi_g\| < \infty$$

is satisfied, see [20, Theorem 5.15] and [8].

Remark 4 Let $\phi(f) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^d} (a^*(k)\hat{f}(k) + a(k)\hat{f}(-k))dk$ be the field operator. It can be seen in a similar way to [18] that $\|e^{\beta\phi(f)^2} \varphi_g\| < \infty$ for $\beta < 1/(2\|f\|)$.

1.2 Main results

Although the models discussed here are defined in terms of self-adjoint operators on a joint particle-field space of functions, for our purposes a Feynman-Kac type approach will be more suitable. Then the related evolution semigroups can be represented in terms of averages over suitable random processes. As it will be seen below in detail, in the relativistic case we have an infinite-dimensional jump process, and in the non-relativistic case an infinite-dimensional diffusion.

Definition 1 (Ground state transform) Let K be a self-adjoint operator acting in $L^2(M, dm)$ with a measurable space (M, m) , and strictly positive ground state ψ of K with $K\psi = \kappa\psi$, where $\kappa = \inf \text{Spec}(K)$. Let M_f denote the multiplication by f . We call the map

$$K \mapsto L_K = M_{\frac{1}{\psi}}(K - \kappa)M_\psi$$

the *ground state transform* of K .

For notational convenience, in what follows we write L_K simply as

$$\frac{1}{\psi}(K - \kappa)\psi.$$

In probability theory this transform is usually called Doob's h -transform. We see that L_K is also self-adjoint on $D(L_K) = \{f \in L^2(M, \psi^2 dm) : f\psi \in D(K)\}$. Informally, a Markov process possessing K as an infinitesimal generator is called a $P(\phi)_1$ -process. The exact definition of $P(\phi)_1$ -processes will be given in Definition 3 below.

We will need below some notations used in probability theory. Let $(X, \mathcal{B}(X), m)$ be a probability space. The expectation of a measurable function $f : X \rightarrow \mathbb{C}$ with respect to m will be denoted by $\mathbb{E}_m[f] = \int_X f(x)dm$. Let $(q_t)_{t \geq 0}$ be an $\mathbb{R}^d$ -valued random process on $(X, \mathcal{B}(X), m^x)$, where m^x , $x \in \mathbb{R}^d$, is a probability measure such that $m^x(q_0 = x) = 1$. In this case we use the notation $\mathbb{E}_m^x$ instead of $\mathbb{E}_{m^x}$. Furthermore, we write $\mathbb{E}^x$ for $\mathbb{E}_m^x$ unless any confusion may arise.

A simple example of a $P(\phi)_1$ -process is the Ornstein-Uhlenbeck process. Let $h = -\frac{1}{2}\Delta + \frac{1}{2}x^2$ be the Hamiltonian of the one-dimensional harmonic oscillator defined on $L^2(\mathbb{R})$. It is known that h has the unique ground state $\psi(x) = \pi^{-\frac{1}{4}}e^{-\frac{x^2}{2}}$, satisfying $h\psi = \frac{1}{2}\psi$. The ground state transform of h is then given by

$$L_h = \frac{1}{\psi}(h - \frac{1}{2})\psi.$$

The operator L_h is self-adjoint acting on $L^2(\mathbb{R}, \psi^2 dx)$, and $\psi^2 dx$ is a probability measure on $\mathbb{R}$. Also, $L_h \mathbb{1} = 0$. The Ornstein-Uhlenbeck process $(y_t)_{t \geq 0}$ is a diffusion process on a suitable probability space, satisfying

$$(f, e^{-tL_h}g)_{L^2(\mathbb{R}, \psi^2 dx)} = \int_{\mathbb{R}} \mathbb{E}^x[\bar{f}(y_0)g(y_t)]\psi^2(x)dx.$$

It is straightforward to see that

$$L_h f = -\frac{1}{2}\Delta f + x \nabla f$$

and $(y_t)_{t \geq 0}$ is the strong solution of the SDE

$$dy_t = y_t dt + dB_t,$$

where $(B_t)_{t \geq 0}$ is Brownian motion.

This can be extended to more general Schrödinger operators H_p acting on $L^2(\mathbb{R}^d)$. Let

$$L_p = \frac{1}{\varphi_p}(H_p - E_p)\varphi_p$$

using the non-relativistic operator (7). Formally we can see that

$$L_p f = -\frac{1}{2}\Delta f - \frac{\nabla \varphi_p}{\varphi_p} \cdot \nabla f.$$

We denote by $(z_t)_{t \geq 0}$ the $P(\phi)_1$ -process associated with H_p on a probability space. $(z_t)_{t \geq 0}$ is a possible solution of the SDE

$$dz_t = \frac{\nabla \varphi_p}{\varphi_p}(z_t)dt + dB_t. \quad (8)$$

For the operators (5)-(6) and more general non-local Schrödinger operators, ground state transformed processes and related stochastic differential equations with jumps have been analysed in [28]. It is, however, in general not trivial to obtain the necessary regularity of φ_p and the existence of a solution of (8) needs a proper study. Nevertheless, in [5, 25] a $P(\phi)_1$ -process associated with non-relativistic Schrödinger operators H_p is constructed, and in [23, 28] for Schrödinger operators. In all these cases we have, with the respective operators L_p , the expressions

$$\begin{aligned} (f, e^{-tL_p}g)_{L^2(\mathbb{R}^d, \varphi_p^2 dx)} &= \int_{\mathbb{R}^d} \mathbb{E}^x[\bar{f}(z_0)g(z_t)]\varphi_p^2(x)dx, \\ (e^{-tL_p}g)(x) &= \mathbb{E}^x[g(z_t)]. \end{aligned} \quad (9)$$

In the relativistic case $(z_t)_{t \in \mathbb{R}}$ is given by a jump process, and in the non-relativistic case by a diffusion. This will be discussed in detail further below, see (13)-(14).

In this paper we consider a $P(\phi)_1$ -process associated with H_N . By a joint use of the ground state-transform of H_p and the Wiener-Segal-Itô isomorphism $\mathcal{F}_b \cong L^2(Q, dG)$ (see Section 2), the components of H_N can be mapped via a unitary map into a suitable space of functions. Then H_N is transformed to the self-adjoint operator

$$H = L_p \otimes \mathbb{1} + \mathbb{1} \otimes \tilde{H}_f + \tilde{H}_i$$

on $L^2(\mathbb{R}^d, \varphi_p^2 dx) \otimes L^2(Q, dG)$. To simplify notation, we write again H_f for $\tilde{H}_f$, and H_i for $\tilde{H}_i$. The free field component H_f gives rise to a Q space-valued Ornstein-Uhlenbeck process $(\xi_t)_{t \in \mathbb{R}}$ on a probability space $(\mathfrak{X}_Q, \mathcal{B}(\mathfrak{X}_Q), \mathcal{G})$ (see (23)), satisfying

$$(\Phi, e^{-tH_f}\Psi) = \mathbb{E}_{\mathcal{G}}[\Phi(\xi_0)\Psi(\xi_t)].$$

The particle operator $L_p = \varphi_p^{-1}(H_p - E_p)\varphi_p$ contributes by a $P(\phi)_1$ -process $(z_t)_{t \geq 0}$ in accordance with (9) explained above. This allows to derive a Feynman-Kac type representation of e^{-tH} involving an infinite-dimensional joint particle-field random process $(z_t, \xi_t)_{t \geq 0}$ generated by

$$H_0 = L_p \otimes \mathbb{1} + \mathbb{1} \otimes H_f. \quad (10)$$

Since the semigroup operators e^{-tH} are not Markovian, the modified process becomes a Markov process only after a ground state transform is performed on H , i.e.,

$$L_N = \frac{1}{\varphi_g}(H - E)\varphi_g.$$

The details will be explained further around (29) below. Our interest in this paper is to construct $P(\phi)_1$ -processes associated with L_N and to derive a counterpart of (9). This leads us then to

Main result 1: *There exists a $P(\phi)_1$ -process associated with L_N .*

This will be stated and proven below in Theorem 1 for the relativistic case and Theorem 2 for the non-relativistic case.

A major difference between the two cases appears in the path regularity (and thus support) properties of the corresponding $P(\phi)_1$ -processes. While in the non-relativistic case we have infinite-dimensional diffusions on a sample space of functions allowing continuous paths, in the relativistic case we have more technical difficulties resulting from paths with jump discontinuities. However, since we are in the context of concrete models, we are able to control the jumps and path oscillations directly instead of using abstract arguments.

1.3 Functional central limit theorem

The classical central limit theorem says that the partial sums

$$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$$

of independently and identically distributed random variables X_i with mean zero and finite variance v weakly converge to a Gaussian random variable with mean zero and variance v . We see that

$$\frac{1}{n} \mathbb{E} \left[\left| \sum_{i=1}^n X_i \right|^2 \right] = v$$

and $\lim_{n \rightarrow \infty} \mathbb{E}[g(S_n)] \rightarrow \mathbb{E}[g(X)]$, where X is a Gaussian random variable with variance v , and g is any measurable function. On the other hand, a functional central limit theorem (FCLT) concerns a random process $(X_t)_{t \geq 0}$ and the limit of

$$\frac{1}{\sqrt{n}} \int_0^{nt} (Lf)(X_s) ds \quad (11)$$

with suitable functions f . Bhattacharya [7, Theorem 2.1] shows that for a stationary and ergodic Markov process $(X_t)_{t \geq 0}$ with self-adjoint generator $-L$ it follows that

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \int_0^{nt} (Lf)(X_s) ds \stackrel{d}{=} 2(f, Lf) B_t, \quad (12)$$

where $(B_t)_{t \geq 0}$ is standard Brownian motion. Here the $\stackrel{d}{=}$ sign means that the random variables at the two sides have the same distribution. Further details will be presented in Section 4 below.

An FCLT for the non-relativistic Nelson model has been first established by Betz and Spohn in a different approach [6]. They have shown that under the Gibbs measure obtained from taking the marginal over the particle-generated component of the path measure, the appropriately scaled process converges in distribution to Brownian motion having reduced diffusion coefficients. This means that the particle increases its effective mass due to the coupling to the boson field. A main observation in their paper is that one can associate a martingale with the functionals (11) of the process, whose long time behaviour can be predicted by using a martingale convergence theorem. This result is originally due to Kipnis and Varadhan [24, Theorem 1.8] for more general Markov processes, see also the ground-breaking work [10], and related problems studied in [7, 9, 11]. For the relativistic Nelson model the problem has not been studied so far. In our approach we use the $P(\phi)_1$ -process to derive FCLT behaviours rather than to construct Gibbs measures obtained after integrating out the quantum field variables. Although this involves working with effectively infinite dimensional processes, it is also more robust, and applicable to a range of other quantum field or solid state physics models.

An offshoot of our construction of $P(\phi)_1$ -processes is that it allows to derive a FCLT for additive functionals of the processes, relating to particle-field operators in the relativistic and non-relativistic Nelson models. We emphasize that such a behaviour results also for the relativistic case. In Section 3 we construct a $P(\phi)_1$ -process $(\tilde{X}_t)_{t \in \mathbb{R}}$, whose self-adjoint generator is $-L_N$ with relativistic H_p , which we use later to study FCLT for $(\tilde{X}_t)_{t \in \mathbb{R}}$. We consider

$$\tilde{F}_t(f) = \int_0^t f(\tilde{X}_s) ds$$

where $f \in D(L_N^{-1/2})$. Here $(\tilde{X}_t)_{t \in \mathbb{R}}$ will be obtained as a stationary and ergodic Markov process with invariant initial measure M . Then we arrive at the second main result of this paper:

Main result 2: *It follows that*

$$\lim_{s \rightarrow \infty} \frac{1}{\sqrt{s}} \tilde{F}_{st}(f) = \varkappa(f) B_t$$

in weak convergence sense, where $\varkappa(f) = 2\|L_N^{-1/2}f\|_{L^2(M)}^2$ and $(B_t)_{t \geq 0}$ is Brownian motion.

This will be rigorously stated in Theorem 3 for both the relativistic and non-relativistic cases.

The remainder of the paper is organized as follows. In Section 2 we briefly discuss the functional integral representations of the non-relativistic and relativistic Nelson models. In Section 3 we construct $P(\phi)_1$ -processes associated with the two models, and present detailed proofs of path regularity and support properties. Section 4 is devoted to proving a FCLT for additive functionals associated with the Nelson models by using the properties of the $P(\phi)_1$ -processes. We show some functionals of special interest for both cases and determine explicitly the variance in the related FCLT.

2 Functional integral representations of the Nelson Hamiltonians

2.1 Functional integral representation of the particle Hamiltonians

Let $V : \mathbb{R}^d \rightarrow \mathbb{R}$ be a Borel-measurable function giving the potential. We denote the multiplication operator defined by V by the same label. The Schrödinger operators (5)-(7) can be defined in the sense of perturbation theory by choosing suitable conditions on V . However, since in this paper we will use methods of path integration, a natural choice is Kato-class, in each case given in terms of the related random processes.

Let $\mathbb{R}^+ = [0, \infty)$ and $d \geq 1$. We will consider the following processes and related path spaces.

Cauchy and relativistic Cauchy processes: Let

$$\tilde{\mathcal{X}} = D(\mathbb{R}^+, \mathbb{R}^d)$$

be the space of càdlàg paths (i.e., continuous functions from right with left limits) on $\mathbb{R}^+$ with values in $\mathbb{R}^d$, and $\mathcal{B}(\tilde{\mathcal{X}})$ the σ -field generated by the cylinder sets of $\tilde{\mathcal{X}}$. The operator $-H_p^0$ in (5) is the infinitesimal generator of a rotationally symmetric relativistic Cauchy process, and in (6) the infinitesimal generator of a rotationally symmetric Cauchy (i.e., 1-stable) process. We will denote by $(b_t)_{t \geq 0}$, $b_t(\omega) = \omega(t)$ for $\omega \in \tilde{\mathcal{X}}$ the coordinate process defined on the probability space $(\tilde{\mathcal{X}}, \mathcal{B}(\tilde{\mathcal{X}}), \tilde{\mathcal{W}}^x)$ with path measure $\tilde{\mathcal{W}}^x$ starting from x at time $t = 0$, and simply $\tilde{\mathcal{W}}$ when $x = 0$. We note that although both of these processes have paths with jump discontinuities, they have important qualitative differences resulting from the fact that the first has a jump measure which decays exponentially, while the second decays polynomially at infinity.

Brownian motion: Let

$$\mathcal{X} = C(\mathbb{R}^+, \mathbb{R}^d)$$

be the space of $\mathbb{R}^d$ -valued continuous functions on $\mathbb{R}^+$. Also, let $(B_t)_{t \geq 0}$ be d -dimensional Brownian motion defined on the probability space $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mathcal{W}^x)$, where $\mathcal{B}(\mathcal{X})$ is the σ -field generated by the cylinder sets of $\mathcal{X}$, and $\mathcal{W}^x$ Wiener measure starting from x at $t = 0$, denoted $\mathcal{W}$ when $x = 0$. Its infinitesimal generator is $-H_p^0$ as in (7).

It is well-known that all three processes have the strong Markov property with respect to their natural filtrations. By using these processes we can define the related spaces for the potentials.

Definition 2 Let $(Z_t)_{t \geq 0}$ be a random process on a probability space $(X, \mathcal{B}(X), m^x)$. We say that $V = V_+ - V_-$ is a *Kato-class potential* with respect to $(Z_t)_{t \geq 0}$ whenever for its positive and negative parts $V_- \in \mathcal{K}^Z$ and $V_+ \mathbb{1}_C \in \mathcal{K}^Z$ for every compact set $C \subset \mathbb{R}^d$ hold, where $f \in \mathcal{K}^Z$ means that

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \mathbb{E}_m^x \left[\int_0^t |f(Z_s)| ds \right] = 0.$$

When $(Z_t)_{t \geq 0} = (B_t)_{t \geq 0}$, we call this space *Kato-class*, and when $(Z_t)_{t \geq 0} = (b_t)_{t \geq 0}$, we call it *relativistic Kato-class*.

By Khasminskii's lemma [25, Lemma 3.37] and its straightforward extension to relativistic Kato-class it follows that the random variables $-\int_0^t V(b_s)ds$ are exponentially integrable for all $t \geq 0$, and thus we can define the Feynman-Kac semigroup

$$T_t f(x) = \mathbb{E}_{\mathcal{W}}^x \left[e^{-\int_0^t V(b_s)ds} f(b_t) \right]$$

Using the Markov property and stochastic continuity of $(b_t)_{t \geq 0}$ it can be shown that $\{T_t : t \geq 0\}$ is a strongly continuous symmetric one-parameter semigroup on $L^2(\mathbb{R}^d)$. The same considerations also hold when the process is Brownian motion. For all three processes respectively, by the semigroup version of Stone's theorem [25, Proposition 3.26] in each case there exists a self-adjoint operator K bounded from below such that $e^{-tK} = T_t$. Using the generator K , we can give a definition to the operators (5)-(7) for Kato-class potentials, however, we keep using the notations H_p for simplicity. In both the non-relativistic and relativistic cases we have then the following Feynman-Kac formulae.

Relativistic case: Let H_p be given by (5) or (6), and consider the related random process $(b_t)_{t \geq 0}$. If V is of relativistic Kato-class, then

$$(f, e^{-tH_p} g)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{W}}^x [\bar{f}(b_0) e^{-\int_0^t V(b_s)ds} g(b_t)] dx. \quad (13)$$

For a proof and many details we refer to [21, Section 4].

Non-relativistic case: Let H_p be given by (7) and consider Brownian motion $(B_t)_{t \geq 0}$. If V is of Kato-class, then

$$(f, e^{-tH_p} g)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{W}}^x [\bar{f}(B_0) e^{-\int_0^t V(B_s)ds} g(B_t)] dx. \quad (14)$$

For a proof we refer to [25, Sections 3.3, 3.6].

From the above we can derive a functional integral representation of $(e^{-tH_p} g)(z)$ as

$$(e^{-tH_p} g)(x) = \mathbb{E}_m^x [e^{-\int_0^t V(X_s)ds} g(X_t)], \quad (15)$$

where $X_t = b_t$ and $m^x = \tilde{\mathcal{W}}^x$ for the relativistic case and $X_t = B_t$, and $m^x = \mathcal{W}^x$ for the non-relativistic case.

2.2 $P(\phi)_1$ -process associated with H_p

Let

$$\tilde{\mathfrak{X}} = D(\mathbb{R}, \mathbb{R}^d)$$

be the space of càdlàg paths on $\mathbb{R}$ with values in $\mathbb{R}^d$, and $\mathcal{B}(\tilde{\mathfrak{X}})$ the σ -field generated by the cylinder sets of $\tilde{\mathfrak{X}}$. Similarly, let

$$\mathfrak{X} = C(\mathbb{R}, \mathbb{R}^d)$$

be the space of $\mathbb{R}^d$ -valued continuous paths on $\mathbb{R}$ and $\mathcal{B}(\mathfrak{X})$ the σ -field generated by the cylinder sets of $\mathfrak{X}$. We will need two-sided processes $(b_t)_{t \in \mathbb{R}}$ and $(B_t)_{t \in \mathbb{R}}$ on $(\tilde{\mathfrak{X}}, \mathcal{B}(\tilde{\mathfrak{X}}), \tilde{\mathcal{W}}^x)$ and $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}), \mathcal{W}^x)$, respectively, i.e., indexed by the time-line $\mathbb{R}$ instead of usually by the semi-axis $\mathbb{R}^+$. In what follows the notations $\tilde{\mathcal{W}}^x$ and $\mathcal{W}^x$, respectively, will be used for the probability measures on either of the measurable spaces $(\tilde{\mathfrak{X}}, \mathcal{B}(\tilde{\mathfrak{X}}))$ or $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}))$, and $(\tilde{\mathfrak{X}}, \mathcal{B}(\tilde{\mathfrak{X}}))$ and $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}))$, respectively.

The properties of the so obtained processes can be summarized as follows; for details we refer to [23]:

- (i) $\tilde{\mathcal{W}}^x(b_0 = x) = 1$, $x \in \mathbb{R}^d$,
- (ii) the increments $(b_{t_i} - b_{t_{i-1}})_{1 \leq i \leq n}$ are independent symmetric relativistic Cauchy (resp. Cauchy) random variables for any $0 = t_0 < t_1 < \dots < t_n$ with $b_t - b_s \stackrel{d}{=} b_{t-s}$ for $t > s > 0$,

- (iii) the increments $(b_{-t_{i-1}} - b_{-t_i})_{1 \leq i \leq n}$ are independent symmetric relativistic Cauchy (resp. Cauchy) random variables for any $0 = -t_0 > -t_1 > \dots > -t_n$ with $b_{-t} - b_{-s} \stackrel{d}{=} b_{s-t}$ for $-t > -s > 0$,
- (iv) the function $\mathbb{R} \ni t \mapsto b_t(\omega) \in \mathbb{R}^d$ is càdlàg for every $\omega \in \tilde{\mathcal{X}}$,
- (v) the random variables b_t and b_s are independent for $t > 0$ and $s < 0$.

An entirely similar construction can be done to obtain two-sided Brownian motion $(B_t)_{t \in \mathbb{R}}$, with simplifications due to path continuity.

Definition 3 ($P(\phi)_1$ -process) Let $(E, \mathcal{F}, P)$ be a probability space and K be a self-adjoint operator on $L^2(E, dP)$, bounded from below. We say that an E -valued random process $(Z_t)_{t \in \mathbb{R}}$ on a probability space $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}}, \mathcal{Q}^z)$ is a $P(\phi)_1$ -process associated with $((E, \mathcal{F}, P), K)$ if the following conditions are satisfied:

- (i) (Initial distribution) $\mathcal{Q}^z(Z_0 = z) = 1$.
- (ii) (Reflection symmetry) $(Z_t)_{t \geq 0}$ and $(Z_t)_{t \leq 0}$ are independent and $Z_t \stackrel{d}{=} Z_{-t}$, $t \in \mathbb{R}$.
- (iii) (Markov property) $(Z_t)_{t \geq 0}$ and $(Z_t)_{t \leq 0}$ are Markov processes with respect to $\sigma(Z_s, 0 \leq s \leq t)$ and $\sigma(Z_s, t \leq s \leq 0)$, respectively. Here $\sigma(Z_s, 0 \leq s \leq t)$ and $\sigma(Z_s, t \leq s \leq 0)$ denote the minimal σ -fields generated by $\{Z_s : 0 \leq s \leq t\}$ and $\{Z_s : t \leq s \leq 0\}$, respectively.
- (iv) (Shift invariance and functional integration) Let $-\infty < t_0 \leq t_1 < \dots \leq t_n < \infty$, $f_j \in L^\infty(E, dP)$, $j = 1, \dots, n-1$, and $f_0, f_n \in L^2(E, dP)$. Then for every $s \in \mathbb{R}$,

$$\int_E \mathbb{E}_{\mathcal{Q}^z} \left[\prod_{j=0}^n f_j(Z_{t_j}) \right] dP(z) = \int_E \mathbb{E}_{\mathcal{Q}^z} \left[\prod_{j=0}^n f_j(Z_{t_j+s}) \right] dP(z) = (\bar{f}_0, e^{-(t_1-t_0)K} f_1 \dots f_{n-1} e^{-(t_n-t_{n-1})K} f_n).$$
(16)

In particular, from the above it follows that for $f, g \in L^2(E, dP)$

$$\begin{aligned} (f, e^{-tK} g)_{L^2(E)} &= \int_E \mathbb{E}_{\mathcal{Q}^z} [\bar{f}(Z_0) g(Z_t)] dP(z), \\ (e^{-tK} g)(z) &= \mathbb{E}_{\mathcal{Q}^z} [g(Z_t)] \end{aligned}$$

hold. Denote

$$dN(y) = \varphi_p^2(y) dy, \quad y \in \mathbb{R}^d. \quad (17)$$

Since by part (iv) of Assumption 1 the function φ_p is square integrable and L^2 -normalized, dN is a probability measure on $\mathbb{R}^d$. Define the unitary operator

$$U_p : L^2(\mathbb{R}^d, dN) \rightarrow L^2(\mathbb{R}^d, dy), \quad f \mapsto \varphi_p f. \quad (18)$$

Recall that $E_p = \inf \text{Spec}(H_p)$. Using that φ_p is strictly positive by the same assumption,

$$L_p = U_p^{-1} (H_p - E_p) U_p, \quad (19)$$

is well-defined and $D(L_p) = \{f \in L^2(\mathbb{R}^d, dN) \mid f \varphi_p \in D(H_p)\}$. Since $e^{-tL_p} \mathbb{1} = \mathbb{1}$ for the identity function $\mathbb{1} \in L^2(\mathbb{R}^d, dN)$, the operator $-L_p$ is the infinitesimal generator of a Markov process.

Proposition 1 ($P(\phi)_1$ -process associated with H_p)

(Relativistic case) Let H_p be either of the relativistic Schrödinger operators (5) or (6), with V in relativistic Kato-class, and consider L_p defined in (19). Then

- (i) there exists a probability measure $\tilde{N}^y$ on the measurable space $(\tilde{\mathcal{X}}, \mathcal{B}(\tilde{\mathcal{X}}))$ such that the coordinate process $(\tilde{Y}_t)_{t \in \mathbb{R}}$ on the probability space $(\tilde{\mathcal{X}}, \mathcal{B}(\tilde{\mathcal{X}}), \tilde{N}^y)$ is a $P(\phi)_1$ -process starting from $y \in \mathbb{R}^d$, associated with $((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), N), L_p)$,
- (ii) the function $t \mapsto \tilde{Y}_t$ is almost surely càdlàg.

(Non-relativistic case) Let H_p be the non-relativistic Schrödinger operator (7), with V in Kato-class, and consider L_p defined in (19). Then

- (i) there exists a probability measure $\mathcal{N}^y$ on the measurable space $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}))$ such that the coordinate process $(Y_t)_{t \in \mathbb{R}}$ on the probability space $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}), \mathcal{N}^y)$ is a $P(\phi)_1$ -process starting from $y \in \mathbb{R}^d$, associated with $((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathbb{N}), L_p)$,
- (ii) the function $t \mapsto Y_t$ is almost surely continuous.

Proof We refer to [23, Theorem 5.1] for the relativistic case, and [25, Theorem 3.106] for the non-relativistic case. $\square$

Proposition 1 yields that

$$(f, e^{-tL_p}g)_{L^2(\mathbb{R}^d, dN)} = \int_{\mathbb{R}^d} \mathbb{E}_{M^y} [\bar{f}(X_0)g(X_t)] dN(y), \quad (20)$$

$$(e^{-tL_p}g)(y) = \mathbb{E}_{M^y} [g(X_t)], \quad (21)$$

where $M = \tilde{\mathcal{N}}$ and $X_t = \tilde{Y}_t$ for the relativistic case, and $M = \mathcal{N}$ and $X_t = Y_t$ for the non-relativistic case. We have functional integrals for H_p and L_p in (15) and (21), respectively. From these expressions it is seen that the infinitesimal generators of the $P(\phi)_1$ -processes above are L_p (while H_p are not Markov generators as they do not preserve full mass).

2.3 Infinite dimensional Ornstein-Uhlenbeck process

Recall $\omega(k) = \sqrt{|k|^2 + \nu^2}$. Let $\mathcal{K}$ be a Hilbert space over $\mathbb{R}$, defined by the completion of $D(1/\sqrt{\omega})$ with respect to the norm determined by $(f, g)_{\mathcal{K}} = (1/2)(\hat{f}/\sqrt{\omega}, \hat{g}/\sqrt{\omega})_{L^2(\mathbb{R}^d)}$, i.e., $\mathcal{K} = \overline{D(1/\sqrt{\omega})}^{\|\cdot\|_{\mathcal{K}}}$. Let $T : \mathcal{K} \rightarrow \mathcal{K}$ be a positive self-adjoint operator with a Hilbert-Schmidt inverse such that $\sqrt{\omega}T^{-1}$ is bounded. Define $\mathcal{K}_n = \overline{\cap_{p=1}^{\infty} D(T^p)}^{\|T^{n/2} \cdot\|_{\mathcal{K}}}$. Identifying $\mathcal{K}_{+2}^* = \mathcal{K}_{-2}$, we construct a triplet $\mathcal{K}_{+2} \subset \mathcal{K} \subset \mathcal{K}_{-2}$. Write $Q = \mathcal{K}_{-2}$, and endow Q with its Borel σ -field $\mathcal{B}(Q)$. Let $\xi(f) = ((\xi_t, f))$ for $f \in \mathcal{K}_{+2}$, where $((\cdot, \cdot))$ denotes the pairing between Q and $\mathcal{K}_{+2}$. We can construct a probability measure G on $(Q, \mathcal{B}(Q))$ such that

$$\mathbb{E}_G[\xi(f)\xi(g)] = (f, g)_{\mathcal{K}}. \quad (22)$$

Note that by (22) every $\xi(f)$ can be uniquely extended to test functions $f \in \mathcal{K}$, which for simplicity we will denote in the same way. Thus we are led to the probability space $(Q, \mathcal{B}(Q), G)$. We will see that $\mathcal{F}_b$ is unitary equivalent to $L^2(Q, dG)$ in Section 2.4 below. Consider the set

$$\mathfrak{X}_Q = C(\mathbb{R}, Q)$$

of continuous functions on $\mathbb{R}$, with values in Q , and denote its Borel σ -field by $\mathcal{B}(\mathfrak{X}_Q)$. We can define a Q -valued Ornstein-Uhlenbeck process $(\xi_t)_{t \in \mathbb{R}}$, $\mathbb{R} \ni t \mapsto \xi_t \in Q$, on the probability space

$$(\mathfrak{X}_Q, \mathcal{B}(\mathfrak{X}_Q), \mathcal{G}) \quad (23)$$

with a probability measure $\mathcal{G}$. Let $\xi_t(f) = ((\xi_t, f))$ for $f \in \mathcal{K}_{+2}$. For every $t \in \mathbb{R}$ and f we have that $\xi_t(f)$ is a Gaussian random variable with mean zero and covariance

$$\mathbb{E}_G[\xi_t(f)\xi_s(g)] = (f, e^{-|t-s|\omega}g)_{\mathcal{K}}. \quad (24)$$

Note that by (24) every $\xi_t(f)$ can be uniquely extended to test functions $f \in \mathcal{K}$, which for simplicity we will denote in the same way. The notation $\mathcal{G}^{\xi}(\cdot) = \mathcal{G}(\cdot | \xi_0 = \xi)$ used below describes the regular conditional probability measure obtained by conditioning the Gaussian measure under the indicated event.

2.4 Functional integral representation for Nelson's Hamiltonian

To obtain a path integral representation of the Nelson Hamiltonian, first we need a representation of H_N in a suitable function space. The particle operator H_p (any of the three above) acting on $L^2(\mathbb{R}^d)$ can be treated by a ground state transform as in (19). To transform the boson field operators, we use the Wiener-Itô-Segal isomorphism

$$U_f : \mathcal{F}_b \rightarrow L^2(Q, dG), \quad U_f \phi(f) U_f^{-1} = \xi(f).$$

Here $\phi(f)$ is the field operator defined in Remark 4. In combination with (17), define the product measure

$$P = N \otimes G, \quad (25)$$

which is a probability measure on the product space

$$(\mathbb{M}, \mathcal{B}_{\mathbb{M}}) = (\mathbb{R}^d \times Q, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(Q)).$$

The map $U_p \otimes U_f : \mathcal{H} \rightarrow L^2(\mathbb{M}, dP)$ establishes a unitary equivalence between $L^2(\mathbb{M}, dP)$ and $\mathcal{H}$, and we make the identification

$$\mathcal{H} \cong L^2(\mathbb{M}, dP). \quad (26)$$

For convenience, hereafter we write simply as

$$L^2(\mathbb{M}, dP) = L^2(P), \quad L^2(\mathbb{R}^d, dN) = L^2(N), \quad L^2(Q, dG) = L^2(G).$$

Thus we have $L^2(P) \cong L^2(N) \otimes L^2(G)$. The images of the free field and interaction Hamiltonians on $L^2(P)$ under this unitary map are given by $H_f \mapsto U_f H_f U_f^{-1}$ and $H_i(y) \mapsto U_f H_i U_f^{-1}(y) = \xi(\tilde{\varphi}(\cdot - y))$, where $\tilde{\varphi}$ is the inverse Fourier transform of $\hat{\varphi}/\sqrt{\omega}$. Let $\tilde{H}_f = U_f H_f U_f^{-1}$. We have

$$(\Phi, e^{-t\tilde{H}_f} \Psi)_{L^2(G)} = \mathbb{E}_G[\Phi(\xi_0) \Psi(\xi_t)] = \int_Q \mathbb{E}_{G^\xi}[\Phi(\xi_0) \Psi(\xi_t)] dG(\xi).$$

To simplify the notations, we write again H_f and H_i for the above images. The Nelson Hamiltonian H_N is unitary equivalent with

$$H = L_p \otimes \mathbb{1} + \mathbb{1} \otimes H_f + H_i \quad (27)$$

acting on $L^2(P)$, where L_p is given by (19).

Using the $P(\phi)_1$ -processes $(\tilde{Y}_t)_{t \in \mathbb{R}}$ and $(Y_t)_{t \in \mathbb{R}}$ respectively on $(\tilde{\mathfrak{X}}, \mathcal{B}(\tilde{\mathfrak{X}}), \tilde{\mathcal{N}}^y)$ and $(\mathfrak{X}, \mathcal{B}(\mathfrak{X}), \mathcal{N}^y)$ associated with $((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), N), L_p)$ according to the relativistic case and non-relativistic cases in Proposition 1, we write

$$d\tilde{\mathcal{N}} = dN(y) d\tilde{\mathcal{N}}^y \quad \text{and} \quad d\mathcal{N} = dN(y) d\mathcal{N}^y,$$

respectively. The probability spaces for the joint system in the relativistic and non-relativistic cases respectively are then the product spaces

$$(\tilde{\mathfrak{X}} \times \mathfrak{X}_Q, \mathcal{B}(\tilde{\mathfrak{X}}) \otimes \mathcal{B}(\mathfrak{X}_Q), \tilde{\mathcal{P}}_0) \quad \text{and} \quad (\mathfrak{X} \times \mathfrak{X}_Q, \mathcal{B}(\mathfrak{X}) \otimes \mathcal{B}(\mathfrak{X}_Q), \mathcal{P}_0),$$

where

$$\tilde{\mathcal{P}}_0 = \tilde{\mathcal{N}} \otimes \mathcal{G} \quad \text{and} \quad \mathcal{P}_0 = \mathcal{N} \otimes \mathcal{G}.$$

Define the shift operator $\tau_s : L^2(G) \mapsto L^2(G)$ by $\tau_s \xi(h) = \xi(h(\cdot - s))$. We have then the following functional integral representation for the Nelson Hamiltonian H in $L^2(P)$.

Proposition 2 *Let $\Phi, \Psi \in L^2(P)$ and suppose that $s \leq 0 \leq t$. Then*

$$(\text{relativistic case}) \quad (\Phi, e^{-(t-s)H} \Psi)_{L^2(P)} = \mathbb{E}_{\tilde{\mathcal{P}}_0}[\Phi(\tilde{Y}_s, \xi_s) e^{-\int_s^t \tau_{\tilde{Y}_r} \xi_r(\tilde{\varphi}) dr} \Psi(\tilde{Y}_t, \xi_t)],$$

$$(\text{non-relativistic case}) \quad (\Phi, e^{-(t-s)H} \Psi)_{L^2(P)} = \mathbb{E}_{\mathcal{P}_0}[\Phi(Y_s, \xi_s) e^{-\int_s^t \tau_{Y_r} \xi_r(\varphi) dr} \Psi(Y_t, \xi_t)].$$

Proof For the non-relativistic Nelson model see [25, Theorem 6.2]. For the relativistic case the proof can be made similarly. $\square$

2.5 Positivity improving

For later use we have the following representation formula using the specific process instead of a $P(\phi)_1$ -process.

Proposition 3 *Let $\Phi, \Psi \in L^2(\mathbb{P})$ and $s \leq 0 \leq t$. Then for both the relativistic and non-relativistic cases we have that*

$$(\Phi, e^{-(t-s)H} \Psi)_{L^2(\mathbb{P})} = \int_{\mathbb{M}} \mathbb{E}_{\mathcal{W} \otimes \mathcal{G}}^{(y, \xi)} \left[\Phi(z_s, \xi_s) \varphi_{\mathbb{P}}(z_s) e^{-\int_s^t \tau_{z_r} \xi_r(\tilde{\varphi}) dr} \Psi(z_t, \xi_t) \varphi_{\mathbb{P}}(z_t) e^{-\int_s^t V(z_r) dr} \right] dy \otimes dG.$$

with the respective processes $(z_t)_{t \in \mathbb{R}}$ and path measures.

Proof For the non-relativistic Nelson model see [25, Theorem 6.3]. For the relativistic case a similar proof can be applied, the details are left to the reader. $\square$

Let (E, ν) be a σ -finite measure space and S be a bounded operator on $L^2(E, d\nu)$. Recall that S is called positivity preserving if $(f, Sg)_{L^2(E)} \geq 0$ for any $f \geq 0$ and $g \geq 0$, and positivity improving if $(f, Sg)_{L^2(E)} > 0$ for any $f \geq 0$ and $g \geq 0$ but $f \not\equiv 0$ and $g \not\equiv 0$. It is direct to see that whenever S is positivity improving, $(Sf)(x) > 0$ for a.e. $x \in E$ for every non-negative function $f \geq 0$ ($f \not\equiv 0$). Furthermore, for a semigroup $S = e^{-tK}$ with a given self-adjoint operator bounded from below, we have the following useful fact.

Proposition 4 *Let (E, ν) be a σ -finite measure space, and K a self-adjoint operator on $L^2(E, d\nu)$, bounded from below. Suppose that e^{-tK} is positivity preserving for all $t > 0$ and $\inf \text{Spec}(K)$ is an eigenvalue. Then the following two statements are equivalent:*

- (i) $\inf \text{Spec}(K)$ is a simple eigenvalue with a strictly positive eigenfunction.
- (ii) e^{-tL} is positivity improving for all $t > 0$.

Proof See, e.g., [32, XIII.12]. $\square$

Remark 5 By (iv) of Assumption 1 and Proposition 4 it is seen that e^{-tH_N} is positivity improving. The following result gives a sufficient condition for the positivity improving property of e^{-tH_N} .

Lemma 1 *Let $z_t = b_t$ and $m = \tilde{\mathcal{W}}$ for the relativistic case, and $z_t = B_t$ and $m = \mathcal{W}$ for the non-relativistic case. If there exists $t > 0$ such that*

$$\int_{\mathbb{R}^d} m^x \left(e^{-\int_0^t V(z_s) ds} = 0 \right) dx = 0,$$

then $\{e^{-tH_N} : t \geq 0\}$ is positivity improving. In particular, if $V_+ \in L^1_{\text{loc}}(\mathbb{R}^d)$ and H_N has a ground state, then it is unique.

Proof It suffices to show that $(\Phi, e^{-tH_N} \Psi) > 0$ for every $\Phi, \Psi \geq 0$ such that $\Phi \not\equiv 0 \not\equiv \Psi$. By [25, Theorem 3.55], e^{-tH_P} is positivity improving. It is also known that e^{-tH_f} is positivity improving. A combination of these shows that $e^{-t(H_P^0 \otimes \mathbb{1} + \mathbb{1} \otimes H_f)}$ is also positivity improving. We have

$$(\Phi, e^{-tH_N} \Psi)_{L^2(\mathbb{P})} = \int_{\mathbb{M}} \mathbb{E}_{m \otimes \mathcal{G}}^{(y, \xi)} \left[\Phi(z_0, \xi_0) e^{-\int_0^t \tau_{z_r} \xi_r(\tilde{\varphi}) dr} \Psi(z_t, \xi_t) e^{-\int_0^t V(z_r) dr} \right] dy \otimes dG \geq 0.$$

Suppose that $(\Phi, e^{-tH_N} \Psi)_{L^2(\mathbb{P})} = 0$. We see that

$$\left\| \int_0^t \tau_{z_r} \xi_r(\tilde{\varphi}) dr \right\|_{L^2(Q)} \leq t(\|\hat{\varphi}/\omega\| + \|\hat{\varphi}/\sqrt{\omega}\|)$$

and the measure of the sets

$$\begin{aligned} \text{(relativistic case)} \quad & \left\{ (y, \omega, \xi) \in \mathbb{R}^d \times \tilde{\mathcal{X}} \times Q : \left| \int_0^t \tau_{y+\tilde{X}_r(\omega)} \xi_r(\tilde{\varphi}) dr \right| = \infty \right\}, \\ \text{(non-relativistic case)} \quad & \left\{ (y, \omega, \xi) \in \mathbb{R}^d \times \mathcal{X} \times Q : \left| \int_0^t \tau_{y+X_r(\omega)} \xi_r(\tilde{\varphi}) dr \right| = \infty \right\} \end{aligned}$$

are zero. Hence $(\Phi, e^{-tH_N}\Psi)_{L^2(P)} = 0$ implies

$$\int_{\mathbb{M}} \mathbb{E}_{m \otimes \mathcal{G}}^{(y, \xi)} [\Phi(z_0, \xi_0) \Psi(z_t, \xi_t)] dy \otimes dG = 0$$

by the assumption, contradicting that

$$0 < (\Phi, e^{-t(H_p^0 \otimes \mathbb{1} + \mathbb{1} \otimes H_f)} \Psi) = \int_{\mathbb{M}} \mathbb{E}_{m \otimes \mathcal{G}}^{(y, \xi)} [\Phi(z_0, \xi_0) \Psi(z_t, \xi_t)] dy \otimes dG.$$

We conclude that $(\Phi, e^{-tH_N}\Psi)_{L^2(P)} > 0$ and the lemma follows. $\square$

3 $P(\phi)_1$ -processes associated with the Nelson Hamiltonians

3.1 $P(\phi)_1$ -process associated with the relativistic Nelson model

In this section we construct the $P(\phi)_1$ -processes associated with the relativistic and non-relativistic Nelson models. The two cases differ significantly by the sample path regularity (and thus also support) properties, requiring different treatment. First we consider the relativistic Nelson model for the particle operators (5) and (6), thus the process $(b_t)_{t \in \mathbb{R}}$ below will mean either a relativistic Cauchy or a Cauchy process.

Since the ground state φ_g of H is strictly positive and L^2 -normalized, we can define the probability measure

$$dM = \varphi_g^2 dP, \quad (28)$$

on $\mathbb{M}$. Also, consider the unitary operator

$$U_g : L^2(\mathbb{M}, dM) \rightarrow L^2(\mathbb{M}, dP), \quad \Phi \mapsto \varphi_g \Phi,$$

and denote

$$L^2(M) = L^2(\mathbb{M}, dM).$$

Recall that $E = \inf \text{Spec}(H_N)$. We define the ground state-transformed operator by

$$L_N = U_g^{-1}(H - E)U_g \quad (29)$$

with the domain

$$D(L_N) = \{f \in L^2(M) \mid f\varphi_g \in D(H)\} = \{g/\varphi_g \mid g \in D(H)\}.$$

First we make the following observation.

Lemma 2 L_N is self-adjoint and e^{-tL_N} is positivity improving for $t > 0$. In particular, $L_N f$ is real-valued whenever $f \in D(L_N)$ is real-valued.

Proof Since $\varphi_g > 0$ we see that $\text{Ker}(L_N \pm i\mathbb{1}) = \text{Ker}(\varphi_g^{-1}(H - E + \pm i\mathbb{1})\varphi_g) = \text{Ker}(H - E + \pm i\mathbb{1}) = \{0\}$. This implies that L_N is self-adjoint. Let $\Psi, \Phi \in L^2(M)$ and $\Psi, \Phi \geq 0$ such that $\Phi \not\equiv 0 \neq \Psi$. We see that

$$(\Psi, e^{-tL_N}\Phi)_{L^2(M)} = (\varphi_g\Psi, e^{-t(H-E)}\varphi_g\Phi)_{L^2(P)} > 0$$

since $\varphi_g\Psi \geq 0$ and $\varphi_g\Phi \geq 0$. Thus e^{-tL_N} is positivity improving for $t > 0$. Let $f \in D(L_N)$ be a real-valued function. We see that $e^{-tL_N}f$ is also real. Since $L_N f$ equals the derivative of $e^{-tL_N}f$ at $t = 0$, $L_N f$ is also real. $\square$

Let

$$\tilde{\mathfrak{X}}_{\mathbb{M}} = D(\mathbb{R}, \mathbb{M})$$

be the space of càdlàg paths indexed by $\mathbb{R}$, with values in $\mathbb{M}$, and $\tilde{\mathcal{C}}_{\mathbb{M}}$ the σ -field generated by cylinder sets. The following is our first main result.

Theorem 1 (*$P(\phi)_1$ -process associated with the relativistic Nelson Hamiltonian*) *Let $(y, \xi) \in \mathbb{M}$. The following hold.*

- (i) *There exists a probability measure $\tilde{\mathcal{P}}^{(y, \xi)}$ on the measurable space $(\tilde{\mathcal{X}}_{\mathbb{M}}, \tilde{\mathcal{C}}_{\mathbb{M}})$ such that the $\mathbb{M}$ -valued coordinate process $(\tilde{X}_t)_{t \in \mathbb{R}}$ on $(\tilde{\mathcal{X}}_{\mathbb{M}}, \tilde{\mathcal{C}}_{\mathbb{M}}, \tilde{\mathcal{P}}^{(y, \xi)})$ is a $P(\phi)_1$ -process associated with $((\mathbb{M}, \mathcal{B}_{\mathbb{M}}, \mathbb{M}), L_N)$. In particular,*

$$(\Psi, e^{-tL_N} \Phi)_{L^2(\mathbb{P})} = \int_{\mathbb{M}} \mathbb{E}_{\tilde{\mathcal{P}}}^{(y, \xi)} [\Psi(\tilde{X}_0) \Phi(\tilde{X}_t)] d\mathbb{M}, \quad (30)$$

$$(e^{-tL_N} \Phi)(y, \xi) = \mathbb{E}_{\tilde{\mathcal{P}}}^{(y, \xi)} [\Phi(\tilde{X}_t)]. \quad (31)$$

- (ii) *The function $t \mapsto \tilde{X}_t$ has a càdlàg version.*

In order to prove this theorem we need a string of lemmas below. Let $\mathcal{B}_{\mathbb{M}}$ be the family of Borel measurable sets of $\mathcal{M}$. Define the family of set functions $\{\mathcal{M}_\Lambda : \Lambda \subset [0, \infty), \#\Lambda < \infty\}$ on the $\#\Lambda$ -fold product $\mathcal{B}_{\mathbb{M}}^{\#\Lambda} = \times_{j=1}^{\#\Lambda} \mathcal{B}_{\mathbb{M}}$ by

$$\mathcal{M}_\Lambda(A_0 \times A_1 \times \cdots \times A_n) = \left(\mathbb{1}_{A_0}, e^{-(t_1-t_0)L_N} \mathbb{1}_{A_1} \cdots \mathbb{1}_{A_{n-1}} e^{-(t_n-t_{n-1})L_N} \mathbb{1}_{A_n} \right)_{L^2(\mathbb{M})} \quad (32)$$

for $\Lambda = \{t_0, \dots, t_n\}$. It is straightforward to show that the family of set functions $\mathcal{M}_\Lambda$ satisfies the Kolmogorov consistency relation

$$\mathcal{M}_{\{t_0, t_1, \dots, t_{n+m}\}}((\times_{i=0}^n A_i) \times (\times_{i=n+1}^{n+m} \mathbb{M})) = \mathcal{M}_{\{t_0, t_1, \dots, t_n\}}(\times_{i=0}^n A_i), \quad m, n \in \mathbb{N}.$$

Define the projection

$$\pi_\Lambda : \mathbb{M}^{[0, \infty)} \rightarrow \mathbb{M}^\Lambda$$

by $w \mapsto (w(t_0), \dots, w(t_n))$ for $\Lambda = \{t_0, \dots, t_n\}$. Then $\mathcal{A} = \{\pi_\Lambda^{-1}(A) : A \in \mathcal{B}_{\mathbb{M}}^{\#\Lambda}, \#\Lambda < \infty\}$ is a finitely additive family of sets, and the Kolmogorov extension theorem yields that there exists a unique probability measure $\mathcal{M}$ on $(\mathbb{M}^{[0, \infty)}, \sigma(\mathcal{A}))$ such that $\mathcal{M}(\pi_\Lambda^{-1}(A_1 \times \cdots \times A_n)) = \mathcal{M}_\Lambda(A_1 \times \cdots \times A_n)$ for all $\Lambda \subset [0, \infty)$ with $\#\Lambda < \infty$ and $A_j \in \mathcal{B}_{\mathbb{M}}$, and

$$\mathcal{M}_{\{t_0, \dots, t_n\}}(A_0 \times \cdots \times A_n) = \mathbb{E}_{\mathcal{M}} \left[\prod_{j=0}^n \mathbb{1}_{A_j}(Z_{t_j}) \right] \quad (33)$$

holds. Here $(Z_t)_{t \geq 0}$ is an $\mathbb{M}$ -valued coordinate process defined by $Z_t(\omega) = \omega(t)$ for $\omega \in \mathbb{M}^{[0, \infty)}$. From (33) the following result is immediate.

Lemma 3 *The random process $(Z_t)_{t \geq 0}$ is time-shift invariant under $\mathcal{M}$, i.e., for every $f_0, \dots, f_n \in \mathcal{K}$ and $s \geq 0$ it follows that*

$$\mathbb{E}_{\mathcal{M}} \left[\prod_{j=0}^n f_j(Z_{t_j+s}) \right] = \mathbb{E}_{\mathcal{M}} \left[\prod_{j=0}^n f_j(Z_{t_j}) \right] = \left(\bar{f}_0, e^{-(t_1-t_0)L_N} f_1 \cdots f_{n-1} e^{-(t_n-t_{n-1})L_N} f_n \right)_{\mathcal{K}}. \quad (34)$$

Now we prove that $(Z_t)_{t \geq 0}$ has a càdlàg version under $\mathcal{M}$. For this purpose, we need the following technical lemmas which make use of the general ideas in [33, pp 59-62]. Let $I \subset [0, \infty)$ and $\varepsilon > 0$. We say that $Z_t(\omega)$, with fixed ω , has ε -oscillation n times in I if there exist $t_0, t_1, \dots, t_n \in I$ such that $t_0 < t_1 < t_2 < \cdots < t_n$ and $\|Z_{t_j} - Z_{t_{j-1}}\|_{\mathbb{M}} > \varepsilon$ for $j = 1, \dots, n$. We say that $Z_t(\omega)$ has ε -oscillation infinitely often in I , if for every n , $Z_t(\omega)$ has ε -oscillation n times in I . Let

$$\Omega' = \left\{ \omega \in \mathbb{M}^{[0, \infty)} : \lim_{s \in \mathbb{Q}, s \downarrow t} Z_s(\omega) \text{ and } \lim_{s \in \mathbb{Q}, s \uparrow t} Z_s(\omega) \text{ exist in } \mathbb{M} \text{ for all } t \geq 0 \right\},$$

$$A_{N,k} = \left\{ \omega \in \mathbb{M}^{[0, \infty)} : Z_t(\omega) \text{ does not have } \frac{1}{k} - \text{oscillation infinitely often in } [0, N] \cap \mathbb{Q} \right\},$$

$$\Omega'' = \cap_{N=1}^{\infty} \cap_{k=1}^{\infty} A_{N,k}.$$

Similarly as in [33, Lemma 11.2], it is seen that $\Omega'' \subset \Omega'$. Define the event

$$U(p, \varepsilon, I) = \left\{ \omega \in \mathbb{M}^{[0, \infty)} : Z_t(\omega) \text{ has } \varepsilon - \text{oscillation } p \text{ times in } I \right\}.$$

Lemma 4 For every $\varepsilon > 0$ we have $\lim_{|t-s| \rightarrow 0} \mathcal{M}(\|Z_t - Z_s\|_{\mathbb{M}} > \varepsilon) = 0$ and $\mathcal{M}(\Omega'') = 1$.

Proof In this proof we write $\varphi_{\text{pg}}(y, \xi) = \varphi_{\text{p}}(y)\varphi_{\text{g}}(y, \xi)$ and $\mathbb{E}[\dots] = \int_{\mathbb{M}} \mathbb{E}_{\tilde{\mathcal{W}} \times \mathcal{G}}^{(y, \xi)}[\dots] dy \otimes dG(\xi)$ for notational simplicity.

Step 1: Write $B^c((y', \xi'), \varepsilon) = \{(y, \xi) \in \mathbb{M} : \|(y, \xi) - (y', \xi')\|_{\mathbb{M}} > \varepsilon\}$, and consider

$$J_\varepsilon = \sup_{(y, \xi) \in \mathbb{M}} (\tilde{\mathcal{W}}^y \otimes \mathcal{G}^\xi)(Z_{t-s} \in B^c((y, \xi), \varepsilon)) = (\tilde{\mathcal{W}} \otimes \mathcal{G})(Z_{t-s} \in B^c((0, 0), \varepsilon)) = (\tilde{\mathcal{W}} \otimes \mathcal{G})(\|Z_{t-s}\|_{\mathbb{M}} > \varepsilon).$$

By Proposition 3 we have $\mathcal{M}(\|Z_t - Z_s\|_{\mathbb{M}} > \varepsilon) = (\mathbb{1}, e^{-(t-s)L_N} \mathbb{1}_{B^c(0, \varepsilon)})_{L^2(\mathbb{M})}$. Also, by Proposition 2 we have

$$\sup_{(y, \xi) \in \mathbb{M}} |\varphi_{\text{g}}(y, \xi)| = \sqrt{K_2} < \infty \quad \text{and} \quad \sup_{y \in \mathbb{R}^d} \mathbb{E}_{\tilde{\mathcal{W}}}^y \left[e^{-4 \int_0^{t-s} V(b_r) dr} \right] \leq C_V.$$

Hence,

$$\begin{aligned} & \mathcal{M}(\|Z_t - Z_s\|_{\mathbb{M}} > \varepsilon) \\ &= \mathbb{E} \left[\varphi_{\text{pg}}(y, \xi) e^{-\int_0^{t-s} \tau_{b_r} \xi_r(\tilde{\varphi}) dr} e^{-\int_0^{t-s} V(b_r) dr} \varphi_{\text{pg}}(b_{t-s}, \xi_{t-s}) \mathbb{1}_{B^c((y, \xi), \varepsilon)}(b_{t-s}, \xi_{t-s}) \right] e^{(t-s)E}. \end{aligned}$$

The right-hand side can be evaluated as

$$\leq K_2 \mathbb{E} \left[\varphi_{\text{p}}(y) e^{-\int_0^{t-s} \tau_{b_r} \xi_r(\tilde{\varphi}) dr} e^{-\int_0^{t-s} V(b_r) dr} \varphi_{\text{p}}(b_{t-s}) \mathbb{1}_{B^c((y, \xi), \varepsilon)}(b_{t-s}, \xi_{t-s}) \right] e^{(t-s)E}.$$

By using Schwarz inequality twice, we obtain

$$\begin{aligned} & \leq K_2 \|\varphi_{\text{p}}\| \mathbb{E} \left[|\varphi_{\text{p}}(y)|^2 \mathbb{1}_{B^c((y, \xi), \varepsilon)}(b_{t-s}, \xi_{t-s}) \right]^{\frac{1}{4}} \times \\ & \quad \times \mathbb{E} \left[|\varphi_{\text{p}}(y)|^2 e^{-4 \int_0^{t-s} \tau_{b_r} \xi_r(\tilde{\varphi}) dr} e^{-4 \int_0^{t-s} V(b_r) dr} \right]^{\frac{1}{4}} e^{(t-s)E}. \end{aligned}$$

Note that

$$\begin{aligned} & \mathbb{E} \left[|\varphi_{\text{p}}(y)|^2 \mathbb{1}_{B^c((y, \xi), \varepsilon)}(b_{t-s}, \xi_{t-s}) \right]^{\frac{1}{4}} \leq J_\varepsilon^{1/4} \|\varphi_{\text{p}}\|^{1/2}, \\ & \mathbb{E} \left[|\varphi_{\text{p}}(y)|^2 e^{-4 \int_0^{t-s} \tau_{b_r} \xi_r(\tilde{\varphi}) dr} e^{-4 \int_0^{t-s} V(b_r) dr} \right] \\ &= \int_{\mathbb{M}} \mathbb{E}_{\tilde{\mathcal{W}}}^y \left[|\varphi_{\text{p}}(y)|^2 e^{-4 \int_0^{t-s} V(b_r) dr} \right] \mathbb{E}_{\mathcal{G}}^\xi \left[e^{-4 \int_0^{t-s} \tau_{b_r} \xi_r(\tilde{\varphi}) dr} \right] dy \otimes dG, \\ & \mathbb{E}_{\mathcal{G}}^\xi \left[e^{-4 \int_0^{t-s} \tau_{b_r} \xi_r(\tilde{\varphi}) dr} \right] \leq e^{2(t-s)\|\tilde{\varphi}/\omega\|^2} = C_{t-s} < \infty. \end{aligned}$$

Hence we deduce that

$$\begin{aligned} & \mathbb{E} \left[|\varphi_{\text{p}}(y)|^2 e^{-4 \int_0^{t-s} \tau_{b_r} \xi_r(\tilde{\varphi}) dr} e^{-4 \int_0^{t-s} V(b_r) dr} \right] \\ & \leq C_{t-s} \int_{\mathbb{M}} |\varphi_{\text{p}}(y)|^2 \mathbb{E}_{\tilde{\mathcal{W}}}^y \left[e^{-4 \int_0^{t-s} V(b_r) dr} \right] dy \otimes dG \leq C_{t-s} C_V \|\varphi_{\text{p}}\|^2, \end{aligned}$$

and obtain

$$\mathcal{M}(\|Z_t - Z_s\|_{\mathbb{M}} > \varepsilon) \leq K_2 C_V^{1/4} \|\varphi_{\text{p}}\|^2 C_{t-s} J_\varepsilon^{1/4} e^{(t-s)E}. \quad (35)$$

Step 2: We show that $\lim_{|t-s| \rightarrow 0} J_\varepsilon = 0$. In fact, we have

$$J_\varepsilon = (\tilde{\mathcal{W}} \otimes \mathcal{G}) \left(\{\omega = (\omega_1, \omega_2) \in \mathbb{M}^{[0, \infty)} : \|b_{t-s}(\omega_1)\|_{\mathbb{R}^d}^2 + \|\xi_{t-s}(\omega_2)\|_Q^2 > \varepsilon^2\} \right).$$

By the stochastic continuity of the Lévy process $(b_t)_{t \geq 0}$ and the OU-process $(\xi_t)_{t \geq 0}$ we have that $\lim_{|t-s| \rightarrow 0} J_\varepsilon = 0$. Thus $\lim_{|t-s| \rightarrow 0} \mathcal{M}(\|Z_t - Z_s\|_{\mathbb{M}} > \varepsilon) = 0$ follows. To see that $\mathcal{M}(\Omega'')=1$, it suffices to show that $\mathcal{M}(A_{N,k}^c) = 0$ for any fixed N and k . We have

$$\begin{aligned} \mathcal{M}(A_{N,k}^c) &= \mathcal{M}\left(\left\{Z_t \text{ has } \frac{1}{k} - \text{oscillation infinitely often in } [0, N] \cap \mathbb{Q}\right\}\right) \\ &\leq \sum_{j=1}^l \mathcal{M}\left(\left\{Z_t \text{ has } \frac{1}{k} - \text{oscillation infinitely often in } \left[\frac{j-1}{l}N, \frac{j}{l}N\right] \cap \mathbb{Q}\right\}\right) \\ &= \sum_{j=1}^l \lim_{p \rightarrow \infty} \mathcal{M}\left(U\left(p, \frac{1}{k}, \left[\frac{j-1}{l}N, \frac{j}{l}N\right] \cap \mathbb{Q}\right)\right). \end{aligned}$$

We enumerate as $[\frac{j-1}{l}N, \frac{j}{l}N] \cap \mathbb{Q} = \{t_1, t_2, \dots\}$ and $\mathcal{M}\left(U\left(p, \frac{1}{k}, \left[\frac{j-1}{l}N, \frac{j}{l}N\right] \cap \mathbb{Q}\right)\right) = \lim_{n \rightarrow \infty} \mathcal{M}(U_n)$, where $U_n = U\left(p, \frac{1}{k}, \{t_1, \dots, t_n\}\right)$. By using Proposition 3 we obtain

$$\mathcal{M}(U_n) = e^{NE/l} \mathbb{E}\left[\varphi_{\text{pg}}(y, \xi) e^{-\int_0^{N/l} \tau_{b_r} \xi_r(\tilde{\varphi}) dr} \varphi_{\text{pg}}(b_{N/l}, \xi_{N/l}) e^{-\int_0^{N/l} V(b_r) dr} \mathbb{1}_{U_n}(b_{N/l}, \xi_{N/l})\right].$$

Hence in the estimate preceding (35) we have

$$\mathcal{M}(U_n) \leq K_2 C_V^{1/4} \|\varphi_p\|^2 e^{\frac{1}{2} C_{N/l}} \left(\sup_{(y, \xi) \in \mathbb{M}} (\tilde{\mathcal{W}}^y \otimes \mathcal{G}^\xi)(U_n)\right)^{\frac{1}{4}} e^{(N/l)E}. \quad (36)$$

By [33, Lemma 11.4], we furthermore have

$$(\tilde{\mathcal{W}}^y \otimes \mathcal{G}^\xi)(U_n) \leq \left(\sup_{\substack{s, t \in [0, N] \\ t-s \in [0, N/l]}} (\tilde{\mathcal{W}} \otimes \mathcal{G})\left(|(b_s, \xi_s) - (b_t, \xi_t)| \geq \frac{1}{4k}\right)\right)^p. \quad (37)$$

Moreover, by stochastic continuity of $((b_t, \xi_t))_{t \geq 0}$ we have uniform stochastic continuity, i.e.,

$$\sup_{\substack{s, t \in [0, N] \\ t-s \in [0, N/l]}} (\tilde{\mathcal{W}} \otimes \mathcal{G})\left(|(b_s, \xi_s) - (b_t, \xi_t)| \geq \frac{1}{4k}\right) \rightarrow 0 \quad (38)$$

as $l \rightarrow \infty$, which is shown in the next step below.

Step 3: To complete the proof, we show (38); for a version of this see [33]. For notational simplicity we write $X'_s = (b_s, \xi_s)$. Fix $a > 0$. For any t there exists $\delta_t > 0$ such that $(\tilde{\mathcal{W}} \otimes \mathcal{G})(|X'_t - X'_s| \geq \varepsilon/2) \leq a/2$ for $|t-s| < \delta_t$ by stochastic continuity. Let $I_t = (t - \delta_t/2, t + \delta_t/2)$. Since I_t is compact, there exists a finite covering I_{t_j} , $j = 1, \dots, n$, such that $\cup_{j=1}^n I_{t_j} \supset [0, N]$. Let $\delta = \min_{j=1, \dots, n} \delta_{t_j}$. If $|s-t| < \delta$ and $s, t \in [0, N]$, then $t \in I_{t_j}$ for some j , hence $|s-t_j| < \delta_{t_j}$ and

$$(\tilde{\mathcal{W}} \otimes \mathcal{G})(|X'_t - X'_s| \geq \varepsilon) \leq (\tilde{\mathcal{W}} \otimes \mathcal{G})(|X'_t - X'_{t_j}| \geq \varepsilon) + (\tilde{\mathcal{W}} \otimes \mathcal{G})(|X'_{t_j} - X'_s| \geq \varepsilon) < a.$$

Hence the lemma follows. $\square$

Lemma 5 *The process $(Z_t)_{t \geq 0}$ has a càdlàg version with respect to $\mathcal{M}$.*

Proof Let $(Z'_t)_{t \geq 0}$ be a càdlàg process defined by

$$Z'_t(\omega) = \begin{cases} \lim_{s \in \mathbb{Q}, s \downarrow t} Z_s(\omega) & \omega \in \Omega'', \\ 0 & \omega \notin \Omega''. \end{cases} \quad (39)$$

By Lemma 4 the process $(Z_t)_{t \geq 0}$ is stochastically continuous, which implies that there exists a sequence $(s_n)_{n \in \mathbb{N}}$ such that $\lim_{s_n \in \mathbb{Q}, s_n \downarrow t} Z_{s_n}(\omega) = Z_t(\omega)$ for $\omega \in \Omega''' = \mathbb{M}^{[0, \infty)} \setminus N_t$ with a null set N_t . We also see by

the definition of $(Z'_t)_{t \geq 0}$ that $\lim_{s_n \in \mathbb{Q}, s_n \downarrow t} Z_{s_n}(\omega) = Z'_t(\omega)$ for $\omega \in \Omega''$, and $\mathcal{M}(\Omega'') = 1$ by Lemma 4. Hence for every t it follows that $Z_t(\omega) = Z'_t(\omega)$ for $\omega \in \Omega'' \cap \Omega'''$, and $\mathcal{M}(\Omega'' \cap \Omega''') = 1$. Thus $(Z'_t)_{t \geq 0}$ is a càdlàg version of $(Z_t)_{t \geq 0}$. $\square$

We denote the càdlàg version of $(Z_t)_{t \geq 0}$ by $(\bar{Z}_t)_{t \geq 0}$, and the set of $\mathbb{M}$ -valued càdlàg paths by

$$\tilde{\mathfrak{X}}_{\mathbb{M}}^+ = D(\mathbb{R}^+, \mathbb{M}).$$

Note that $(\bar{Z}_t)_{t \geq 0}$ is a random process on the probability space $(\mathbb{M}^{[0, \infty)}, \sigma(\mathcal{A}), \mathcal{M})$, and the map

$$\bar{Z} : (\mathbb{M}^{[0, \infty)}, \sigma(\mathcal{A}), \mathcal{M}) \rightarrow (\tilde{\mathfrak{X}}_{\mathbb{M}}^+, \tilde{\mathcal{C}}^+)$$

is measurable, where $\tilde{\mathcal{C}}^+$ denotes the σ -field generated by cylinder sets. This map induces the image measure $\mathcal{P}_+ = \mathcal{M} \circ \bar{Z}^{-1}$ on $(\tilde{\mathfrak{X}}_{\mathbb{M}}^+, \tilde{\mathcal{C}}^+)$. The coordinate process $(X_t^+)_{t \geq 0}$ on $(\tilde{\mathfrak{X}}_{\mathbb{M}}^+, \tilde{\mathcal{C}}^+, \mathcal{P}_+)$ satisfies $\bar{Z}_t \stackrel{d}{=} X_t^+$ for $t \geq 0$. Let $(y, \xi) \in \mathbb{M}$ and define a regular conditional probability measure on $(\tilde{\mathfrak{X}}_{\mathbb{M}}^+, \tilde{\mathcal{C}}^+)$ by

$$\mathcal{P}_+^{(y, \xi)}(\cdot) = \mathcal{P}_+(\cdot \mid X_0^+ = (y, \xi)).$$

Since the distribution of X_0^+ is $\mathbb{M}$, we see that $\mathcal{P}_+(A) = \int_{\mathbb{M}} \mathbb{E}_{\mathcal{P}_+}^{(y, \xi)}[\mathbb{1}_A] d\mathbb{M}(y, \xi)$. Using $\mathcal{P}_+^{(y, \xi)}$ we have

$$(f, e^{-tL_N} g)_{L^2(\mathbb{M})} = \int_{\mathbb{M}} \mathbb{E}_{\mathcal{P}_+}^{(y, \xi)}[\bar{f}(X_0^+) g(X_t^+)] d\mathbb{M}(y, \xi), \quad (40)$$

$$(e^{-tL_N} g)(y, \xi) = \mathbb{E}_{\mathcal{P}_+}^{(y, \xi)}[g(X_t^+)]. \quad (41)$$

Lemma 6 *The random process $(X_t^+)_{t \geq 0}$ on $(\tilde{\mathfrak{X}}_{\mathbb{M}}^+, \tilde{\mathcal{C}}^+, \mathcal{P}_+^{(y, \xi)})$ has the Markov property with respect to the natural filtration $\sigma(X_s^+, 0 \leq s \leq t)$.*

Proof In this proof we set $z = (y, \xi)$, $z_j = (y_j, \xi_j) \in \mathbb{M}$ for notational simplicity. Let

$$p_t(z, A) = (e^{-tL_N} \mathbb{1}_A)(z), \quad A \in \mathcal{B}_{\mathbb{M}}. \quad (42)$$

Notice that $p_t(z, A) = \mathbb{E}_{\mathcal{P}_+}^z[\mathbb{1}_A(X_t^+)] = \mathbb{E}_{\mathcal{P}_+}[\mathbb{1}_A(X_t^+) \mid X_0^+ = z]$. We show that $p_t(z, A)$ is a probability transition kernel, i.e., (a) $p_t(z, \cdot)$ is a probability measure on $\mathcal{B}(Q)$, (b) the function $z \mapsto p_t(z, A)$ is Borel measurable, and (c) the Chapman-Kolmogorov identity $\int_{\mathbb{M}} p_t(z, A) p_s(x, dz) = p_{s+t}(x, A)$ is satisfied. First note that by (40) it is easy to see that e^{-tL_N} is positivity improving. For every function $f \in \mathcal{H}$ such that $0 \leq f \leq \mathbb{1}$, we have

$$(e^{-tL_N} f)(z) = \mathbb{E}_{\mathcal{P}_+}^z[f(X_t^+)] \leq \mathbb{E}_{\mathcal{P}_+}^z[\mathbb{1}] = 1.$$

Thus $0 \leq e^{-tL_N} f \leq \mathbb{1}$ and $e^{-tL_N} \mathbb{1} = \mathbb{1}$, and (a)-(b) follow. We can also show that the finite dimensional distributions are given by

$$\mathbb{E}_{\mathcal{P}_+} \left[\prod_{j=1}^n \mathbb{1}_{A_j}(X_{t_j}^+) \right] = \int_{\mathbb{M}^n} \prod_{j=1}^n \mathbb{1}_{A_j}(z_j) \prod_{j=1}^n p_{t_j - t_{j-1}}(z_{j-1}, dz_j).$$

This implies (c), and thus $(X_t^+)_{t \geq 0}$ is a Markov process by [25, Proposition 2.17]. $\square$

Now we extend $(X_t^+)_{t \geq 0}$ to a Markov process indexed by the real line $\mathbb{R}$. Consider the product probability space $(\hat{\mathfrak{X}}_{\mathbb{M}}, \hat{\mathcal{C}}, \hat{\mathcal{P}}^{(y, \xi)})$ with $\hat{\mathfrak{X}}_{\mathbb{M}} = \tilde{\mathfrak{X}}_{\mathbb{M}}^+ \times \tilde{\mathfrak{X}}_{\mathbb{M}}^+$, $\hat{\mathcal{C}} = \tilde{\mathcal{C}}^+ \otimes \tilde{\mathcal{C}}^+$ and $\hat{\mathcal{P}}^{(y, \xi)} = \mathcal{P}_+^{(y, \xi)} \otimes \mathcal{P}_+^{(y, \xi)}$, and let $(\hat{X}_t)_{t \in \mathbb{R}}$ be the random process on the product space defined by

$$\hat{X}_t(\omega) = \begin{cases} X_t^+(\omega_1) & t \geq 0, \\ X_{-t}^+(\omega_2) & t \leq 0, \end{cases} \quad \omega = (\omega_1, \omega_2) \in \hat{\mathfrak{X}}_{\mathbb{M}}.$$

Lemma 7 *It follows that*

(i) $\hat{X}_0 = (y, \xi)$, $\hat{\mathcal{P}}^{(y, \xi)}$ -almost surely,

- (ii) $\hat{X}_t, t \geq 0$ and $\hat{X}_s, s < 0$ are independent,
- (iii) $\hat{X}_t \stackrel{d}{=} \hat{X}_{-t}$ for all $t \in \mathbb{R}$,
- (iv) $(\hat{X}_t)_{t \geq 0}$ and $(\hat{X}_t)_{t \leq 0}$ are Markov processes for the filtrations $\sigma(\hat{X}_s, 0 \leq s \leq t)$ and $\sigma(\hat{X}_s, t \leq s \leq 0)$, respectively,
- (v) for $f_0, \dots, f_n \in L^2(\mathbb{M})$ and $-t = t_0 \leq t_1 \leq \dots \leq t_n = t$, we have

$$\mathbb{E}_{\hat{\mathcal{P}}_+} \left[\prod_{j=0}^n f_j(\hat{X}_{t_j}) \right] = \left(\bar{f}_0, e^{-(t_1+t)L_N} f_1 \dots f_{n-1} e^{-(t-t_{n-1})L_N} f_n \right)_{L^2(\mathbb{M})}. \quad (43)$$

Proof (i)-(iii) are straightforward. (iv) follows from Lemma 6. (v) follows from (33) and a simple limiting argument. $\square$

Proof of Theorem 1: We need to show that the random process $(\hat{X}_t)_{t \in \mathbb{R}}$ defined on $(\hat{\mathfrak{X}}_{\mathbb{M}}, \hat{\mathcal{C}}^+, \hat{\mathcal{P}}^{(y, \xi)})$ is a $P(\phi)_1$ -process associated with $((\mathbb{M}, \mathcal{B}_{\mathbb{M}}, \mathbb{M}), L_N)$. The Markov property, reflection symmetry and shift invariance follow from Lemma 7. The càdlàg property of $t \mapsto \hat{X}_t, t \geq 0$, (resp. càglàd property of $t \mapsto \hat{X}_t, t \leq 0$) was shown in Lemma 5. Thus the map $\hat{X} : (\hat{\mathfrak{X}}_{\mathbb{M}}, \hat{\mathcal{C}}^+, \hat{\mathcal{P}}^{(y, \xi)}) \rightarrow (\tilde{\mathfrak{X}}_{\mathbb{M}}, \tilde{\mathcal{C}}_{\mathbb{M}})$ is measurable and the image measure $\tilde{\mathcal{P}}^{(y, \xi)} = \hat{\mathcal{P}}^{(y, \xi)} \circ \hat{X}^{-1}$ defines a probability measure on $(\tilde{\mathfrak{X}}_{\mathbb{M}}, \tilde{\mathcal{C}}_{\mathbb{M}})$. Hence the coordinate process $(\tilde{X}_t)_{t \in \mathbb{R}}$ on $(\tilde{\mathfrak{X}}_{\mathbb{M}}, \tilde{\mathcal{C}}_{\mathbb{M}}, \tilde{\mathcal{P}}^{(y, \xi)})$ satisfies $\tilde{X}_t \stackrel{d}{=} \hat{X}_t$, and is a $P(\phi)_1$ -process associated with $((\mathbb{M}, \mathcal{B}_{\mathbb{M}}, \mathbb{M}), L_N)$. $\square$

3.2 $P(\phi)_1$ -process associated with the non-relativistic Nelson model

Next we consider the non-relativistic case. As before, formulae (28)-(29) hold similarly for the non-relativistic setting. Let now

$$\mathfrak{X}_{\mathbb{M}} = C(\mathbb{R}, \mathbb{M})$$

be the set of continuous paths with values in $\mathbb{M}$ and indexed by the real line $\mathbb{R}$, and $\mathcal{C}_{\mathbb{M}}$ the σ -field generated by the cylinder sets. A second main result of this section is the following.

Theorem 2 (*$P(\phi)_1$ -process associated with the non-relativistic Nelson Hamiltonian*) *Let $(y, \xi) \in \mathbb{M}$. Then the following hold.*

- (i) *There exists a probability measure $\mathcal{P}^{(y, \xi)}$ on the measurable space $(\mathfrak{X}_{\mathbb{M}}, \mathcal{C}_{\mathbb{M}})$ such that the coordinate process $(X_t)_{t \in \mathbb{R}}$ on $(\mathfrak{X}_{\mathbb{M}}, \mathcal{C}_{\mathbb{M}}, \mathcal{P}^{(y, \xi)})$ is a $P(\phi)_1$ -process associated with $((\mathbb{M}, \mathcal{B}_{\mathbb{M}}, \mathbb{M}), L_N)$.*
- (ii) *The function $t \mapsto X_t$ is almost surely continuous.*

The construction involving the counterpart of the probability measure (32) is similar as in Theorem 1, and we leave the details to the reader. We only discuss path continuity, which makes a substantial difference from the previous case. We write $Z_t = (x_t, \xi_t)$, where $x_t \in \mathbb{R}^d$ and $\xi_t \in Q$ are the coordinate processes $x_t(\omega) = \omega_1(t)$ and $\xi_t(\omega) = \omega_2(t)$ for all $t \geq 0$ and $\omega = (\omega_1, \omega_2) \in \mathbb{M}^{[0, \infty)}$.

Lemma 8 *The random process $(Z_t)_{t \geq 0}$ on $(\mathbb{M}^{[0, \infty)}, \sigma(\mathcal{A}))$ has a continuous version.*

Proof Define $\|Z_t\|_{\mathbb{M}} = \sqrt{\|x_t\|_{\mathbb{R}^d}^2 + \|\xi_t\|_Q^2}$. Using the Kolmogorov-Čentsov theorem, the estimate

$$\mathbb{E}_{\mathcal{M}}[\|Z_t - Z_s\|_{\mathbb{M}}^4] \leq D|t - s|^2 \quad (44)$$

with a suitable $D > 0$ implies that $(Z_t)_{t \geq 0}$ has a continuous version. Since $\|Z_t\|_{\mathbb{M}}^4 \leq 2(\|x_t\|_{\mathbb{R}^d}^4 + \|\xi_t\|_Q^4)$, it will suffice to prove the bounds

$$\mathbb{E}_{\mathcal{M}}[\|x_t - x_s\|_{\mathbb{R}^d}^4] \leq D_1|t - s|^2, \quad (45)$$

$$\mathbb{E}_{\mathcal{M}}[\|\xi_t - \xi_s\|_Q^4] \leq D_2|t - s|^2. \quad (46)$$

To obtain (45), we use the formula in Proposition 3 and the moments $\mathbb{E}[|B_t - B_s|^{2n}] = K_n |t - s|^n$, where K_n is a constant. Let $x_t = (x_t^1, \dots, x_t^d)$. We have for all $1 \leq i, j \leq d$ that

$$\begin{aligned} \mathbb{E}_{\mathcal{M}}[(x_t^i)^n (x_s^j)^m] &= ((x^i)^n \varphi_g, e^{-(t-s)(H-E)} (x^j)^m \varphi_g)_{L^2(\mathbb{P})} \\ &= \int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{W} \times \mathcal{G}}^x \left[(B_0^i)^n (B_{t-s}^j)^m \varphi_{\text{pg}}(B_0, \xi_0) \varphi_{\text{pg}}(B_{t-s}, \xi_{t-s}) e^{-\int_0^{t-s} \tau_{B_r} \xi_r(\tilde{\varphi}) dr} e^{-\int_0^{t-s} V(B_r) dr} \right] dx. \end{aligned}$$

Using the eigenvalue equations and the Feynman-Kac formula for the particle and the full Nelson Hamiltonians, it follows that

$$\sup_{x \in \mathbb{R}^d} |\varphi_{\text{p}}(x)| = C_1 < \infty \quad \text{and} \quad \sup_{x \in \mathbb{R}^d} \|\varphi_g(x, \cdot)\|_{L^2(\mathbb{G})} C_2 < \infty. \quad (47)$$

We have

$$\begin{aligned} \mathbb{E}_{\mathcal{M}}[|x_t - x_s|^4] &= \int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{W} \times \mathcal{G}}^x \left[|B_0 - B_{t-s}|^4 \varphi_{\text{pg}}(B_0, \xi_0) \varphi_{\text{pg}}(B_{t-s}, \xi_{t-s}) e^{-\int_0^{t-s} \tau_{B_r} \xi_r(\tilde{\varphi}) dr} e^{-\int_0^{t-s} V(B_r) dr} \right] dx \\ &\leq C_2^2 \left(\int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{W} \times \mathcal{G}}^x \left[|\varphi_{\text{p}}(B_{t-s})|^2 \right] dx \right)^{1/2} \\ &\quad \times \left(\int_{\mathbb{R}^d} |\varphi_{\text{p}}(x)|^2 \mathbb{E}_{\mathcal{W}}^x \left[|B_0 - B_{t-s}|^8 e^{-2 \int_0^{t-s} V(B_r) dr} \mathbb{E}_{\mathcal{G}} \left[e^{-2 \int_0^{t-s} \tau_{B_r} \xi_r(\tilde{\varphi}) dr} \right] \right] dx \right)^{1/2} \end{aligned}$$

and $\mathbb{E}_{\mathcal{G}}[e^{-\int_0^{t-s} \tau_{B_r} \xi_r(\tilde{\varphi}) dr}] \leq e^{\frac{t-s}{2} \|\tilde{\varphi}/\omega\|^2} = C$. Hence we obtain

$$\begin{aligned} \mathbb{E}_{\mathcal{M}}[|x_t - x_s|^4] &\leq C_2^2 C \|\varphi_{\text{p}}\|^2 \left(\mathbb{E}_{\mathcal{W}}^x \left[|B_0 - B_{t-s}|^{16} \right] \right)^{1/4} \left(\mathbb{E}_{\mathcal{W}}^x [e^{-4 \int_0^{t-s} V(B_r) dr}] \right)^{1/4} \\ &\leq C_2^2 C K_8^{1/4} \sup_{x \in \mathbb{R}^d} \left(\mathbb{E}_{\mathcal{W}}^x [e^{-4 \int_0^{t-s} V(B_r) dr}] \right)^{1/4} |t - s|^2. \end{aligned}$$

Thus (45) follows. Next we prove (46). Let $f \in \mathcal{K}_{+2}$. In a similar way as in the proof of (45), we have

$$\mathbb{E}_{\mathcal{M}}[\xi_t(f)^n \xi_s(f)^m] = \int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{W} \times \mathcal{G}}^x \left[\xi_0(f)^n \xi_s(f)^m \varphi_{\text{pg}}(B_0, \xi_0) \varphi_{\text{pg}}(B_{t-s}, \xi_{t-s}) e^{-\int_0^{t-s} \tau_{B_r} \xi_r(\tilde{\varphi}) dr} e^{-\int_0^{t-s} V(B_r) dr} \right] dx.$$

Hence

$$\begin{aligned} \mathbb{E}_{\mathcal{M}}[|\xi_t(f) - \xi_s(f)|^4] &= \int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{W} \times \mathcal{G}}^x \left[|\xi_0(f) - \xi_{t-s}(f)|^4 \varphi_{\text{pg}}(B_0, \xi_0) \varphi_{\text{pg}}(B_{t-s}, \xi_{t-s}) e^{-\int_0^{t-s} \tau_{B_r} \xi_r(\tilde{\varphi}) dr} e^{-\int_0^{t-s} V(B_r) dr} \right] dx \\ &\leq C_2^2 \int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{W}}^x \left[\varphi_{\text{p}}(x) \varphi_{\text{p}}(B_{t-s}) e^{-\int_0^{t-s} V(B_r) dr} \left(\mathbb{E}_{\mathcal{G}} \left[|\xi_0(f) - \xi_{t-s}(f)|^8 \right] \mathbb{E}_{\mathcal{G}} \left[e^{-2 \int_0^{t-s} \tau_{B_r} \xi_r(\tilde{\varphi}) dr} \right] \right)^{1/2} \right] dx. \end{aligned}$$

By Lemma 9 below we have

$$\mathbb{E}_{\mathcal{M}}[|\xi_t(f) - \xi_s(f)|^4] \leq C_2^2 |t - s|^2 \|f\|_{\mathcal{K}_{+2}}^2 C \|\varphi_{\text{p}}\|^2 \sup_{x \in \mathbb{R}^d} \left(\mathbb{E}_{\mathcal{W}}^x [e^{-2 \int_0^{t-s} V(B_r) dr}] \right)^{1/2}$$

and

$$\mathbb{E}_{\mathcal{M}} \left[\frac{|\xi_t(f) - \xi_s(f)|^4}{\|f\|_{\mathcal{K}_{+2}}^2} \right] \leq D_2 |t - s|^2 \quad (48)$$

with a constant D_2 . Since $\|\xi_t - \xi_s\|_Q = \sup_{f \neq 0} |\xi_t(f) - \xi_s(f)| / \|f\|_{\mathcal{K}_{+2}}$, for every $\varepsilon > 0$ there exists $f_\varepsilon \in \mathcal{K}_{+2}$ such that $\|\xi_t - \xi_s\|_Q \leq |\xi_t(f_\varepsilon) - \xi_s(f_\varepsilon)| / \|f_\varepsilon\|_{\mathcal{K}_{+2}} + \varepsilon$. Thus together with (48) we have $\mathbb{E}_{\mathcal{M}}[|\xi_t - \xi_s|^4] - \varepsilon \leq D_2 |t - s|^2$ and (46) follows. $\square$

Lemma 9 *We have $\mathbb{E}_{\mathcal{G}}[|\xi_t(f) - \xi_s(f)|^{2n}] \leq D_n |t - s|^n \|f\|_{\mathcal{K}_{+2}}^n$.*

Proof Since $\mathbb{E}_G[e^{iz(\xi_t(f) - \xi_s(f))}] = e^{-\frac{1}{2}cz^2}$ with $c = (\hat{f}/\sqrt{\omega}, (1 - e^{-|t-s|\omega})\hat{f}/\sqrt{\omega})$, by taking derivatives $2n$ times at $t = 0$ on both sides, we obtain

$$\mathbb{E}_G[|\xi_t(f) - \xi_s(f)|^{2n}] \leq D_n(\hat{f}/\sqrt{\omega}, (1 - e^{-|t-s|\omega})\hat{f}/\sqrt{\omega})^n \leq D_n|t-s|^n \|\hat{f}\|_{L^2}^n \leq D'_n|t-s|^n \|f\|_{\mathcal{K}_{+2}}^n.$$

Here we used that the embedding $i : \mathcal{K}_{+2} \rightarrow L^2(\mathbb{R}^d)$ is bounded [25, p288]. $\square$

4 Functional central limit theorems

4.1 FCLT for additive functionals

In this section we obtain a FCLT for the Nelson Hamiltonians as an immediate consequence. Let $(B_t)_{t \geq 0}$ be one-dimensional Brownian motion.

We need some further concepts from the general theory of random processes, which we recall for convenience. A random process $(Z_t)_{t \in \mathbb{R}}$ is said to be stationary if $(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n})$ has the same distribution as $(Z_{s+t_1}, Z_{s+t_2}, \dots, Z_{s+t_n})$ for all $t_1, \dots, t_n, s \in \mathbb{R}$, and every $n \in \mathbb{N}$. Also, recall that for a given σ -finite measure space (E, ρ) a semigroup $\{P_t : t \geq 0\}$ of bounded operators is called ergodic if for all $f, g \in L^2(E)$ with $f, g \geq 0$ such that $f, g \not\equiv 0$, there is $t \geq 0$ with $(f, P_t g) \neq 0$.

Remark 6 Let $\{e^{-tL} : t \geq 0\}$ be a semigroup of self-adjoint positivity preserving operators. Then the semigroup is ergodic if and only if it is positivity improving.

A fundamental tool used in this section is as follows:

Proposition 5 ([7, Theorem 2.1]) *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space and $(\mathbb{R}, \mathcal{F}_{\mathbb{R}}, \pi)$ a probability space. Let $Z = (Z_t)_{t \geq 0}$ be an $\mathbb{R}$ -valued Markov process with respect to $(\mathcal{F}_t)_{t \geq 0}$. Assume that*

- (i) *f is a non-constant bounded function for $f \in D(L^{-1/2})$, and $\int_{\mathbb{R}} f d\pi = 0$,*
- (ii) *$(Z_t)_{t \geq 0}$ is a stationary ergodic Markov process with a positive self-adjoint generator L on $L^2(\mathbb{R}, d\pi)$,*
- (iii) *π is an invariant initial distribution for $(Z_t)_{t \geq 0}$, i.e., $\mathbb{E}_P[f(Z_t)] = \int_{\mathbb{R}} \mathbb{E}_{P_z}[f(Z_t)] d\pi(z) = \int_{\mathbb{R}} f(z) d\pi(z) = \mathbb{E}_P[f(Z_0)]$ holds.*

Then the random process $\left(\frac{1}{\sqrt{s}} \int_0^s f(Z_r) dr\right)_{t \geq 0}$ is weakly convergent to $(\varkappa(f)B_t)_{t \geq 0}$ as $s \rightarrow \infty$, where $(B_t)_{t \geq 0}$ is Brownian motion and the coefficient is given by $\varkappa(f) = 2\|L^{-1/2}f\|^2$.

We define probability measures on sets $\mathbb{M} \times \tilde{\mathfrak{X}}_{\mathbb{M}}$ and $\mathbb{M} \times \mathfrak{X}_{\mathbb{M}}$ by

$$d\tilde{\mathcal{P}} = d\tilde{\mathcal{P}}^{(y, \xi)} dM(y, \xi) \quad \text{and} \quad d\mathcal{P} = d\mathcal{P}^{(y, \xi)} dM(y, \xi), \quad (49)$$

respectively. We constructed the $P(\phi)_1$ -processes $(\tilde{X}_t)_{t \in \mathbb{R}}$ on $(\tilde{\mathfrak{X}}_{\mathbb{M}}, \tilde{\mathcal{C}}_{\mathbb{M}}, \tilde{\mathcal{P}}^{(y, \xi)})$ for the relativistic case and $(X_t)_{t \in \mathbb{R}}$ on $(\mathfrak{X}_{\mathbb{M}}, \mathcal{C}_{\mathbb{M}}, \mathcal{P}^{(y, \xi)})$ for the non-relativistic case. Both $(\tilde{X}_t)_{t \in \mathbb{R}}$ and $(X_t)_{t \in \mathbb{R}}$ are Markov processes with semigroup $\{e^{-tL_N} : t \geq 0\}$ and transition probability densities

$$\begin{aligned} p_t(z, A) &= (e^{-tL_N} \mathbb{1}_A)(z) = \mathbb{E}_{\tilde{\mathcal{P}}}^z[\mathbb{1}_A(\tilde{X}_t)] = \mathbb{E}_{\tilde{\mathcal{P}}}[\mathbb{1}_A(\tilde{X}_t) | \tilde{X}_0 = z], \\ p_t(z, A) &= (e^{-tL_N} \mathbb{1}_A)(z) = \mathbb{E}_{\mathcal{P}}^z[\mathbb{1}_A(X_t)] = \mathbb{E}_{\mathcal{P}}[\mathbb{1}_A(X_t) | X_0 = z], \end{aligned}$$

with $z = (y, \xi)$ and $A \in \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(Q)$, respectively. We define the additive functionals

$$\tilde{F}_t(f) = \int_0^t f(\tilde{X}_s) ds \quad \text{and} \quad F_t(f) = \int_0^t f(X_s) ds, \quad (50)$$

for f satisfying the conditions in 5. The purpose of this section is to obtain the limits $\tilde{F}_{st}(f)/\sqrt{s} \rightarrow \varkappa(f)B_t$ and $F_{st}(f)/\sqrt{s} \rightarrow \varkappa(f)B_t$ as $s \rightarrow \infty$ in weak sense.

Lemma 10 *A probability measure M associated to the $P(\phi)_1$ -processes $(\tilde{X}_t)_{t \geq 0}$ (resp. $(X_t)_{t \geq 0}$) is an invariant measure in the sense that*

$$\int_{\mathbb{M}} \mathbb{E}_{\tilde{\mathcal{P}}}^{(y, \xi)} [f(\tilde{X}_t)] dM(y, \xi) = \int_{\mathbb{M}} f(y, \xi) dM(y, \xi)$$

(resp. $\int_{\mathbb{M}} \mathbb{E}_{\mathcal{P}}^{(y, \xi)} [f(X_t)] dM(y, \xi) = \int_{\mathbb{M}} f(y, \xi) dM(y, \xi)$), for every $f \in L^1(M)$.

Proof Since 0 is the lowest eigenvalue of L_N , we have

$$\int_{\mathbb{M}} \mathbb{E}_{\tilde{\mathcal{P}}}^{(y, \xi)} [f(\tilde{X}_t)] dM(y, \xi) = (\mathbb{1}, e^{-tL_N} f)_{L^2(M)} = (e^{-tL_N} \mathbb{1}, f)_{L^2(M)} = (\mathbb{1}, f)_{L^2(M)} = \int_{\mathbb{M}} f(y, \xi) dM(y, \xi).$$

Similarly we can see that $\int_{\mathbb{M}} \mathbb{E}_{\mathcal{P}}^{(y, \xi)} [f(X_t)] dM(y, \xi) = \int_{\mathbb{M}} f(y, \xi) dM(y, \xi)$. $\square$

An FCLT for the process $(F_t)_{t \geq 0}$ under $\mathcal{P}$ is now immediate. Using Proposition 5, we have the following result for both the relativistic and non-relativistic Nelson models.

Theorem 3 *Suppose that $f \in D(L_N^{-1})$ and $\int_{\mathbb{M}} f dM = 0$. Let $\varkappa(f) = 2\|L_N^{-1/2} f\|_{L^2(M)}^2$. Then the following holds:*

$$\begin{aligned} \text{(Relativistic case)} \quad & \lim_{s \rightarrow \infty} \frac{1}{\sqrt{s}} \tilde{F}_{st}(f) = \varkappa(f) B_t. \\ \text{(Non-relativistic case)} \quad & \lim_{s \rightarrow \infty} \frac{1}{\sqrt{s}} F_{st}(f) = \varkappa(f) B_t. \end{aligned}$$

Here $(B_t)_{t \in \mathbb{R}}$ is Brownian motion and the convergence is in the weak sense.

Proof We give the argument for the relativistic case only, for the non-relativistic case it is similar. Suppose that $f \in D(L_N^{-1/2})$ is $\mathbb{R}$ -valued. We have that $(\tilde{X}_t)_{t \geq 0}$ is a Markov process with the positive self-adjoint generator L_N . By Lemma 2 e^{-tL_N} , for every $t \geq 0$ is positivity improving. Hence, by Remark 6 the random process $(X_t)_{t \geq 0}$ is ergodic. By Lemma 10, the probability measure M is an invariant measure. Stationarity follows from the reflection symmetry and shift invariance of $P(\phi)_1$ processes. Thus all the assumptions of Proposition 5 are verified and we deduce that $(\tilde{F}_t(f))_{t \geq 0}$ satisfies a functional central limit theorem relative to $\tilde{\mathcal{P}}$, and the variance is given by $2\|L_N^{-1/2} f\|_{L^2(M)}^2$. $\square$

4.2 Examples of diffusion constants

Here we give some examples of functions $f \in D(L_N^{-1/2})$, $f : \mathbb{M} \ni (x, \xi) \mapsto f(x, \xi) \in \mathbb{C}$, $\mathbb{E}_M[f] = 0$, of direct interest in the FCLT. For simplicity we assume $\omega(k) = \sqrt{|k|^2 + \nu^2}$ with $\nu > 0$.

Lemma 11 *Let f be a function on $\mathbb{M}$ such that $f\varphi_g \in L^2(P)$. Suppose that $(\varphi_g, f\varphi_g)_{L^2(P)} = 0$. Then $\mathbb{E}_M[f] = 0$ and $f \in D(L_N^{-1/2})$, equivalently $f\varphi_g \in D((H - E)^{-1/2})$.*

Proof In case $\nu > 0$ it is known [2, 12] that $\inf \text{Spec}(L_N) = 0$ is discrete. Write $\inf(\text{Spec}(L_N) \setminus \{0\}) = \lambda > 0$; then it follows that $\text{Spec}(L_N) \subset \{0\} \cup [\lambda, \infty)$. Let E be the spectral projection of L_N , and split the Hilbert space $L^2(M)$ as

$$L^2(M) = E(\{0\})L^2(M) \oplus E([\lambda, \infty))L^2(M).$$

Since the ground state $\mathbb{1}$ of L_N is unique, $E(\{0\})L^2(M)$ is a one-dimensional subspace spanned by $\mathbb{1}$. Hence $(\varphi_g, f\varphi_g)_{L^2(P)} = 0$ implies that $f \in L^2(M)$ and $(\mathbb{1}, f)_{L^2(M)} = 0$. We have $f \in E([\lambda, \infty))L^2(M)$. Note that $E([\lambda, \infty))L^2(M) \subset D(L_N^{-1/2})$, and $f \in D(L_N^{-1/2})$ follows. Thus $\mathbb{E}_M[f] = (\mathbb{1}, f)_{L^2(M)} = (\varphi_g, f\varphi_g)_{L^2(P)} = 0$. $\square$

We also introduce the notation

$$[f] = f - (\varphi_g, f\varphi_g)_{L^2(P)}.$$

Example 1 Let $f(x, \xi) = [\gamma \cdot x]$. By Remark 3, we have $\varphi_g \in D(\gamma \cdot x)$. We see that $(\varphi_g, [\gamma \cdot x]\varphi_g) = 0$, which implies that $\mathbb{E}_M[[\gamma \cdot x]] = 0$ and $[\gamma \cdot x]\varphi_g \in D(L_N^{-1/2})$ by Lemma 11. We get

$$\kappa([\gamma \cdot x]) = \|(H - E)^{-1/2}[\gamma \cdot x]\varphi_g\|^2.$$

Example 2 Let $f(x, \xi) = \gamma \cdot x$. Let $R_p F(x) = F(-x)$ for $F \in L^2(M)$, and $r_f g(k) = g(-k)$, and consider the second quantization $\Gamma(r_f)$. Put $R = R_p \otimes \Gamma(r_f)$, and suppose that $V(x) = V(-x)$ and $\hat{\varphi}(-k) = \hat{\varphi}(k)$. Then $L_N R F = R L_N F$ for $F \in D(L_N)$. We can see that φ_g also has reflection symmetry, i.e., $R\varphi_g = \varphi_g$. Reflection symmetry implies that

$$(\varphi_g, \gamma \cdot x \varphi_g)_{L^2(P)} = (R\varphi_g, R\gamma \cdot x \varphi_g)_{L^2(P)} = -(\varphi_g, \gamma \cdot x \varphi_g)_{L^2(P)} = 0.$$

This then implies

$$\kappa(\gamma \cdot x) = \|(H - E)^{-1/2}\gamma \cdot x \varphi_g\|^2.$$

Example 3 Let R be the reflection defined in Example 2 and $P_f = \int_{\mathbb{R}^d} k a^*(k) a(k) dk$ be the field momentum. Suppose that $V(x) = V(-x)$ and $\hat{\varphi}(-k) = \hat{\varphi}(k)$. Reflection symmetry implies that

$$(\varphi_g, \gamma \cdot P_f \varphi_g)_{L^2(P)} = (R\varphi_g, R\gamma \cdot P_f \varphi_g)_{L^2(P)} = -(\varphi_g, \gamma \cdot P_f \varphi_g)_{L^2(P)} = 0.$$

This gives

$$\kappa(\gamma \cdot P_f) = \|(H - E)^{-1/2}\gamma \cdot P_f \varphi_g\|^2.$$

A related example has been given in [6].

Example 4 Let $f(x, \xi) = \xi(h)$ with $\|h\|_{L^2(\mathbb{R}^d)} < \infty$ and $\|h/\sqrt{\omega}\|_{L^2(\mathbb{R}^d)} < \infty$. We have $\varphi_g \in D(\xi(h))$ by the estimate $\|\xi(h)F\| \leq (\|h/\sqrt{\omega}\| + \|h\|)\|(H_f + \mathbb{1})^{1/2}F\|$. We see that $(\varphi_g, [\xi(h)]\varphi_g) = 0$, which implies that $\mathbb{E}_M[[\xi(h)]] = 0$ and $[\xi(h)]\varphi_g \in D(L_N^{-1/2})$. We get

$$\kappa([\xi(h)]) = \|(H - E)^{-1/2}[\xi(h)]\varphi_g\|^2.$$

Example 5 Let $f(x, \xi) = (\gamma \cdot x)\xi(h)$. We also see that $\varphi_g \in D((\gamma \cdot x)\xi(h))$ and

$$\kappa([\gamma \cdot x]\xi(h)) = \|(H - E)^{-1/2}[(\gamma \cdot x)\xi(h)]\varphi_g\|^2.$$

Example 6 Let $f(z, \xi) = e^{z\xi(h)}$, where $z \in \mathbb{C}$. By Remark 4 it is seen that $\varphi_g \in D(e^{z\xi(h)})$. We get

$$\kappa([e^{z\xi(h)}]) = \|(H - E)^{-1/2}[e^{z\xi(h)}]\varphi_g\|^2.$$

Example 7 Consider the non-relativistic or massive relativistic Nelson Hamiltonians. Let $f(x, \xi) = e^{\varepsilon|x|}$, where $\varepsilon > 0$ is sufficiently small. By Remark 3 we see that $\varphi_g \in D(e^{\varepsilon|x|})$. We get

$$\kappa([e^{\varepsilon|x|}]) = \|(H - E)^{-1/2}[e^{\varepsilon|x|}]\varphi_g\|^2.$$

Furthermore, under the same assumptions on V and $\hat{\varphi}$ as in Example 2 we see that $\varphi_g \in D(\gamma \cdot x e^{\varepsilon|x|})$ and $(\varphi_g, \gamma \cdot x e^{\varepsilon|x|}\varphi_g)_{L^2(P)} = 0$ by reflection symmetry. We get

$$\kappa(\gamma \cdot x e^{\varepsilon|x|}) = \|(H - E)^{-1/2}\gamma \cdot x e^{\varepsilon|x|}\varphi_g\|^2.$$

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