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RESEARCH ARTICLE

Frictional contact behaviour of the tyre: The effect of tread slip on the in-plane structural deformation and stress field development

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The analysis of the in-plane deformation of the tyre in relation to the frictional contact between the road and the tread is a crucial first step in the understanding of its contribution to the longitudinal dynamics of a vehicle. In this work the physical mechanism of the generation of the two-dimensional contact pressure distribution for a non-rolling tyre is studied. Towards this aim, a physical tyre model is constructed, consisting of an analytical ring under pre-tension, a non-linear sidewall foundation, and a discretised foundation of viscoelastic elements, representing the tread. Tread behaviour is examined first, with focus on the development of shear micro-slip. The tread simulation is enhanced with the combination of radial and tangential tread elements and the benefits of such an approach are identified. Subsequently, the contact of the complete model is examined by implementing an algorithm for transient simulations in the time domain. The effects of the imposed vertical load and sidewall non-linearity on the contact stress and strain fields are identified. The modelling approach is validated by comparison to published experimental results. The physical mechanism that couples the torsional and horizontal/vertical deformations of the carcass with the frictional forces at the tread is identified and discussed in detail. The proposed modelling approach is found appropriate for the description of the development of the two dimensional contact pressure field, as a function of the frictional potential of the contact.

Keywords: transient tyre simulation; tyre ring model; contact pressure distribution; tyre frictional contact; tyre tread model; tyre contact area;

1. Introduction

The tyre-road interaction is a crucial aspect of any vehicle dynamics study, as all the forces defining the handling potential of a vehicle are generated in the developed contact area. The shape/size of the interface area and the vertical pressure distribution largely influence the shear force development of the tyre. The tyre's shape and structural anisotropy impose severe non-linearity on the contact problem, preventing its analytical formulation. As a result, the frictional contact phenomena can only be studied through a numerical modelling approach. Moreover, the effects of the ever-changing contact area, the structural viscoelasticity and the **micro-slip development (shear displacement of tread on the contact surface, locally developed along the contact area)** can only be fully identified when the contact problem is examined from a dynamic point of view.

Historically, a primal contribution to any tyre contact related problem can be attributed to Hertz [1], who analytically derived the contact solution of an elastic cylinder on a frictionless flat surface. Later, Cattaneo [2] and Mindlin [3] proposed

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the superposition of the tangential stress field for elliptical and circular contact areas, respectively. Both identified the co-existence of adhesional and slipping zones within the same contact area, an important remark for the actual tyre-road interaction. The difference between the contact behaviour of a rigid disc body and a shell structure was discussed by Savkoor in [4, 5] and experimentally studied by Pottinger [6].

A significant step towards the accurate simulation of the tyre contact behaviour is the separate representation of the belt and the tread, with the latter one modelled through a discretised stiffness foundation (the so called Winkler bedding, after [7]). Such a foundation was combined with a tyre belt model consisting of an inextensible, cord-reinforced, membrane shell by Akasaka and Kabe in [8, 9]. The resulting two-dimensional contact area development and pressure distribution showed good correlation with experimental work.

The well-established ring on elastic foundation carcass model [10] was implemented in the tyre contact studies of Yamagishi and Jenkins [11] and Akasaka *et al* [12]. In these studies the ring was assumed inextensible and was combined with a linear stiffness tread foundation. In the first study the sidewall foundation was assumed to act only in the radial direction and the contact was frictionless. Typical convex/trapezoidal vertical pressure distributions were predicted. In the second study the sidewall foundation incorporated tangential stiffness while the contact was regarded as frictional, through an analytical harmonic-shaped tangential stress distribution. The experimentally well-established, inwards radial deformation of the belt in the vicinity of the centre of the contact patch and the resulting concave-shaped vertical stress distribution were investigated.

The development of finite element modelling techniques led to numerous contact deformation and stress distribution studies, showing very good correlation with the actual tyre behaviour. Finite element-based tyre contact solutions can be found in [13–16]. Such methods, though, still impose a large computational load, which limits their use to specific application studies and prevents their incorporation in general tyre simulation algorithms.

The complexity of the tyre-road contact problem renders the experimental approach as the most reliable one for obtaining realistic results. Many works have been published on the topic, a review of which can be found in the classic work edited by Clark [17](chap. 5). Although the contact area and its shape can be easily identified through optical observation (a survey of the contact area modelling and measuring techniques can be found in [18]), this is not the case for the stress distribution (and especially the tangential one) which requires advanced measurement techniques. These techniques may be categorized as:

- (1) pressure sensing contact surfaces (implemented in [6, 12, 15, 19, 20])
- (2) light absorption/reflection techniques(e.g. [21–23])
- (3) ultrasonic measurements (e.g. [24])

In the present study, the concept of in-plane frictional contact simulation is thoroughly investigated for a non rolling tyre. The approach is based on the development of a transient contact algorithm, calculating the contact behaviour through a time-domain simulation procedure. In an effort to further advance the analysis presented in the above studies, the contact boundary conditions are applied through a physical loading procedure and the mechanism of interaction between the structural deformation and pressure distribution development is investigated.

The adoption of separate modelling approaches for the belt and the tread zones of the tyre, enables the identification and isolation of the contribution of tread slip to the above mechanism. As a result, the development of horizontal pressure

Figure 1. An overview of the modelling approach, where the belt, sidewall and tread sub-models are presented.

distribution is incorporated in the above interaction, in relation to the frictional behaviour of the contact surface. The associated physical mechanism is used to explain the contradictory results presented in a number of published experimental works where opposite directions of the horizontal stress field are identified. Moreover, the combination of contact conditions and tyre properties that may lead to one direction or another, is discussed.

The above process of **non-rolling** contact development is regarded as the first step towards the simulation of the corresponding frictional interaction in the transition from static to rolling conditions and transient, in general, tyre behaviour.

2. The concept of dynamic contact simulation

The essential elements of a tyre in-plane contact simulation algorithm may be identified as:

- The effect of the vertical load on the length of the contact patch and the development of the vertical and horizontal stress fields.
- The role of the carcass structural deformation.
- The nonlinearity of the radial stiffness of the sidewall.
- The stiffening effect of inflation pressure on the belt and sidewall.
- The structural viscoelasticity (energy dissipation mechanism) of the belt and sidewall tyre zones, introducing the dependency of the contact behaviour on the rate of load application and the stress relaxation process.
 - The viscoelastic behaviour of the tread, related to the dynamic tribological characteristics of the tyre.
 - The formation of slipping/adhesional zones in the contact area.

The tyre belt is modelled using the ring on elastic foundation model, an established in-plane tyre dynamics simulation tool [10, 25, 26]. Its deformation is calculated through the analytical modal expansion method (Appendix B), although the solution is established in the time domain (Appendix C). The belt response, according to this method, is expressed as a dynamic (over time) superposition of a subgroup of certain mode shapes of the model, calculated in advance. The method enables the computational decoupling between the space domain discretisation used for the accurate derivation of the contact excitation and the - reduced in the modal domain - calculation of the response.

The frequently adopted assumption of inextensibility -see [27, 28] and [29](p. 167, 189)- is disregarded in this study for two reasons: Firstly, belt extensibility allows for the simulation of the effects of inflation pressure, which are vital to the contact behaviour, as will be explained later. Secondly, the modal range required to simulate accurately the contact behaviour extends to an associated frequency range in which the tyre cannot be assumed to be inextensible, as will be later

(a) Radial tread element and its contact behaviour. (b) Radial-tangential element and its contact behaviour.

Figure 2. The radial and the radial-tangential bristle elements and their contact deformation.

proved. The extensible ring equations of motion are given in Appendix B.

The effect of the non-linear radial sidewall stiffness is accounted for by a discretised inextensible string connecting the belt to the rim in the radial direction (Appendix D), which, under the inflation induced tension, generates the ring supportive force. The tangential stiffness in this case conserves its linear behaviour and is provided by the ring elastic foundation. The discretisation of the sidewall coincides with that of the belt and tread, so one sidewall and one tread element exist for every belt node.

The structural viscoelasticity of the carcass is incorporated through the viscous damping coefficients of all modes in which the expansion solution is performed (Appendix C). The process physically corresponds to a distributed viscous damping mechanism, parallel to the stiffness one.

The tread is represented by a combination of discretised radial and tangential Voigt elements. The concept originates from the "brush" models [30], but here the belt deformation imposes a contact angle different from vertical. A friction law is applied on the elements in-contact, governing their micro-slip motion. A similar tread representation, including only the stiffness property, was presented in [26], where a steady state rolling contact study was presented or in [27], where the concept was used for the calculation of the contact effect on the modal behaviour. A viscoelastic tread representation can be found in [31], where a transient simulation approach is followed.

A summary of the modelling approach is presented in fig. 1. As the contact behaviour of the tread foundation is crucial for the present study, the contact behaviour of a single bristle is examined prior to the incorporation of the tread model in the full tyre model.

3. The tread element

3.1. The radial tread element

The contact element concept is presented in fig. 2. A simple, initially radial, viscoelastic Voigt element (fig. 2(a)) serves as the starting point. No moment reaction is assumed to be transferred to the belt from the contact.

A certain combination of belt deformation and road profile define kinematically the deformation of a tread element and its corresponding time rate. The resulting contact force lies on the element's axial direction. This kinematic definition imposes the existence of tangential to the road deformation, only as a component of the total axial deformation of the bristle. This approach does not allow for any independent shear slip motion to be considered. Friction cannot be incorporated in the bristle-road interaction, since any additional slip motion of the bristle's tip would violate the geometrical definition of the element's orientation, which is prescribed by the

angular deformation of the belt at the attachment point (see Appendix D). As a result, the horizontal component of an element's deformation cannot be associated with the concept of frictional **micro-slip**.

Such a tread model, though, can be used for the calculation of vertical pressure distribution, assuming that all elements within the contact remain vertical. This assumption is used, for example, in the simulation study by Shiobara *et al* [19]. Moreover, a theoretical decoupling of the bristle's orientation from the deformation of the belt, entitles the development of the typical brush, shear force generation, models (described in [30]). In that case the vertical pressure distribution is a-priori imposed to the model, as an input to the friction mechanism; any deviation of the bristles' orientation is assumed to be, together with their axial deformation, the result of the adhesion between its tip and the road.

The contact geometry (fig. 2(a)) prescribes:

$$DY = Y_r - (Y_b + l_{tread} \cdot \sin \alpha) \quad (1)$$

$$D\dot{Y} = \dot{Y}_r - (\dot{Y}_b + l_{tread} \cdot \cos \alpha \cdot \dot{\alpha}) \quad (2)$$

In the above relationship the term $l_{tread} \cdot \cos \alpha \cdot \dot{\alpha}$ can be neglected since the imposed rate of approach of the road in the present study is relatively low ($\dot{\alpha}$ is correspondingly small) and $\alpha \simeq \pi/2$. In case of a fast rolling tyre, possibly subjected to braking/traction, the time rate of the angle will increase, however the rate of vertical approach ($\dot{Y}_r - \dot{Y}_b$) will also increase and at the end the contribution of the angle rate term remains proportionally small, since $\alpha \simeq \pi/2$.

The resulting force can be calculated through the axial deformation components:

$$DR = DY \cdot \sin \alpha \quad (3)$$

$$D\dot{R} = D\dot{Y} \cdot \sin \alpha \quad (4)$$

$$F_c = K_r \cdot DR + C_r \cdot D\dot{R} \Rightarrow \quad (5)$$

$$F_{cX} = F_c \cdot \cos \alpha \quad (6)$$

$$F_{cY} = F_c \cdot \sin \alpha \quad (7)$$

where :

DY	the vertical deformation of the element
Y_r	the road level
Y_b	the vertical global position of the attachment node
l_{tread}	the undeformed tread element length
α	the angle of contact
DR	the axial deformation of the tread element
F_c	the axial contact force
F_{cX}	the horizontal contact force
F_{cY}	the vertical contact force
K_r	the stiffness of the element
C_r	the viscous damping of the element
$(\dot{})$	time derivative

3.2. The radial-tangential tread element

The potential of friction-dictated shear deflection is included through an additional contact element, which is perpendicular to the radial one. This element is assumed to be initially dimensionless and attached to the free tip of the radial element, in tangential orientation (fig. 2(b)). Now, the contact problem becomes kinematically unconstrained and the degree of freedom associated with the the tip of the tangential element varies according to the frictional interaction between the tread element and the ground. As a result, friction-regulated **micro-slip** motion may be developed.

The study of a tread element's total deformation is better performed using global, vertical and horizontal, coordinates, instead of the radial and tangential ones. The vertical contact deformation (and deformation velocity) remain kinematically constrained, as in the previous case and are therefore calculated using eqn. 1. The horizontal (shear) deformation, though, is dynamically governed by a differential equation. A typical contact situation is illustrated in fig. 2(b). The transformation of the stiffness and damping coefficients from radial/tangnetial to vertical/horizontal coordinates is presented in Appendix A, while a discussion on the transformation angle is included in Appendix D.

In the general case where the tread element is considered as inertia-equipped, the horizontal motion of the tread tip is governed by the dynamic equation:

$$\Sigma F_X = m_{tread} \cdot D\ddot{X} \Rightarrow F_{treadX} + F_{friction} = m_{tread} \cdot D\ddot{X} \quad (8)$$

where :

DX	horizontal displacement (micro-slip) of the tip, measured from the zero tread deformation position
m_{tread}	the mass of the tread element
F_{tread}	the force applied to the tread tip by the viscoelastic foundation of the element (see Appendix A)
F_{treadX}, F_{treadY}	the horizontal and vertical components of the above force
$F_{friction}$	the friction force

Under the assumption of an inertialess tread, the eqn. 8 reads:

$$\Sigma F = 0 \Rightarrow F_{treadX} + F_{friction} = 0 \Rightarrow \quad (9)$$

$$C_{11} \cdot D\dot{X} + C_{12} \cdot D\dot{Y} + K_{11} \cdot DX + K_{12} \cdot DY + F_{friction} = 0 \Rightarrow \quad (10)$$

$$D\dot{X} = -(C_{12} \cdot D\dot{Y} + K_{11} \cdot DX + K_{12} \cdot DY + F_{friction})/C_{11} \quad (11)$$

where additionally:

K_{ij}	stiffness coefficients (see Appendix A)
C_{ij}	damping coefficients (see Appendix A) $i, j = 1, 2$

The horizontal deflection of the element is calculated by the time integration of the relative horizontal velocity between the tip and the corresponding belt point (eqn. 11), for the duration of a time step.

$$DX = \int_{time\ step} D\dot{X} dt \quad (12)$$

The contact forces transmitted to the belt are calculated by the following equa-

tions:

$$F_{cY} = -F_{treadY} = -K_{21} \cdot DX - K_{22} \cdot DY - C_{21} \cdot D\dot{X} - C_{22} \cdot D\dot{Y} \quad (13)$$

$$F_{cX} = -F_{treadX} = F_{friction} \quad (14)$$

3.3. Initiation of contact

A search algorithm is implemented in order to establish the origination of contact between a tread element and the ground. The algorithm compares the global vertical position of the tread tip with the road level, taking into consideration the requirement for non-penetration of the road. To improve the accuracy of the algorithm, the calculation of the position of a tread element's tip takes into account possible residual deformations induced by recent contacts. This results in the following equation for the calculation of the position of the tip:

$$Y_{tread} = Y_b + (l_{tread} + Drad) \sin a + Dtang \cdot \cos a \quad (15)$$

where :

$Drad$ the residual deformation of the radial element

$Dtang$ the residual deformation of the tangential element

When a tread tip location dictates penetration into the road, its vertical position is corrected in order to coincide with the road level and the corresponding vertical deformation is assumed. The horizontal global velocity of the tip is initially assumed to be zero, i.e. the tip momentarily comes to a halt as it establishes contact with the road. Its further shear motion depends on the development of the friction force. This process is captured by the following relationship:

$$Y_{tread} < Y_r \Rightarrow contact \Rightarrow \left\{ \begin{array}{l} Y_{tread} = Y_r \\ \dot{X}_{tread} = 0 \end{array} \right\} \quad (16)$$

where additionally $\dot{X}_{tread}$ is the global horizontal velocity of the tread tip.

3.4. Loss of contact

A force criterion is used for the termination of contact; the contact reaction force must satisfy the following inequality for contact to persist after its initial kinematic establishment:

$$F_{cY} \geq 0 \quad (17)$$

On termination of contact, both radial and tangential tread elements return to their free length gradually (l_{tread} for the radial element and 0 for the tangential one), as prescribed by the first order differential equations governing the deflection of inertialess Kelvin-Voigt elements. Since the motion is not affected by the global orientation angle (a) of the tread element, the differential equations are solved in their corresponding local radial/tangential axes in the interest of computational efficiency. The transformation from the global horizontal/vertical system of axis to the radial/tangential one follows the analysis of Appendix A and offers the initial conditions for the differential equations governing the expansion of the bristles back

to their free length:

$$C_r \cdot D\dot{r} + K_r \cdot D\dot{r} = 0 \quad (18)$$

$$C_t \cdot D\dot{t} + K_t \cdot D\dot{t} = 0 \quad (19)$$

The elements are considered to have reached their free length when their local deformation reduces below a pre-specified limit and the above equations are made redundant for the following steps.

3.5. Friction force

A number of friction laws can be adopted for the horizontal interaction between the tread and the road. However, the classical law due to Coulomb is implemented in this study as a starting point. The law can be described as:

$$F_{friction} = \begin{cases} F_{static}, & \dot{X}_{tread} = 0, \quad F_{static} \leq (\mu \cdot F_{cY}) \\ \text{sign}(F_{static}) \mu \cdot F_{cY}, & \dot{X}_{tread} = 0, \quad F_{static} > (\mu \cdot F_{cY}) \\ -\text{sign}(\dot{X}_{tread}) \mu \cdot F_{cY}, & \dot{X}_{tread} \neq 0, \end{cases} \quad (20)$$

where μ is the coefficient of sliding friction, while the global horizontal velocity of the tip, $\dot{X}_{tread}$, can be calculated according to:

$$\dot{X}_{tread} = \dot{X}_b + D\dot{X} \quad (21)$$

$\dot{X}_b$ being the horizontal global velocity of the corresponding belt node ([Appendix C](#)).

The value of static friction, F_{static} , is calculated through the equilibrium equation of the tread tip (eqn. 11), assuming $\dot{X}_{tread} = 0$:

$$(11) \xrightarrow{(21)} \dot{X}_{tread} - \dot{X}_b = \frac{-(C_{12} \cdot D\dot{Y} + K_{11} \cdot DX + K_{12} \cdot DY + F_{friction})}{C_{11}} \quad (22)$$

$$\frac{\dot{X}_{tread}=0}{F_{friction}=F_{static}} F_{static} = C_{12} \cdot D\dot{Y} + K_{11} \cdot DX + K_{12} \cdot DY - C_{11} \cdot \dot{X}_b \quad (23)$$

Although the adopted friction law is a simple one, its non-linear nature adds significant complications to the numerical simulation of the tread motion. The magnitude of the friction force is a function of the vertical load but independent of the slip velocity. As a result, it is possible that the friction force imposes a change in the expected direction of slip motion, which violates the physical behaviour and introduces solution instability problems. The tread slip velocity is calculated as:

$$(11) \xrightarrow{(20),(21)} \dot{X}_{tread} = \frac{D_{11} \cdot \dot{X}_b - C_{12} \cdot D\dot{Y} - K_{11} \cdot DX - K_{12} \cdot DY + \text{sign}(\dot{X}_{tread})\mu \cdot F_{cY}}{C_{11}} \quad (24)$$

Given that $C_{11} < 0$, the case :

$$\dot{X}_{tread} \cdot (C_{11} \cdot \dot{X}_b - C_{12} \cdot D\dot{Y} - K_{11} \cdot DX - K_{12} \cdot DY + \text{sign}(\dot{X}_{tread})\mu \cdot F_{cY}) > 0 \quad (25)$$

induces an artificial self excited oscillation at the tread tip, as a result of a self-conserved loop of successive changes of velocity and friction directions, always opposing each other. In other words the horizontal velocity of the element cannot reach a zero value, an essential requirement for changing between slipping/adhesion zones. A similar, friction-induced, stability problem in tyre tread simulation has been described by Oertel [32] and by Zimmer and Otter [33].

The problem of the numerically induced oscillation is typically overcome in the above studies through the regularized friction function, according to which a zone of progressive transition between the two friction directions is introduced in the zero velocity area. The solution has been extensively described, also in regards to its implementation in numerical simulations, by Karnopp in [34]. Since such an approach would artificially alter the model's behaviour in the critical for this study standstill equilibrium phase, a simpler solution is required. In order not to introduce any deviation in the friction law, an additional criterion is introduced, according to which a tread tip cannot change its direction of shear global velocity in two successive simulation steps. Every value of tread global velocity, $\dot{X}_{tread}$, predicted by the eqn. 24 is compared with the corresponding value of the previous simulation step. If a change of direction is identified, the tip is considered to adhere to the road and its horizontal position is constrained for the present step:

$$\dot{X}_{tread} \cdot \dot{X}_{tread}^* \begin{cases} \geq 0 \Rightarrow \dot{X}_{tread}: \text{accepted}, & F_{friction} = -\text{sign}(\dot{X}_{tread})\mu \cdot F_{cY} \\ < 0 \Rightarrow \dot{X}_{tread} = 0, & F_{friction} = F_{static} \end{cases} \quad (26)$$

where $\dot{X}_{tread}^*$ refers to the velocity value at the previous time-step.

The tangential interaction between the tread tip and the road surface can be apparently captured by more advanced friction models, incorporating the crucial effect of the sliding velocity on the friction coefficient. Such models were initially proposed by Savkoor [35], inspired by the corresponding viscoelastic materials behaviour, while the same dependency, modelled through a different concept and combined with the stick slip behaviour is precisely represented by the LuGre family of models [36, 37]. Such modelling approaches may result in more accurate numerical results, but as both phenomena (velocity dependency and stick slip behaviour) depend on the tread material properties and its deformation, their use is avoided. The present study focus in the interaction between the structural (belt and tread) deformation and the friction potential; the use of friction models incorporating structural deformation phenomena would obstruct the above aim.

4. Behaviour of a single tread element

Prior to incorporating the above described contact algorithm to a tyre carcass model, a deeper insight into its contact simulation potential is attempted through the examination of two simple contact case studies:

- (1) A vertical, harmonic velocity excitation is applied at the belt attachment point, the node, while the free end of the bristle is initially assumed in road contact, under zero deformation condition.

Table 1. Tread element properties^a

Property	K_r	K_t	C_r	C_t	c	l_{tread}
units	kgm/sec^2	kgm/sec^2	kgm/sec	kgm/sec		mm
Value	3.02×10^5	1.01×10^5	$c \cdot K_r$	$c \cdot K_t$	0.025, single element 0.0025, tyre model	100, single element 10, tyre model

^a stiffness values according to the distributed tread properties given in [28], assuming distribution of each property in 200 tread elements.

(a) The node vertical velocity and displacement of the imposed excitation. (b) The horizontal shear displacement (**micro-slip**) of the tread tip, measured from the initial contact position.

(c) The vertical and horizontal reaction forces. (d) The recovery of the radial and tangential degrees of freedom after the termination of the contact.

Figure 3. The behaviour of the contact element for the 1st case study under contact angle ($\alpha = \pi/4$), for three different values of the friction coefficient ($\mu = 0, 0.2, 0.5$).

- (2) The attachment point follows a circular trajectory with constant velocity. The initially not in contact element establishes and then loses contact.

The second case study corresponds to rotating (spinning) tyre and although the present study deals with a non rotating one, it was chosen as it characteristically offers the tread-road interaction development along the whole contact length. The tread properties used for both the single tread behaviour study and for the complete tyre model are given in table 1. The difference between the damping and free length values between the two cases is introduced deliberately in order for the deformation and viscoelastic behaviour of the single tread element to be exaggerated, so that their influence is easily depicted in the corresponding plots. Their values in the complete tyre model case correspond to more realistic tyre tread properties.

4.1. Case study 1: Vertical excitation of the belt node

The contact behaviour of the tread element, under constant contact angle ($\alpha = \pi/4$), is depicted in fig. 3 for various levels of friction coefficients. The developed slip

Figure 4. The behaviour of the contact element for the 2nd case study, for three different friction coefficients and two values of the base rotation velocity. All subfigures follow the legend of (a), while subfigure (d) follows the additional SDOF legend of (c).

is maximum for the non-frictional surface case (fig. 3(b)), as can be also predicted from eqn. 11 for $F_{friction} = 0$. For a positive coefficient of friction ($\mu = 0.2$) the tip also slides away from the initial contact position, but the developed slip is smaller, as it is deteriorated by the action of the friction. For friction coefficients higher than a critical value the tip remains at its initial contact position, as the value of the static friction is high enough to prevent sliding motion. This critical value is a function of the contact angle and the viscoelastic properties of the tread.

Zero coefficient of friction not only results in zero shear force but also in a lower vertical reaction force, compared to the frictional cases (fig. 3(c)). This is a direct consequence of the interaction between vertical and horizontal tread deformation (see Appendix A). Although both vertical and horizontal reactions increase with the coefficient of friction, the values predicted by eqns. 1-7 for the pure radial element (SDOF) are even higher. Interestingly, the low coefficient of friction permits a change of horizontal force direction towards the end of the contact. Contact loss is strongly affected by the damping behaviour and for this reason the contact duration does not follow the time symmetry of the excitation.

4.2. Case study 2: Circular motion of the belt node

A constant velocity rotation is imposed on the circular base and the developed **micro-slip** and contact forces are examined for two values of anti-clockwise rotation velocity ($\omega = 1 \text{ rad/sec}$, 5 rad/sec) and three values of friction coefficient ($\mu = 0$, 0.2 and 0.5). The contact behaviour is presented in fig. 4. The friction force, as expected, incommodes tread slip movement and the developed global slip velocity, $\dot{X}_{tread}$, in the first stage of the contact decreases for higher coefficient of friction

values (fig. 4(a)). Higher friction potential leads to increased values of negative slip, as the tread end cannot follow the horizontal displacement of the node (fig. 4(b)). Interestingly, this higher deformation results in higher slip velocity at the second part of the total contact duration (upwards vertical motion of the node). During that, the vertical load and the associated friction value decrease, enabling the recovery motion of the tip.

The effect of varying, during the interaction, contact angle is evident in figure 4(b), where for the non frictional motion, the -relative to the belt- slip, DX , changes direction in the second part of the contact. The high deformation level in the higher friction cases, though, prevent such a change of relative slip direction. The damping effect is apparent, as higher angular velocity shortens the angular range of contact and at the same time imposes a higher shear velocity on the tip, as a higher damping force makes it follow the base motion (fig. 4(a)). For this reason, lower **micro-slip** levels develop in the first part of the motion in the higher angular velocity cases (fig. 4(b)).

The developed shear force is presented in fig. 4(c) for all cases, also in comparison to the radial element model. The role of friction potential on the magnitude of the shear forces is crucial, while the damping mechanism exerts slightly higher (in absolute terms) force values in the first part and slightly lower in the second, for the higher velocity case. An equivalent effect is also apparent in the development of the vertical force (fig. 4(d)), where, though, a closer correlation between all the cases is observed. We should highlight the completely different shear force prediction of the purely radial contact element, as the strict dependance of the force on the contact angle causes a harmonic variation to the force value, positive in the first stage of the contact and negative in the second one.

5. The transient contact algorithm of the tyre model

The presented transient tread model can be incorporated in any in-plane tyre belt model, in order to simulate the total contact behaviour of the tyre. The physical properties of the analytical ring model used in the present study are given in tables B1 and B2 of the Appendix B.

The flowchart of the combined belt-tread contact algorithm is presented in fig. 5. The operations carried out within one time step of the simulation may be summarized as:

- (1) The global position (Y_{tread}) of the out-of-contact ($c = 0$) tread elements is calculated (eqn. 15).
- (2) Possible contact is examined through the criterion described by eqn. 16 and the initial contact conditions of the same equation are applied to the elements for which the criterion is satisfied.
- (3) The vertical deformation and deformation velocity (DY and $D\dot{Y}$ in eqns. 1 and 2 respectively), the friction force ($F_{friction}$, eqn. 20) and the tip shear relative velocity ($D\dot{X}$, eqn. 11) are initially calculated, for all the in-contact ($c = 1$) tread elements.
- (4) The contact continuation criterion (eqn. 17) is applied to all the in-contact elements and possible contact termination is defined.
- (5) The transformation from the horizontal/vertical (DX , DY) to the radial/tangential (D_{rad} , D_{tang}) deformation system is applied to elements that step out of the contact (eqn. A1) and the residual deformation is calculated (eqns. 18 and 19) for the elements that still exhibit significant deformation.
- (6) For the in-contact elements, the tread **micro-slip** displacement (relatively to

Figure 5. The flowchart of the total contact algorithm.

the theoretical zero tread deformation position) is derived by the integration of the shear velocity (eqn. 12).

(7) The contact reaction forces applied to the corresponding belt points (F_{cY} and F_{cX} in eqns. 13 and 14 correspondingly) are calculated for the contact elements and the excitation space domain vector ($\mathbf{F}^{space}$, eqn. C5) is formed including the contribution of the inflation pressure and the possible contribution of the non linear sidewall (T_r , eqn. E3) in the corresponding case (Appendix E).

(8) The modal force corresponding to each mode is derived (F_{modal}^j , eqn. C7) and the modal participation factors (η_j) are calculated through the equation C1.

(9) The belt response ($\mathbf{u}$) and deformation velocity ($\dot{\mathbf{u}}$) in the space domain are derived (eqn. C8) so that they can be used in the tread contact calculations of the next time step.

Prior to every contact simulation, the inflation induced deformation is calculated. Inflation pressure is applied as a step excitation and sufficient time is provided until the model reaches its equilibrium condition. It should be noted that the non-linear effect of inflation pressure on the belt behaviour is incorporated through the pretension term in the ring equations of motion. The inflation process leads to a minor radius increase, which could be analytically calculated in the linear sidewall case. The process is important, though, in the case of a non linear sidewall, for the identification of the inflated geometry of the sidewall string elements (Appendix

Figure 6. Vertical deformation as a function of the vertical load for the linear and non linear sidewall cases.

E).

In the contact simulation process, the deformation is achieved by imposing a constant rate of approach between the wheel and the road, until a prescribed final value has been achieved. A common deformation time period (0.4 *sec*) is assumed for all the examined cases. A further time period (0.1 *sec*) is provided for the attenuation of any oscillations and the establishment of equilibrium.

6. Contact behaviour of the model

6.1. Total contact stiffness and the effect of sidewall non-linearity

It is well known from published experimental studies, that, although the ring model offers an excellent prediction of low frequency modal behaviour, the calculated contact stiffness does not correlate with the measured one. The main reason for this deviation is the non-linear behaviour of the radial sidewall foundation. The geometrical buckling of the sidewall structure as deformation increases, reduces the radial force in comparison to the linearly predicted value. The incorporation of a physical string-based mechanism in the ring model for studying the nonlinear contact behaviour was presented by Böhm in [10]. Mousseau and Hulbert use the same concept in [38], while Miege and Popov in [39] notice a 44% higher contact stiffness value predicted by a ring model, compared to the experimental results gained from a truck tyre. A discretised inextensible string physical foundation (Appendix E), simulates the sidewall non linear mechanism in this study. The string dimensions are given in table B2.

The [wheel displacement - total vertical load](#) relationship, for both the linear and non-linear sidewall equipped ring models, is presented in fig. 6. Both models predict an exponential dependance, while the sidewall nonlinearity results in a noticeable load reduction, ranging from 10% for 5mm to 30% for 50mm deformations. The moderately increasing contact stiffness with load, as predicted by the simulation of both sidewall cases, has been experimentally verified in numerous experimental studies [6, 8, 16, 30, 38, 40] and [17](p.559-577).

The predicted two-dimensional contact pressure distributions, under the linear and non linear sidewall assumptions, are correspondingly presented in figs. 7(a) and 8(a), together with the deformed belt, for a typical $d = 40 \text{ mm}$ vertical deformation and $\mu = 0.5$ coefficient of friction. The vertical and horizontal fields are represented by vertical and horizontal lines, respectively, with the vertical ones starting from the nodes of the deformed belt and the horizontal ones starting from the edges of the vertical ones. Common scaling ($1\text{mm} = 10^{-4}\text{kgr mm/sec}^2$) is used in their plotting, so that a sense of their ratio can be acquired. A generally parabolic/trapezoidal vertical stress distribution is common in both sidewall cases. Such pressure distributions have been experimentally measured in numerous stud-

Figure 7. The two dimensional reaction force field and a 4X magnification of the horizontal one, as developed under the linear sidewall assumption ($d = 40mm$, $\mu = 0.5$).

Figure 8. The two dimensional reaction force field and a 4X magnification of the horizontal one, as developed under the linear sidewall assumption ($d = 40mm$, $\mu = 0.5$).

ies [12, 19, 22, 23, 41] and [17](p. 271-292). The non linear sidewall assumption, though, leads to lower force values, while a force reduction area is observed in the central part of the contact length.

This concave-shaped vertical distribution and the associated belt deformation pattern have been analytically discussed and experimentally measured by Akasaka *et al* [12]. According to them, this behaviour is associated primarily with the bending behaviour of the belt and secondarily with the inwards direction of the shear force in the neighboring contact area. Under the modelling approach followed here, this characteristic deformation shape develops only for the non-linear sidewall model. Its crucial effect on the frictional interaction within the contact will be discussed later in more detail.

The horizontal distribution shows a harmonic, full period-shaped variation along the contact length, maximizing (in absolute values) near the edges and reducing down to zero around the symmetry axis. Using a higher magnification and assigning the vertical axis to the horizontal forces for better clarity, the horizontal reaction force fields are given in figs. 7(b) and 8(b) for the linear and non linear case, respectively. The inwards stress direction has been experimentally verified in multiple studies presented in [4] and [17](p.p. 293-307). The non-linear sidewall assumption results in slightly lower horizontal force values, because of the lower deformation field, but the distribution shape is similar for both cases. The concave belt deformation shape affects the horizontal contact pressure distribution through a mechanism that will be later discussed, as part of the friction-deformation interaction.

6.2. The effect of the modal range of the model

For the successful simulation of the tyre contact behaviour it is vital to ensure that the belt modelling approach offers an accurate belt representation. Since the analytical ring model offers an infinite number of vibration modes (Appendix B), the

Figure 9. Contact pressure distributions predicted by different modal reduction subsets ($d = 40mm$, $\mu = 0.5$).

Figure 10. The modal participation factors of the radial and the tangential modes, presented as percentage ratio to the maximum participation values ($d = 40mm$, $\mu = 0.5$).

incorporation in the solution of a reduced subset of them introduces an unavoidable computational error. The selected modal range must be able to represent not only the transition from a circular to a straight belt shape along the contact length, but also the concave shape near the center. Moreover, the belt representation should be valid for a wide range of vertical loads.

Contact fields calculated incorporating different numbers of modes are presented in fig. 9, where the non linear radial sidewall is also implemented. Although the total vertical reaction force is hardly affected by the modal range in use, the predicted distribution shows significant shape variations. Different numbers of radial modes seem to have a pronounced effect on the contact pressure distribution. In contrast, the distribution is largely insensitive to the tangential modes, apart from the zero mode (breathing), which is associated with the inflation procedure. The comparison between fig. 8(a) -radial modal number range 0 to 20 and tangential one from 0 to 4- and fig. 9(d) -0 to 25 and 0 to 6 respectively- reveals that the first one corresponds to acceptable accuracy and these modal ranges are the ones applied to the rest of the study.

The participation of each mode in the final equilibrium deformation of fig.9(d) is presented in fig. 10, where all the participation factors are normalized by dividing with the maximum value and presented as percentages of that. Interestingly, the

Figure 11. Vertical and horizontal contact stress distributions for various values of the imposed deformation ($\mu = 0.5$).

maximum factor corresponds to the 1st radial mode, associated with the rigid ring relative motion of the belt to the wheel. Although the participation factors of radial modes with modal number above 10 are relatively small, their contribution to the accurate stress prediction is vital, as it is evident from fig. 9.

The selected range of radial modes (0th to 20th mode) corresponds to a frequency range up to 934Hz, while the range of the tangential modes (0th to 4th mode) corresponds to a frequency range up to 2485Hz. In the examined case, the solution accuracy depends on the extent to which the linear combination of the participating in the solution mode shapes is able to describe the belt conformity and the variations in its equilibrium deformation shape.

When, though, such a modal expansion/reduction approach is used for the simulation of the high frequency transient dynamic behaviour of the belt, additional factors affect the accuracy of the calculations, apart from the conformity between the imposed excitation pattern and the corresponding mode shapes. The frequency content of the excitation magnifies the participation factor of certain modes, while the step nature of it broadens the range of the participating modes. Furthermore, the effect of the damping coefficient of each mode is crucial for the derivation of a realistic response and these coefficients should be experimentally determined or derived through a space distribution model corresponding to a physical dissipation mechanism. In the present simplified approach, though, the assumption of equal damping coefficients can be justified by the relatively slow nature of the loading procedure.

6.3. The effect of the load level and the belt deformation mechanism

The predicted stress field distributions for a range of vertical deformations is presented in fig. 11 for the non linear sidewall case. It is experimentally verified in [41], the increase in vertical load does not result directly in an increase in stress magnitude, since most of the additional load is compensated for by an increase in the size of the contact area. The latter, though, has a major effect on the shape of the pressure distribution [6, 8, 22, 41] and [17](p. 290). The transition from a parabolic, convex-shaped distribution, towards a trapezoidal one, as the load

Figure 12. The structural deformation of the belt for different values of the imposed vertical deformation ($\mu = 0.5$).

increases, is documented in the above literature. The development of the central concave belt deformation pattern and the associated effect on the distribution are also found to increase with the load. The interaction between the load and the distribution shape is well simulated by the proposed model, as it is apparent in fig. 11.

The resulting structural deformations along the whole circumference of the belt are presented in figs. 12(a) (horizontal) and 12(b) (vertical) for the whole range of vertical load, where the corresponding contact length is also marked. The horizontal deformation follows in general a symmetrical, outwards directed distribution. The horizontal deformation within the contact length is, though, minimal and the predominant deformation is the vertical one in this area (fig. 12(b)). The maximum horizontal deformation occurs at an angle $\pm\pi/4$, measured from the vertical line of symmetry. This deformation lies well outside the contacting part of the belt.

The vertical deformation decreases rapidly towards the edges of the contact length (fig. 12(b)), after reaching its maximum value in the centre of it. A small increase is noticed at angles $\pm\pi/2$ from the symmetry line, a result also confirmed by the maximum modal participation factor corresponding to the vertical rigid ring mode. This effect is intensified by the lower value of tangential sidewall stiffness compared to the radial one. Interestingly, the minimum vertical deformation is predicted at a point symmetrical to the centre of the contact, a direct effect of the increasing stiffness with the radial wheel to belt distance. The general shape of the predicted vertical deformation is in good correlation with the experimental ones presented in [42].

A more detailed insight into the vertical deformation of the belt in the region of the patch is accomplished by presenting the vertical distance of the belt relative to the road-level (fig. 12(c)). The radially inwards deformation pattern of the belt is

Figure 13. Shear displacement (slip) of selected tread tips, measured from the initial contact position, for three different values of the friction coefficient (μ).

present even for minor loads, although it is limited to a small angle range around the axis of symmetry and the convex deformation is dominant. As the load increases, the concave section increases and the dominant deformation shape transforms from convex to concave. This transformation affects not only the vertical but also the horizontal contact pressure distribution in the area, as will be discussed after the role of the tread behaviour becomes clear.

7. Tread related effects

7.1. Friction potential and *micro-slip* development

Moving the discussion from belt behaviour towards tread friction and shear displacement, the transient slip motion of the tread element tips along the contact surface is investigated as the contact conditions are progressively applied. Starting from the initial condition of the lowest, perpendicular to the road, tread element just in contact with the ground, a $d = 40\text{mm}$ vertical displacement is applied to the wheel, over a 0.4sec period and further time is provided for the equilibrium establishment, following the process described in section 5. The non frictional case is compared to two frictional cases ($\mu = 0.01$ and $\mu = 0.1$) and the displacement history -relatively to the initial point of contact- of selected tread elements along the contact semi-length is presented in fig. 13. The numbers next to each slip plot refer to the order of the element, as it can be identified from fig. 8(a) (e.g. no.1 refers to the element at the centre of the contact, while no.13 to the one at the left edge).

The slip displacement increases, in general, as we move from the centre to the edges of the contact patch, reaching its maximum value for elements in the latter zone. The displacement has an outwards direction for the whole patch. In the non-

Figure 14. The deformation of the belt in the area of the contact.

frictional case (fig. 13(a)), a rebound motion can be identified, especially for the elements close to the contact centre, as the displacement reaches a maximum value but later stabilizes to a lower level. The final value is slightly negative for the 2nd element, as the tip stabilizes at a position closer to the centre than the initial contact point.

The presence of friction is in general associated with lower shear displacement and the rebound motion is also prevented (fig. 13(b)). Although the maximum slip values for the first elements (2nd, 4th, 6th) are lower than the corresponding ones of the non-frictional case, the final equilibrium values are higher, as the rebound motion is prevented. For the elements that the rebound motion was small or insignificant (9th, 11th) the final slip displacements appear to be lower, compared to the non-frictional case.

As the coefficient of friction increases further (fig. 13(c)), the first elements lock at their initial contact positions, as the developed friction is high enough to prevent their **micro-slip** motion, even from the first moments after the contact initiation. For $\mu = 0.1$ the 2nd, 4th and 6th elements lock at their initial contact positions, while the rest of the elements develop displacements, which **are, though, much** smaller compared to the lower friction cases.

In general, for every tyre/road interaction case, there is a coefficient of friction value for which the tread shear **micro-slip** motion is prevented for the whole contact length. In result, any further increase of the coefficient does not introduce any variation in the deformation and stress development as no change in the dynamic interaction exists. It may be concluded that the areas close to the edges of the contact patch may develop shear displacement for a broader range of tribological conditions, as the frictional locking appears firstly in the close to the centre area of the contact.

The above described behaviour is in agreement to that presented in [43], where the areas towards the edges of the contact show a higher tendency to slip. Furthermore, a number of experimental and simulation studies (presented in [17, 30]) associate **micro-slip** primarily with the edges of the contact area and adhesion/slip successions with the central one, under typical tyre operating conditions.

7.2. The belt–tread interaction mechanism

Tread element behaviour is not only determined by the vertical load and the frictional potential, but also interacts with the belt deformation. The tread tendency

Figure 15. The tread **micro-slip** and the belt deformation corresponding to the 2nd contact point, for different values of the friction coefficient.

for outwards slip motion (fig. 13) is a direct result of a corresponding deformation of the belt (fig. 12(a)). The value of the belt horizontal deformation increases as we move from the centre to the edge of the contact semi-length as may be observed in fig. 14(a), in accordance to the similar increase of the tread slip (fig. 13).

An insight into the interaction between the belt deformation and the tread **micro-slip** is attempted by the observation of the deformation of the corresponding nodes to the previously examined tread elements. This is accomplished in fig. 14(b), for the non-frictional case, where the horizontal deformation of each node is presented as the percentage ratio to its final, equilibrium corresponding value. The rebound motion of the 2nd, the 4th and the 6th nodes is apparent. Moreover the 2nd node exhibits an inwards final deformation. No rebound motion is identified for the rest of the nodes. This rebound motion of the central belt zone is associated with the concave formation in the same area (fig. 12(c), as the upwards belt movement results in a horizontally inwards one (shrinkage).

A friction induced discrepancy between the rebound motion of the belt and the corresponding one of the tread foundation should be highlighted. Although the friction existence may suspend the rebound motion of the tread, this is not true for the corresponding one of the belt nodes. The friction force cannot prevent the formation of the concave pattern, although it can slightly affect its magnitude. The bending mechanism which governs the concave formation, also discussed in [12], is present for any tribological conditions. It is apparent, though, that certain tribological conditions exist for which the tip rebound or even its complete shear displacement may be suspended, although the belt nodes bounce back, as the concave deformation pattern is formed. This is presented in fig. 15, where the tread tip and the belt node displacements of the second contact element are examined (figs. 15(a) and 15(b) respectively), for various values of friction coefficient.

The above discussed interaction offers the development mechanism of the tangential contact reaction field of the examined model. In the central zone, the road-imposed reaction force becomes outwards pointing (positive), as the rebound (inwards) motion of the tread zone is suspended. Moving along the contact half-length, the point where no further tread rebound motion is observed in the frictionless case, coincides with the change in the contact force direction, for any value of the coefficient of friction. From that point the tangential contact field receives its generally observed inwards direction, in correlation to the measurements presented in [17]. This mechanism, associated with the change in the contact force direction along the contact semi-length, deteriorates in the presence of high friction, where the positive **micro-slip** motion of the elements is in general suspended (fig. 13(c)). In such cases the contact reaction is directed inwards along almost the full contact length, as will be later discussed.

(a) The orientation of the tread elements of the contact area, for various levels of deformation. (b) The angular deviation history of the orientation of the 2nd tread element from the initial, radial, orientation.

Figure 16. The contact effect on the orientation of the tread elements ($d = 40\text{mm}$, $\mu = 0$).

The present analysis offers an explanation for an observed contradiction between published experimental studies. Although a good agreement among them can be identified for the shape of the vertical pressure distribution and the vertical belt deformation, this is not true for the corresponding horizontal values. As discussed by Pottinger [6], the frictional contact of a solid cylinder would result in pure inwards directed contact forces emerging from the conformation of its circular shape to the plane surface. In the case of the tyre, though, some experimental results of static and slow rolling contact indicate an outwards directed forces field ([6, 41] while others studies, summarized in [4, 17], indicate the contrary. As it has been highlighted, the contact stress direction is dictated by the belt contact deformation mechanism. When the dominant mechanism is the solid disk-like one, the belt deformation is outwards directed and a contrary contact stress field is measured. When the dominant mechanisms, though, are the relaxation of the radial, inflation-induced, tension and the bending, then the deformation is directed towards the centre of the contact and radially inwards, resulting in outwards tangential contact stresses, even for the whole contact area (e.g. [6, 42]).

The behaviour of the model presented in this study is controlled mainly by the first of the two mechanisms, in correlation to the majority of the experimental measurements [17], while the second mechanism can be identified only in the central concave zone. In the case of a real tyre, though, the mechanical properties of its materials, the carcass design and the belt-tread interaction will define the predominant mechanism and the relative amplitude of the two zones. An experimental evidence of the transition between these zones can be found in the measurements published in [8].

7.3. The effect of the contact angle

Up to this point, the effect of the contact angle in this interaction has been omitted from the analysis. It is apparent, though, that the belt deformation, from circular shaped to linear as it conforms to the road plane, transforms the initially radial orientation of the tread elements to an *almost* perpendicular one (Appendix D). Their total horizontal and vertical stiffness and damping characteristics vary, though, as a function of the contact angle (Appendix A). The actual effect of the orientation change on the contact phenomena is directly related to the relative radial and tangential properties of the tread model.

The equilibrium contact angle of the 14 tread elements of the contact area (2nd to 15th) is presented in figure 16(a) for five load cases. In the same plot the elements

participating in the contact are identified. Apart from the apparent orientation change towards the perpendicular one, a larger deviation from the latter can be observed for the elements located near the edge of the contact length in every case, as the imposed load increases. In other words, an increase in contact length results in a broader range of the participating contact angles.

Interestingly, the orientation of the contact elements is also affected by the concave pattern. In fig. 16(b) the angular deviation history of the 2nd contact element orientation is presented. The deviation is calculated from the initial radial orientation, while the deviation value corresponding to perpendicular orientation is also plotted for comparisons reasons. As the belt node participates in the concave pattern, the element's orientation may decrease further than $-\pi/2$. Its final value, though, depends on the -relative to the concave's central plateau- placement of the node.

8. The friction dictated mechanism of the horizontal contact pressure distribution

It is evident from the above analysis that the horizontal contact stress field development is a highly non linear function of the interaction between the two-dimensional belt deformation and the frictional behaviour of the tread, in regards to its possible shear slip deformation. In fig. 17 the evolution of the horizontal contact stress field is presented as a function of the friction coefficient, for a constant vertical load (corresponding to $d = 40mm$). The mechanism of the stress field generation can be summarized in the following conclusions:

- For limited friction potential an almost symmetrical stress distribution is predicted for the contact half-length. The distribution corresponds to outwards-directed stresses in the inner part and inwards-directed stresses in the outer part. During the initial period of contact an outwards directed **micro-slip** displacement is developed along the whole contact length. This is imposed by a similar deformation pattern of the belt, as its circular shape conforms to the contact plane. The initially moderate vertical load cannot generate friction forces high enough to suspend this slip. As the deformation increases, the developing concave pattern of the belt induces an inwards motion of the central nodes. The friction force now is higher, as the vertical load has increased, and may resist to the belt-induced, rebound motion of the tread. The resulting contact stress for the affected part of the belt is outwards directed. In contrast, the outer part of the contact half length, is not affected by the concave deformation pattern. The belt displacement in this case remains outwards-directed for the whole duration of the contact realization. Friction force reacts to this motion, resulting in the inwards stress direction.
- As the coefficient of friction increases, the tread **micro-slip** diminishes and the stress values in both zones (inwards and outwards) are in general pronounced. In the central zone, slip may still be entitled during the initial stage, but the later suspension of the rebound motion leads to a slight increase in the stress values, compared to lower friction coefficient cases. The stress increase is higher, though, in the outer zone, as the outwards directed deformation of the belt is directly retarded by the enhanced friction potential.
- A further increase in the coefficient of friction results in the radical expansion of the zone associated with the inwards directed stress field along the contact semi-length. The tread **micro-slip** starts to be arrested (frictional lock) for the central elements and the effect is extended to the rest of the contact area, as the

Figure 17. The development of the shear reaction forces along the contact semi-length, as a function of the friction coefficient.

coefficient raises, emerging in the sound increase of the inwards directed stress.

- The stress field converges to a final distribution, when the coefficient of friction reaches a value that prevents slip along the whole contact length. From that point onwards, the stress field becomes insensitive to any further increase in friction.

9. Conclusions

A contact simulation algorithm has been developed, appropriate to represent the transient interaction phenomena between the tyre deformation and the tread **micro-slip**. The tyre tread is modelled as a combination of radial and tangential viscoelastic elements, the contact behaviour of which was extensively investigated prior to their integration into the belt model. The tread foundation was coupled to the well-established, analytical ring on elastic foundation model, the solution of which was acquired by the modal reduction/expansion method. The algorithm was enriched with a string-based, non linear sidewall foundation. Initially, the general contact behaviour of the model was presented and published experimental work was used for its evaluation.

The tread **micro-slip** has been found to be dictated by the belt deformation with regards to both general behaviour and local, non-linear phenomena. The level of convention between the belt deformation and the tread slip is heavily influenced by the friction potential of the surface, which reacts to the tread shear displacement. Two contradicting contact deformation mechanisms of the belt were identified: the solid disk-like one and the combination of the bending and the relaxation of the, inflation-induced, radial stress. The first mechanism results in a horizontally expanding belt while the second one results in a horizontally shrinking belt and the characteristic concave formation. The relative to each other level of conformation to each mechanism is depicted on the contact stress distribution of the tyre and associates it with the belt mechanical properties. The mechanism explains a frequent

contradiction of experimental results observed in the literature.

In general, the proposed approach can be easily extended in order to incorporate the rotational and horizontal translational degrees of freedom of the wheel, additionally to the vertical one assigned here. The wheel motion (kinematically or dynamically induced) will impose a dynamic excitation to the belt through the sidewall structure, altering the contact interaction of the tread elements with the ground. Accordingly, the study is believed to constitute an initial basis for the study of a number of transient phenomena related to the interaction between the tyre structural deformation and its shear force development potential.

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Appendix A. System of axes transformation for the contact force calculation

The transformation from the radial/tangential to the global horizontal/vertical expressions of the deformation vector for each tread element reads:

$$\begin{bmatrix} D_{rad} \\ D_{tang} \end{bmatrix} = T(a) \cdot \begin{bmatrix} DX \\ DY \end{bmatrix} \quad (A1)$$

where: $\mathbf{T}(a) = \begin{bmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{bmatrix}$ and a is the relative angle between the two systems of axes (Appendix D). Assuming $\mathbf{K} = \begin{bmatrix} -K_r & 0 \\ 0 & -K_t \end{bmatrix}$ is the matrix of the radial and tangential stiffness properties (K_r and K_t respectively), the forces can be expressed as :

$$\begin{bmatrix} F_{rad}^s \\ F_{tang}^s \end{bmatrix} = \mathbf{K} \cdot \begin{bmatrix} D_{rad} \\ D_{tang} \end{bmatrix} = \mathbf{K} \cdot \mathbf{T}(a) \cdot \begin{bmatrix} DX \\ DY \end{bmatrix} \quad (A2)$$

$$\begin{bmatrix} F_X^s \\ F_Y^s \end{bmatrix} = \mathbf{T}^T(a) \cdot \begin{bmatrix} F_{rad}^s \\ F_{tang}^s \end{bmatrix} = \mathbf{T}^T(a) \cdot \mathbf{K} \cdot \mathbf{T}(a) \cdot \begin{bmatrix} DX \\ DY \end{bmatrix} = \begin{bmatrix} K_{11}(a) & K_{12}(a) \\ K_{21}(a) & K_{22}(a) \end{bmatrix} \begin{bmatrix} DX \\ DY \end{bmatrix} \quad (A3)$$

The damping forces are calculated by an equivalent procedure. The deformations' velocities are given by the time derivation of eqn. A1:

$$\begin{bmatrix} \dot{D}_{rad} \\ \dot{D}_{tang} \end{bmatrix} = \dot{T}(\alpha) \cdot \begin{bmatrix} DX \\ DY \end{bmatrix} + T(\alpha) \cdot \begin{bmatrix} D\dot{X} \\ D\dot{Y} \end{bmatrix} \quad (A4)$$

Applying a simplification similar to the one applied to eqn. 2, the term $\dot{T}(\alpha) \cdot \begin{bmatrix} DX \\ DY \end{bmatrix}$

Figure B1. The ring on elastic foundation model with its basic properties and dimensions.

can be neglected since the imposed rate of approach of the road in the present study is relatively low, resulting in correspondingly low values of $\dot{\alpha}$. In case of a fast rolling tyre, possibly subjected to braking/traction, the above rate of the angle will increase, however the rate of vertical and shear deformations ($D\dot{X}$, $D\dot{Y}$) will also increase and at the end the contribution of the above angle rate term remains proportionally small.

If the damping properties matrix is: $\mathbf{C} = \begin{bmatrix} -C_r & 0 \\ 0 & -C_t \end{bmatrix}$, the damping forces (under the above simplification) can be written as:

$$\begin{bmatrix} F_X^d \\ F_Y^d \end{bmatrix} = \mathbf{T}^T(a) \cdot \mathbf{C} \cdot \mathbf{T}(a) \cdot \begin{bmatrix} D\dot{X} \\ D\dot{Y} \end{bmatrix} = \begin{bmatrix} C_{11}(a) & C_{12}(a) \\ C_{21}(a) & C_{22}(a) \end{bmatrix} \begin{bmatrix} D\dot{X} \\ D\dot{Y} \end{bmatrix} \quad (\text{A5})$$

where $(\dot{})$ corresponds to the time derivative of the deformations. The force expressions used in section 3.2 are derived as:

$$\begin{bmatrix} F_{treadX} \\ F_{treadY} \end{bmatrix} = \begin{bmatrix} F_X^s \\ F_Y^s \end{bmatrix} + \begin{bmatrix} F_X^d \\ F_Y^d \end{bmatrix} \quad (\text{A6})$$

It should be noticed that the horizontal/vertical expression of the stiffness or damping force value develops as function not only of both radial and tangential coefficients but also of both radial and tangential deformations.

Appendix B. The ring on elastic foundation model

The equations of motion for the complete extensible ring on elastic foundation under inflation pretension [29, 44] may be written as:

$$\frac{D}{R^4} \left(\frac{\partial^4 u_r}{\partial \theta^4} - \frac{\partial^3 u_t}{\partial \theta^3} \right) + \frac{K}{R^2} \left(u_r + \frac{\partial u_t}{\partial \theta} \right) + \frac{N}{R} \left(2 \frac{\partial u_t}{\partial \theta} - \frac{\partial^2 u_r}{\partial \theta^2} \right) + k_r \cdot u_r + b \cdot d \cdot \rho \frac{\partial^2 u_r}{\partial t^2} = q_r + \frac{N}{R} \quad (\text{B1a})$$

$$\frac{D}{R^4} \left(\frac{\partial^3 u_r}{\partial \theta^3} - \frac{\partial^2 u_t}{\partial \theta^2} \right) - \frac{K}{R^2} \left(\frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_t}{\partial \theta^2} \right) - \frac{N}{R} \left(2 \frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_t}{\partial \theta^2} \right) + k_t \cdot u_t + b \cdot d \cdot \rho \frac{\partial^2 u_t}{\partial t^2} = q_t \quad (\text{B1b})$$

where :

Table B1. Ring model properties^a

Property	elastic modulus (E)	density (ρ)	inflation pressure (P_0)	ring radius (R)	cross section width (b)	cross section height (d)	viscous damping factor (λ)
units	$\text{kg}/(\text{mm sec})$	kg/mm^3	$\text{kg}/(\text{mm sec}^2)$	mm	mm	mm	kg/mm^2
Value	10.4×10^6	8.1×10^{-6}	220	300	152	3.1	0.01

^a after [27].

Table B2. Sidewall properties

Property	radial stiffness ^a	tangential stiffness ^a	string element length ^b	rim to belt initial distance ^b
k_r	k_t	l_{string}	l_{string}	$R - \frac{D_{wheel}}{2}$
units	$\text{kg}/(\text{mm sec}^2)$	$\text{kg}/(\text{mm sec}^2)$	mm	mm
Value	1.929×10^3	648.7	122	109.5

^a linear sidewall case, after [27].^b non linear sidewall case, assuming wheel diameter $D_{wheel} = 15\text{in}$ and radial stiffness in the non deformed case equal to the linear one.

$u_r(\theta)$	radial deformation
$u_t(\theta)$	tangential deformation
θ	the central angle of a point on the circumference
b	width of the ring cross section
d	height of the ring cross section
ρ	ring material density
R	ring radius
$D = E \cdot b \cdot d^3/12$	bending stiffness of the ring
$K = E \cdot b \cdot d$	membrane stiffness of the ring
E	elastic modulus of the ring material
P_0	inflation pressure
$N = P_0 \cdot b$	inflation tension force
k_r	radial stiffness of the sidewall foundation
k_t	tangential stiffness of the sidewall foundation
$q_r(\theta)$	radial external force excitation
$q_t(\theta)$	tangential external force excitation

The values of k_r , k_t , q_r and q_t are normalized to the length of the circumference of the ring. A comprehensive discussion on the variants of the ring equations of motion, their derivation and the modal behaviour characteristics may be found in [44]. Tyre modelling approaches based on analytical ring models have been presented, among others, in [10, 25–28] and [17](p.757-784). The ring model properties used in this work are given in tables B1 and B2.

The application of the modal expansion theory on a structure of double axis symmetry, such as the ring, prescribes harmonic mode shape deformation functions such as:

$$U_j^r(\theta, t) = R_j(\theta)e^{i\omega_j t} \quad (\text{B2})$$

$$U_j^t(\theta, t) = \Theta_j(\theta)e^{i\omega_j t}, \quad 1 \leq j \leq m \quad (\text{B3})$$

where :

Figure B2. The $n = 0$ to $n = 4$ mode shapes of the ring.

$R_j(\theta) = A_j \cos(n_j(\theta - \varphi))$	mode shape functions
$\Theta_j(\theta) = B_j \sin(n_j(\theta - \varphi))$	
ω_j	natural frequency
A_j, B_j	mode shape amplitudes
n_j	modal number
φ	arbitrary angle which depends on the excitation orientation
j	the examined mode
m	total number of modes

The substitution of the deformation expressions B2 to the equations of motion B1 leads to the calculation of the natural frequencies (ω_j) and the associated mode shape amplitudes (A_j, B_j) as functions of the modal number (n_j). This modal expansion procedure offers two different modes with each modal number, n ; one of them, typically the lower frequency one, is associated with primarily radial deformation while the second one, the high frequency one, with tangential deformation. The $n=0$ to $n=4$ radial and tangential modes are presented in fig. B2.

Each of the derived modes is incorporated twice in the solution basis (Φ), with an angle relative to each other $\varphi = \frac{\pi}{2n_j}$, so that every possible orientation of the corresponding to a certain couple mode shape can be reproduced. The application of the modal expansion method on the analytical ring model is comprehensively presented in the majority of the publications on tyre modeling based on the analytical ring equations, as also in relative textbooks [44],[29](chap. 8) and further analysis of the subject is unnecessary.

Since the derivation of tread contact forces presented in section 3 uses a horizontal/vertical frame of reference, a corresponding transformation should be applied to the deformation predicted by the mode-shape functions. The transformation from the radial/tangential deformation system to the horizontal/vertical one for a point of the belt circumference at an angle θ (fig. B1) may be written as:

$$\begin{bmatrix} X_j(\theta) \\ Y_j(\theta) \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} R_j(\theta) \\ \Theta_j(\theta) \end{bmatrix} \quad (\text{B4})$$

Moreover, the adoption of a constant discretisation for the belt, assigns a constant angle to every belt node. The above analytical mode shape functions may be written

in the following vector form, assuming that the first node lies on the vertical axis:

$$\Phi_j = \begin{bmatrix} X_j^1(\theta_1) \\ Y_j^1(\theta_1) \\ \vdots \\ X_j^z(\theta_z) \\ Y_j^z(\theta_z) \\ \vdots \\ X_j^{n_b}(\theta_{n_b}) \\ Y_j^{n_b}(\theta_{n_b}) \end{bmatrix} \quad (\text{B5})$$

where the node angle θ_z is calculated as a function of the total number of nodes (n_b) as:

$$\theta_z = -\frac{\pi}{2} + (z-1)\frac{2 \cdot \pi}{n_b}, \quad 1 \leq z \leq n_b \quad (\text{B6})$$

The mode shape matrix is synthesized by the mode shape vectors as:

$$\Phi = [\Phi_1 \ \cdots \ \Phi_j \ \cdots \ \Phi_m] \quad (\text{B7})$$

Appendix C. Belt response calculation and its space-to-modal transformation

Although the belt deformation is calculated through its modally expanded representation, the contact algorithm necessitates the expression of the belt deformation in space coordinates, so that the interaction with the road can be modelled. For this reason, the transformation from the modal to the space domain is necessary in every step of the calculation.

The differential equation governing the belt vibrations, expressed in the modal domain (after [29]) is:

$$\frac{\partial^2 \eta_j}{\partial t^2} + 2 \cdot \zeta_j \cdot \omega_j \cdot \frac{\partial \eta_j}{\partial t} + \omega_j^2 \cdot \eta_j = F_j^{modal}(t), \quad 1 \leq j \leq m \quad (\text{C1})$$

where :

- $\eta_j(t)$ the modal participation factor of the j th mode
- ζ_j the modal damping coefficient
- ω_j the natural frequency of the mode
- F_j^{modal} the modal force

The modal damping coefficient is given by the equation:

$$\zeta_j = \frac{\lambda_j}{2 \cdot \rho \cdot d \cdot \omega_j} \quad (\text{C2})$$

where λ_j is the viscous damping factor while the modal of the j th mode. The modal force in the general, analytical case is given by the equation:

$$F_j^{modal}(t) = \frac{1}{M_j} R \cdot b \cdot \int_0^{2\pi} (q_r(\theta) \cdot U_j^r(\theta) + q_t(\theta) \cdot U_j^t(\theta)) d\theta \quad (C3)$$

where:

M_j is the modal mass of the j th mode, given by the equation:

$$M_j = \rho \cdot h \cdot R \cdot b \int_0^{2\pi} \left((U_j^r(\theta))^2 + (U_j^t(\theta))^2 \right) d\theta \quad (C4)$$

As the space discretisation is constant, the force vector in the space domain has constant dimensions and a simpler modal force equation can be derived. Since every node is associated with two degrees of freedom (horizontal and vertical global deformations) it may be proposed:

$$\mathbf{F}_{2n_b \times 1}^{space}(t) = \mathbf{F}_{inflation}(t) + \mathbf{F}_{contact}(t) + \{\mathbf{F}_{sidewall}(t)\} \quad (C5)$$

where n_b is the number of the belt nodes. The sidewall term is incorporated in the force vector calculation only when a non-linear sidewall is incorporated and leads to the elimination of the radial term (k_r) of the linear sidewall foundation from the ring equations of motion motion(B1).

Following the discretisation introduced in Appendix B for the mode shape functions, it may be written:

$$\mathbf{F}^{space}(t) = \begin{bmatrix} F_X^1(\theta_1) \\ F_Y^1(\theta_1) \\ \vdots \\ F_X^z(\theta_z) \\ F_Y^z(\theta_z) \\ \vdots \\ F_X^{n_b}(\theta_{n_b}) \\ F_Y^{n_b}(\theta_{n_b}) \end{bmatrix} \quad (C6)$$

The analytical expression for the modal force (eqn. C5) can be substituted by the corresponding one based on the discretised model:

$$F_j^{modal}(t) = \frac{1}{M_j} (\mathbf{\Phi}_j)^T \cdot \mathbf{F}_{space}(t) \quad (C7)$$

The transformation of the deformation and velocity responses from modal to space coordinates is given by the equations:

$$\mathbf{u}(t) = \mathbf{\Phi} \cdot \begin{bmatrix} \eta_1(t) \\ \vdots \\ \eta_m(t) \end{bmatrix}, \quad \dot{\mathbf{u}}(t) = \mathbf{\Phi} \cdot \begin{bmatrix} \frac{\partial \eta_1}{\partial t} \\ \vdots \\ \frac{\partial \eta_m}{\partial t} \end{bmatrix} \quad (C8)$$

where the deformation and deformation velocity vectors ($\mathbf{u}$ and $\dot{\mathbf{u}}$ respectively) follow the same space discretisation pattern as the mode shapes.

Figure E1. The geometry of the inextensible string sidewall element.

The global position of a belt node (z), as used in section 3 for the calculation of the tread contact forces, may be expressed as:

$$X_b = R \cdot \cos \theta_z + u_X^z \quad (C9)$$

$$Y_b = R \cdot \sin \theta_z + u_Y^z \quad (C10)$$

Appendix D. Orientation of the tread elements

The tread orientation angle, a , of eqn. 15, which is also used for the calculation of the tread's global stiffness and damping values (Appendix A), is primarily a function of the rotation of the belt node around the rim centre, $a_{rotation}$.

A secondary possible contribution to the angle a may exist, expressed as the deviation (torsion) of the tread's orientation from the perpendicular to the belt's tangent at the attachment point. This angle corresponds to the rotation of the tread tip around the node and for the ring model (Appendix B), can be expressed as :

$$a_{torsion} = -\frac{1}{R} \frac{\partial u_r}{\partial \theta} \quad (D1)$$

The total angle, a , can be calculated as :

$$a = a_{rotation} + a_{torsion} \quad (D2)$$

Appendix E. The inextensible string based, non linear, sidewall model

The tension force of a string is proportional to the inflation pressure and its radius, and its direction is always tangential to the string. Its magnitude is given by the equation:

$$T = P_0 \cdot r \quad (E1)$$

It follows from the geometry presented in fig. E:

$$\left. \begin{aligned} \frac{\pi-2w}{l_{string}} &= \frac{1}{r} \\ \cos w &= \frac{L}{2 \cdot r} \end{aligned} \right\} \rightarrow L(\pi - 2w) = 2l_{string} \cdot \cos w \quad (\text{E2})$$

where :

T	string force, circumference length normalized
T_r	in-plane component of the string force
T_n	out-of-plane component of the string force
L	distance between wheel and belt corresponding points
P_0	inflation pressure
r	radius of the string
l_{string}	length of the string
w	angle between the wheel axis of symmetry and the direction, vertical to the in-plane projection of the string (see fig. E)

The nonlinear equation E2 is solved with an iteration procedure in every calculation step. The angle w is calculated for the instantaneous distance, L , which depends on the deformation of the belt.

The in-plane component of the string force is given by the equation:

$$T_r = P_0^* \cdot l_{string} \frac{\sin w}{\pi - 2w} \quad (\text{E3})$$

where P_0^* is the discretisation-normalized inflation pressure, eliminating the discretisation effect on the force calculation.

$$P_0^* = P_0 \frac{2 \cdot \pi \cdot R}{n_b} \quad (\text{E4})$$

The in-plane component of the string force lies on the direction of the string element, so it contributes to both vertical and horizontal node forces, according to the angle $\alpha_{rotation}$ (fig. E). This angle expresses the relative angular orientation of the belt node to the corresponding wheel attachment point:

$$T_{rX}^z = T_r \cdot \cos \alpha_{rotation} \quad (\text{E5})$$

$$T_{rY}^z = T_r \cdot \sin \alpha_{rotation}, \quad 1 \leq z \leq n_b \quad (\text{E6})$$

Finally, the sidewall force is expressed in the complete belt discretised (vector) form, following the pattern of eqn. C6.