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Abstract For addressing the Allis-type anomalies, a fractional degree reference dependent stochastic dominance rule is developed which is a generalization of the integer degree reference dependent stochastic dominance rules. This new rule can effectively explain why the risk comparison does not satisfy translational invariance and scaling invariance in some cases. The rule also has a good property that it is compatible with the endowment effect of risk. This rule can help risk-averse but not absolute risk-averse decision makers to compare risks relative to reference points. We present some tractable equivalent integral conditions for the fractional degree reference dependent stochastic dominance rule, as well as some practical applications for the rule in economics and finance.

Keywords Reference dependent stochastic dominance · Consumption utility · Reference point

Mathematics Subject Classification (2000) 60E15

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1 Introduction

The stochastic dominance (SD) via the expected utility theory has been proved a powerful tool for ranking the distributions ([15], [30]) and it is usually employed in various risky prospect selection problem ([1], [20], [9], [36]). Fractional degree stochastic dominance ($(1+\gamma)$-SD) rules have been developed in [25] to reveal a preference of the investor whose utility function is not concave everywhere by defining a new class of differential utility function as:

$$U_\gamma = \{u: \; 0 \leq \gamma u'(y) \leq u'(x), \text{ for all } x, y \in \Re, x \leq y\} \quad \gamma \in [0,1] \tag{1}$$

The $(1+\gamma)$-SD rules play an important role in several fields, particularly in economic and financial researches, and effectively provide economic interpretations to some sensible risk preferences pioneered by [12] and [7].

However, many recent literatures provide some nontrivial empirical observations on decision behaviours which are inconsistent at all with the classical expected utility theory framework. For example, [27] present the famous Allais-type anomalies, [18] introduce the endowment effect for risk, and the risk comparison does not satisfy translational invariance and scaling invariance in some cases. Thus, the fractional degree stochastic dominance rule based on expected utility theory does not have the abilities to resolve Allais-type anomalies, to capture the violation of translational invariance and scaling invariance, and to accommodate the endowment effect for risk. This makes it meaningful to furthermore develop the fractional degree stochastic dominance rule such that it possesses these appealing properties.

Reference dependency theory has been introduced in [34]. It can explain why many risk attitudes are inconsistent with the classical expected utility theory with some different specifications of the reference point, including the status quo, lagged status quo, and the mean of the chosen risk. The reference dependent utility theory has been developed in [18]. For a wealth level x and a fixed reference point r, the utility function $v(x;r,u)$ in their theory is separated into two terms

$$v(x;r,u) = u(x) + \mu\left(u(x) - u(r)\right) \tag{2}$$

where the term $u(x)$ is intrinsic "consumption utility" usually assumed relevant in economics, and the term $\mu\left(u(x) - u(r)\right)$ is the reference-dependent gain-loss utility. This separation and interdependence of economic refer to assumptions made previously by [2], [33], [18]. It has been widely used in modeling agents preferences with reference points.

We develop a fractional degree reference dependent stochastic dominance rule based on the model (2) in this paper. Reference dependent stochastic dominance is not a new notion. Integer degree reference dependent stochastic dominance rules has been proposed in [13] under the assumption that the reference-dependent gain-loss utility $\mu(\cdot)$ in model (2) satisfies

$$\mu(x) = \begin{cases} \eta x, & x > 0, \\ \eta\lambda x, & x \leq 0. \end{cases} \tag{3}$$

where $\eta > 0$ is the relative weight of gain-loss utility, and $\lambda > 1$ measures the magnitude of loss aversion. With reference point r, the preference of investor whose consumption utility is increasing is given by the first degree reference dependent stochastic dominance (denoted by FSD^r), and the increasing concave consumption utility is given by the second degree reference dependent stochastic dominance (denoted by SSD^r). The fractional degree reference dependent stochastic dominance corresponds to the consumption utility in U_γ. It can help risk-averse but not absolute risk-averse decision makers to compare risks relative to reference points. This paper focused on relating this new stochastic dominance rule to some simple and tractable equivalent conditions and discussing some practical applications of the rule in the economic and financial fields.

This paper is organized as follows. In section 2, we introduce the basic definition of fractional degree reference-dependent stochastic dominance and devote to find out its equivalent integral conditions. In section 3, we explore the relationships between the fractional degree reference dependent stochastic dominance and the fractional degree stochastic dominance, and analyze the effects of reference point on risk choice. In section 5 and section 6, the fractional degree stochastic dominance rule is extended to stochastic reference points. Two applications to financial field are illustrated. Throughout paper, we assume that all distributions have a finite mean and that all expected utilities are finite.

2 Definition and basic properties

2.1 The definitions

We begin with the concept of fractional degree reference dependent stochastic dominance rules. For a risky asset X and utility function $v(x;r,u)$ with the expression (2), define the expected reference-dependent utility function as

$$E\left[v\left(X;r,u\right)\right] = E\Big[u(X)\Big] + E\Big[\mu\big(u(X) - u(r)\big)\Big] \tag{4}$$

In this paper, we assume that the reference-dependent gain-loss utility $\mu(\cdot)$ satisfies

$$\mu(x) = \begin{cases} \eta x, & x > 0, \\ \eta\lambda x, & x \le 0. \end{cases}$$

and assume that $\eta^* \ge 0$ and $\lambda^* \ge 1$ are pre-specified lower bounds for the parameters η and λ, respectively.

Definition 1 Let X and Y be two random variables. The reference point r is given. Then X is stochastically dominated by Y relative to r in $(1+\gamma)$ order, denoted by $X \le_{(1+\gamma)\text{-}SD}^{r,\eta^*,\lambda^*} Y$, if for all consumption utility $u \in U_\gamma$, $\eta \ge \eta^*$, $\lambda \ge \lambda^*$,

$$E\left[v\left(X;r,u\right)\right] \le E\left[v\left(Y;r,u\right)\right] \tag{5}$$

Similar as Definition 2 in [13], we impose a specific condition that r should be given in Definition 1, which can limit the generality of the stochastic dominance in the sense that the ranking results are no longer consistent with different reference points, see the following Proposition 2. But it is rational from the practical viewpoint because the reference point is generally a benchmark or a target followed by individual. When the reference point tends to infinity, the $(1+\gamma)$-SD^r rules are reduced to $(1+\gamma)$-SD rules.

However, $(1+\gamma)$-SD^r can not be defined for any $\gamma > 1$. Because U_γ is empty except it contains zero function in this case. But we can define the high-order fractional reference dependent stochastic dominance, denoted by $(n+\gamma)$-SD^r, by using the consumption utility function class introduced in [4]. In this paper, we focus on the development of $(1+\gamma)$-SD^r.

In Definition 1, γ provides a bound on how much marginal utility can decrease as x decreases. It is obvious that

$$U_0 = \{u : u'(x) \geq 0 \text{ for all } x \in \Re\}$$

and

$$U_1 = \{u :\ 0 \leq u'(y) \leq u'(x),\ \text{ for all } x, y \in \Re, x \leq y\}.$$

Thus, 1-SD^r is equivalent to FSD^r, while 2-SD^r is equivalent to SSD^r. The $(1+\gamma)$-SD^r establishes an interpolating between FSD^r and SSD^r. It is also clear that for any $0 \leq \gamma_1 \leq \gamma_2 \leq 1$ and given reference point r, $X \preceq^{r,\lambda^*,\eta^*}_{(1+\gamma_1)\text{-}SD} Y$ implies $X \preceq^{r,\lambda^*,\eta^*}_{(1+\gamma_2)\text{-}SD} Y$ because $U_{\gamma_1} \subseteq U_{\gamma_2}$.

The differentiability condition in Definition 1 is not critical as Definition 2.3 in [25], we can replace it by U^*_γ defined as

$$U^*_\gamma = \left\{u : 0 \leq \gamma \frac{u(x_4) - u(x_3)}{x_4 - x_3} \leq \frac{u(x_2) - u(x_1)}{x_2 - x_1}, \forall x_1 \leq x_2 \leq x_3 \leq x_4\right\} \quad (6)$$

[5] has considered U^*_γ in term of the generalization of expected utility model. They defined a index of greediness of utility function $u(\cdot)$ as

$$g_u = \sup_{x_1<x_2<x_3<x_4} \left\{\frac{u(x_4) - u(x_3)}{x_4 - x_3} \Big/ \frac{u(x_2) - u(x_1)}{x_2 - x_1}\right\} \quad (7)$$

U^*_γ and U_γ are both the class of utility function with $g_u \leq \frac{1}{\gamma}$.

2.2 Basic properties

We provide an equivalent integral condition for (1+γ)-SD^r in the following. Define a function

$$f_{\eta^*,\lambda^*}(x; r) = \begin{cases} \frac{1+\eta^*\lambda^*}{1+\eta^*}, & \text{if } x \leq r \\ 1, & \text{if } x > r. \end{cases} \quad (8)$$

Theorem 1 *Let X and Y be two random variables with cumulative distribution functions (cdfs) $F(x)$ and $G(x)$, respectively. The following three conditions are equivalent:*

[1]. $X \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)\text{-}SD} Y$;
[2]. $E[v(X;r,u)] \leq E[v(Y;r,u)]$ *for all functions* $u \in U_\gamma{}^*, \eta \geq \eta^*, \lambda \geq \lambda^*$;
[3]. *For all* $t \in R$,

$$\int_{-\infty}^{t} \left[(G(x) - F(x))_+ - \gamma\,(F(x) - G(x))_+\right] f_{\eta^*,\lambda^*}(x,r)dx \leq 0 \qquad (9)$$

where $(x)_+ = \max\{x, 0\}$.

Theorem 1 offers a precise characterization of $(1+\gamma)$-SD^r. From Theorem 1, we can know that the fractional degree reference dependent stochastic dominance rules has no translational invariance and scaling invariance, and can obtain the following immediate consequences:

(1). Assume that l_X and l_Y are the left support points of X and Y, respectively. If $X \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)\text{-}SD} Y$, then $l_X \leq l_Y$.
(2). Let G_i be the cdfs of random variables Y_i, $i = 1, 2$. For $\alpha \in (0, 1)$, let $\alpha G_1 + (1 - \alpha)G_2$ be the cdf of random variable Z. If $Y_i \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)\text{-}SD} X$, $i = 1, 2$, then $Z \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)\text{-}SD} X$.
(3). Let Y_i, $i = 1, 2, \ldots$ be a random variable sequence. Assume that $Y_i \to_d Y$ as $i \to \infty$. If $X \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)-SD} Y_i$, $i = 1, 2, \ldots$, then $X \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)-SD} Y$.

The integral condition of Theorem 1 can be simple when F is a simple spread of G. First, we recall the concept of simple spreads. Let X and Y be two random variables with cdfs $F(x)$ and $G(x)$, respectively. X is called a simple spread of Y if there exists a single crossing $x_0 \in \Re$ such that

$$F(x) \geq G(x) \text{ on } (-\infty, x_0] \text{ and } F(x) \leq G(x) \text{ on } (x_0, \infty).$$

Corollary 1 *Suppose F is a simple spread of G at x_0. Given reference point r, $\eta^* > 0$, $\lambda^* > 1$, denote*

$$A = \int_{-\infty}^{x_0} (F(x) - G(x))\, f_{\eta^*,\lambda^*}(x,r)dx \qquad (10)$$

and

$$B = \int_{x_0}^{\infty} (G(x) - F(x))\, f_{\eta^*,\lambda^*}(x,r)dx. \qquad (11)$$

Then, $F \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)-SD} G$ *if and only if*

$$\gamma \geq \frac{B}{A}.$$

Proof Define the function:

$$\phi^{\eta^*,\lambda^*}_{\gamma,r}(x) = \int_{-\infty}^{x} \left[\gamma\,(F(t) - G(t))_+ - (G(t) - F(t))_+\right] f_{\eta^*,\lambda^*}(t,r)dt.$$

If F and G satisfy the single spread condition with A and B defined in formulas (10) and (11), the function $\phi_{\gamma,r}^{\eta^*,\lambda^*}(x)$ is increasing in $(-\infty, x_0)$ and decreasing in (x_0, ∞) with

$$\lim_{x\to-\infty} \phi_{\gamma,r}^{\eta^*,\lambda^*}(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} \phi_{\gamma,r}^{\eta^*,\lambda^*}(x) = \gamma A - B.$$

Therefore, $\phi_{\gamma,r}^{\eta^*,\lambda^*}(x)$ is nonnegative if and only if $\gamma A - B \geq 0$.

Note that if $A = B$, then the condition holds for $\gamma = 1$ and $F \leq_{2-SD}^{r,\eta^*,\lambda^*} G$. If $B = 0$, which means $F(x) \geq G(x)$ everywhere, thus, $F \leq_{1-SD} G$. When $0 < B < A$ but $F(x) < G(x)$ for some x, then $F \leq_{(1+\gamma)-SD}^{r,\eta^*,\lambda^*} G$ for $\gamma \in [\frac{B}{A}, 1]$. Therefore, when $F \leq_{2-SD}^{r,\eta^*,\lambda^*} G$, but $F \nleq_1 G$, we can always determine the smallest $\gamma = \gamma(F, G, r)$ such that $F \leq_{(1+\gamma)-SD}^{r,\eta^*,\lambda^*} G$. This $\gamma(F, G, r)$ can be interpreted as an index of risk aversion that makes it necessary for a decision maker to prefer G to F relative to reference point r. From corollary 1, in the case of a single spread of the distribution functions, $\gamma(F, G, r) = B/A$.

The following corollary generalizes the single-crossing result to the situation where F and G cross multiple times. The proof is easy. we omit it in this paper.

Corollary 2 *Suppose that there exist $n \in \mathbb{N}$ and $x_1 < x_2 < \ldots < x_n$ with $x_0 = -\infty$ and $x_{n+1} = \infty$ such that $F(x) \geq G(x)$ for $x_{i-1} < x < x_i$ if i is odd and $F(x) \leq G(x)$ for $x_{i-1} < x < x_i$ if i is even, $i = 1, \ldots, n+1$. For $i = 1, \ldots, n+1$, define*

$$A_i = \int_{x_{i-1}}^{x_i} (F(x) - G(x))_+ f_{\eta^*,\lambda^*}(x, r)dx$$

and

$$B_i = \int_{x_{i-1}}^{x_i} (G(x) - F(x))_+ f_{\eta^*,\lambda^*}(x, r)dx.$$

Then, $F \leq_{(1+\gamma)-SD}^{r,\eta^*,\lambda^*} G$ if and only if, for all $j = 0, 1, \ldots, n$,

$$\gamma \geq \frac{\sum_{i=1}^{j+1} B_i}{\sum_{i=1}^{j+1} A_i}.$$

2.3 Examples

In the following, we provide some examples to illustrate our results.

Example 1 (Binary distribution). Let X and Y be two binary random variables with probability mass functions given by

$$P(X = x_1) = p = 1 - P(X = x_2) \quad \text{and} \quad P(Y = y_1) = q = 1 - P(Y = y_2),$$

where $x_1 < x_2$ and $y_1 < y_2$. Let F and G be two cdfs of X and Y, respectively.

– If $x_1 < x_2 \leq y_1 < y_2$, then $F(x) > G(x)$ for all $x \in \Re$. Thus, $X \leq_{1-SD} Y$.
– If $x_1 < y_1 < x_2 < y_2$, then

$$G(x) - F(x) = \begin{cases} -p, & x_1 \leq x < y_1, \\ q-p, & y_1 \leq x < x_2, \\ q-1, & x_2 \leq x < y_2, \\ 0, & otherwise. \end{cases}$$

Thus, for given reference point r and $\gamma \in [0,1]$, $X \leq^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} Y$ if and only if

$$\gamma \geq \frac{(q-p)\int_{y_1}^{x_2} f_{\eta^*,\lambda^*}(x,r)dx}{p\int_{x_1}^{y_1} f_{\eta^*,\lambda^*}(x,r)dx}.$$

– If $x_1 \leq y_1 < y_2 \leq x_2$, similarly, we have $X \leq^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} Y$ if and only if

$$\gamma \geq \frac{(q-p)_+\int_{y_1}^{y_2} f_{\eta^*,\lambda^*}(x,r)dx + (1-p)\int_{y_2}^{x_2} f_{\eta^*,\lambda^*}(x,r)dx}{p\int_{x_1}^{y_1} f_{\eta^*,\lambda^*}(x,r)dx + (p-q)_+\int_{y_1}^{y_2} f_{\eta^*,\lambda^*}(x,r)dx}.$$

Example 2 (Uniform distribution). Let X and Y be two random variables uniformly distributed over the intervals (a,b) and (c,d), respectively. Assume that for any reference point r and $\gamma \in [0,1]$, $X \leq^{\gamma,\lambda^*,\eta^*}_{(1+\gamma)-SD} Y$. Based on the formula (9), we have $a \leq c$.

– If $a < c$ and $b < d$, then $X \leq_{1-SD} Y$.
– If $a < c < d \leq b$, then

$$G(x) - F(x) = \begin{cases} -\frac{x-a}{b-c}, & a \leq x < c, \\ \frac{x-c}{d-c} - \frac{x-a}{b-a}, & c \leq x < d, \\ 1 - \frac{x-a}{b-a}, & d \leq x < b, \\ 0 & otherwise. \end{cases}$$

Clearly, F singlely crosses G at $x_0 \in (c,d)$ from the above, where

$$x_0 = c + \frac{(d-c)(c-a)}{(b+c-a-d)}.$$

Let

$$A := \int_a^{x_0} (F(x) - G(x))\, f_{\eta^*,\lambda^*}(x;r)dx,$$

and

$$B := \int_{x_0}^{b} (G(x) - F(x))\, f_{\eta^*,\lambda^*}(x;r)dx.$$

From Corollary 1, we have $X \leq^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} Y$ if and only if

$$\gamma \geq \frac{B}{A}.$$

Example 3 (Normal distribution). Let X and Y be two normal random variables with means μ_1, μ_2 and standard derivations σ_1, σ_2, respectively. Assume that, for any reference point r and $\gamma \in [0,1]$, $X \leq^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} Y$. The cdfs are single-crossing at

$$x_0 = \frac{\mu_2\sigma_1 - \mu_1\sigma_2}{\sigma_1 - \sigma_2}.$$

Let

$$A := \int_{-\infty}^{x_0} \left[\Phi\left(\frac{x-\mu_1}{\sigma_1}\right) - \Phi\left(\frac{x-\mu_2}{\sigma_2}\right)\right] f_{\eta^*,\lambda^*}(x,r)dx,$$

and

$$B := \int_{x_0}^{\infty} \left[\Phi\left(\frac{x-\mu_2}{\sigma_2}\right) - \Phi\left(\frac{x-\mu_1}{\sigma_1}\right)\right] f_{\eta^*,\lambda^*}(x,r)dx.$$

Thus, $X \leq^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} Y$ if and only if

$$\gamma \geq \frac{B}{A}.$$

3 Advanced properties

3.1 Compared with fractional degree stochastic dominance

[25] have provided an equivalent integral condition for $(1+\gamma)$-SD, that is,

$$X \leq_{(1+\gamma)-SD} Y$$

if and only if, for any $t \in \Re$,

$$\int_{-\infty}^{x} \gamma \left(F(x) - G(x)\right)_+ - \left(G(x) - F(x)\right)_+ dx \geq 0 \tag{12}$$

And since $f_{\eta^*,\lambda^*}(x;r)$ is positive and decreasing for any fixed reference point r and $\lambda^* > 1$ and $\eta^* > 0$, The formula (12) implies that

$$\int_{-\infty}^{x} \left[\gamma \left(F(x) - G(x)\right)_+ - \left(G(x) - F(x)\right)_+\right] f_{\eta^*,\lambda^*}(x;r)dx \geq 0.$$

Therefore, $X \leq_{(1+\gamma)\text{-}SD} Y$ implies that $X \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)\text{-}SD} Y$. But we can prove the converse is not true.

First, we recall the definition of a γ -transfer. [25] have proved that a decision maker's preferences satisfy $(1+\gamma)$-SD if and only if any γ -transfer are acceptable.

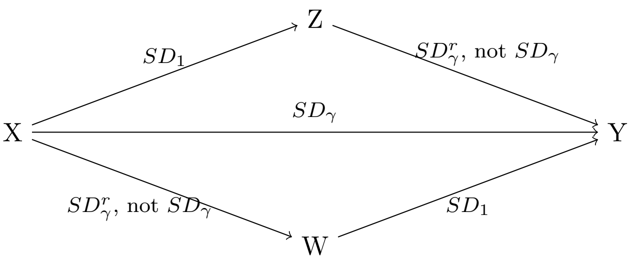
Definition 2 ([25]) Let X and Y be two discrete random variables with cdfs F and G, respectively. G is obtained from F via a γ - transfer if there exist $x_1 < x_2 \leq x_3 < x_4$ and $\zeta_1, \zeta_2 \geq 0$ with $\gamma\zeta_1(x_2 - x_1) = \zeta_2(x_4 - x_3)$ such that

$$G(x) - F(x) = \begin{cases} -\zeta_1, & x_1 \leq x \leq x_2, \\ \zeta_2, & x_3 \leq x \leq x_4, \\ 0, & \text{for all other values } x. \end{cases} \tag{13}$$

Proposition 1 *Let X and Y be two discrete random variables with cdfs F and G, respectively. For any $\gamma \in [0,1]$, if the condition (13) holds , then when the reference point satisfies $x_1 < r < x_4$, we have*

[1]. *There exists a random variable Z, which satisfies that $X \leq_{1-SD} Z$ and $Z \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)-SD} Y$ but not $Z \leq_{(1+\gamma)-SD} Y$.*
[2]. *There exists a random variable W, which satisfies that $W \leq_{1-SD} Y$ and $X \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)-SD} W$ but not $X \leq_{(1+\gamma)-SD} W$.*

The Proposition 1 illustrates that $(1+\gamma)$- SD is just a sufficient but not a necessary condition of $(1+\gamma)$-SD^r. If we simplify the symbols $(1+\gamma)$-SD^r and $(1+\gamma)$-SD as SD^r_γ and SD_γ, respectively. Then, the above relationship can be summarized as follows:



Further, we will give an example to illustrate Proposition 1.

Example 4 Suppose $r = 0$, Let $X = (-10, 0.5; 10, 0.5)$ and $Y \equiv 10\alpha$, $0 < \alpha < 1$. For $\gamma = \frac{1-\alpha}{\alpha+1}$, we have $X \leq_{(1+\gamma)\text{-}SD} Y$. Set

$$Z = (-10, 0.5; z, 0.5) \text{ with } z = 10\gamma\left(\alpha + \frac{1+\eta^*\lambda^*}{1+\eta^*}\right) + 10\alpha.$$

We can verify that $Z \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)\text{-}SD} Y$ but not $Z \leq_{(1+\gamma)\text{-}SD} Y$. Obviously, $z > 10$. Hence $X \leq_{1\text{-}SD} Z$. Set

$$W = (-10, 0.5 - \zeta_1^*; 10\alpha, 0.5 + \zeta_1^*) \text{ with } \zeta_1^* = \frac{0.5(1-\alpha)(1+\eta^*)}{\gamma\left[1 + \eta^*\lambda^* + \alpha(1+\eta^*)\right]},$$

we can verify that $X \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)\text{-}SD} W$ but not $X \leq_{(1+\gamma)\text{-}SD} W$, and $W \leq_{1\text{-}SD} Y$.

3.2 Influences of reference points

Let us continue with Example 4. Let $G(x)$ and $F_1(x)$ be the cdfs of Y and Z, respectively. Then, we have

$$\gamma \int_{-10}^{10\alpha} (F_1(x) - G(x))_+ f_{\eta^*,\lambda^*}(x;0)dx = \int_{10\alpha}^{z} (G(x) - F(x))_+ f_{\eta^*,\lambda^*}(x;0)dx.$$

Since $f_{\eta^*,\lambda^*}(x;0) > f_{\eta^*,\lambda^*}(x;-1)$ on $[-1,0]$ and $f_{\eta^*,\lambda^*}(x;0) = f_{\eta^*,\lambda^*}(x;-1)$ on otherwise, it holds that

$$\gamma \int_{-10}^{10\alpha} (F_1(x) - G(x))_+ f_{\eta^*,\lambda^*}(x;-1)dx$$
$$< \int_{10\alpha}^{z} (G(x) - F(x))_+ f_{\eta^*,\lambda^*}(x;-1)dx.$$

That is, taking $r = -1$, $Z \not\leq^{r,\eta^*,\lambda^*}_{(1+\gamma)\text{-}SD} Y$. This illustrates that the investor's preference is no longer preserved with different reference points. Furthermore, we confirms that, given two different reference points, we always find out two risks such that the investors cannot agree with each other on the risk choice (see the next proposition 2).

Proposition 2 *For two fixed reference points r_1 and r_2 where $r_1 \neq r_2$, two risks X and Y exist such that $X \leq^{r_1,\eta^*,\lambda^*}_{(1+\gamma)-SD} Y$ but $X \not\leq^{r_2,\eta^*,\lambda^*}_{(1+\gamma)-SD} Y$.*

4 Generating process of $(1+\gamma) - SD^r$

In Theorem 1, we use the integral conditions to characterize $(1+\gamma)$-SD relative to the reference point r. In this section, we use the similar way in [30] and [25] to characterize the stochastic dominance relations with dependent reference point by some transfers of probability. For given cdfs F, G and the reference point r, define the function $H_{\gamma,r}(x)$ with its right derivative

$$H'_{\gamma,r}(x) = \begin{cases} \frac{1+\eta^*\lambda^*}{1+\eta^*}, & F(x) \leq G(x) \text{ and } x \leq r, \\ 1, & F(x) \leq G(x) \text{ and } x > r, \\ \gamma\frac{1+\eta^*\lambda^*}{1+\eta^*}, & F(x) > G(x) \text{ and } x \leq r, \\ \gamma, & F(x) > G(x) \text{ and } x > r. \end{cases}$$

Then, the sufficient and necessary condition of $X \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)-SD} Y$ can be rewritten as

$$\int_{-\infty}^{x} F(t)dH_{\gamma,r}(t) \geq \int_{-\infty}^{x} G(t)dH_{\gamma,r}(t), \quad \text{for all} \quad x \in \Re \tag{14}$$

Definition 3 Let X and Y be two discrete random variables with respective cdfs F and G. r is a given reference point.

(a) G is obtained from F via an increasing transfer if there exist $x_1 < x_2$ and $\zeta > 0$ such that

$$G(x) - F(x) = -\zeta \text{ on } [x_1, x_2],$$
$$G(x) - F(x) = 0 \text{ for all other values } x.$$

(b) G is obtained from F via a (γ, r) - transfer if there exist $x_1 < x_2 \le x_3 < x_4$ and $\zeta_1,\ \zeta_2 \ge 0$ with $\zeta_1\,[H_{\gamma,r}(x_2) - H_{\gamma,r}(x_1)] = \zeta_2\,[H_{\gamma,r}(x_4) - H_{\gamma,r}(x_3)]$ such that

$$\begin{aligned} G(x) - F(x) &= -\zeta_1 \text{ on } [x_1, x_2],\\ G(x) - F(x) &= \zeta_2 \text{ on } [x_3, x_4],\\ G(x) - F(x) &= 0 \text{ for all other values } x. \end{aligned}$$

In Definition 3, if we take $\lambda^* = 2$, $\eta^* = \frac{1}{2}$ and the reference point $r = 0$, then $(1/3, 0)$ - transfer between the mass function F and G for $x_1 = -2, x_2 = x_3 = 1, x_4 = 2$ is illustrated as Fig. 1. For the general (γ, r) - transfer, the difference $G - F$ is a positive constant between x_3 and x_4, a negative constant between x_1 and x_2, and zero otherwise in such a way that the area between the graph of $(G - F) * H^{'}_{\gamma,r}$ in $[x_3, x_4]$ and the x-axis is same as the area between the graph of $(G - F) * H^{'}_{\gamma,r}$ in $[x_1, x_2]$ and the x-axis. If $r = \infty$ and $r = -\infty$, then our (γ, r)- transfer is the γ-transfer, see Definition 2. [25] employ a family of γ-transfers to characterize their $(1+\gamma)$-SD. Theorem 3 below is in the same spirit.

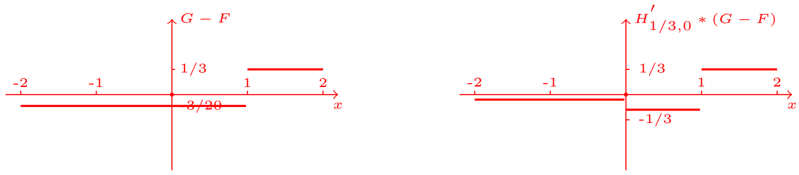


Fig. 1 The difference $G-F$ corresponding to $(1/3, 0)$-transfer (left panel) and the difference $(G - F) * H^{'}_{1/3,0}$ corresponding to $(1/3, 0)$-transfer (right panel).

Theorem 2 *Suppose that X and Y have only a finite number of values. Then $X \le^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} Y$ if and only if G can be obtained from F via a finite sequence of (γ, r)-transfers and increasing transfers.*

For continuous random variables we first show that the order $\le^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD}$ behaves well with respect to the convergence in distribution. The notation $X_n \Rightarrow X$ indicates that X_n converges to X in distribution and $E[X_n] = E[X]$.

Theorem 3 *Assume that X and Y are random variables with finite means. Then $X \le^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} Y$ if and only if there exist two sequences $\{X_n\}$ and $\{Y_n\}$ with finite supports such that*

$$X_n \Rightarrow X, \quad Y_n \Rightarrow Y, \quad \textit{and} \quad X_n \le^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} Y_n.$$

Combing Theorem 2 and Theorem 3 yields the following corollary.

Corollary 3 *Given two random variables X and Y, $X \leq^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} Y$ if and only if there exist two sequence $\{X_n\}$ and $\{Y_n\}$ such that for all n the distribution of Y_n can be obtained from the distribution of X_n via a finite sequence of (γ, r)-transfers and increasing transfers, $X_n \Rightarrow X$ and $Y_n \Rightarrow Y$.*

5 Extension to stochastic reference points

The generalization of model (4) is motivated by the casual observations and psychological research. To evaluate an outcome, investors can make multiple comparisons with several reference points such as indices of different markets, portfolios of their friends and so on. [18] proposed that taking the reference as a distribution is more suitable than simply assuming references to be a single and deterministic point. In this section, we will extend our $(1+\gamma)$-SD^r stochastic dominance rules to stochastic reference points.

As introduced by [18], we assume the investor takes reference random variable R with cdf H, which is independent to the risky wealth X with cdf F. Then the most general investor's expected reference-dependent utility can be written as

$$\begin{aligned} E\left[v(X;R,u)\right] &= E\left[u(X) + \mu\left(u(X) - u(r)\right)\right] \\ &= \int\int \left[u(x) + \mu\left(u(x) - u(r)\right)\right] dF(x) dH(r) \end{aligned} \tag{15}$$

Similar to Definition 1, we can define $(1+\gamma)$ degree stochastic dominance relation relative to reference random variable R, abbreviated $(1+\gamma)$-SD^R.

Definition 4 Let X and Y be two random variables. The reference random variable R with cdf H is given. Then X is stochastically dominated Y relative to R in $(1+\gamma)$ order, denoted by $X \leq^{R,\eta^*,\lambda^*}_{(1+\gamma)-SD} Y$, if for all consumption utility $u \in U_\gamma$, $\eta \geq \eta^*$, $\lambda \geq \lambda^*$,

$$E\left[v\left(X;R,u\right)\right] \leq E\left[v\left(Y;R,u\right)\right] \tag{16}$$

Define

$$f_{\eta^*,\lambda^*}(x;H) = \frac{1+\eta^*\lambda^*}{1+\eta^*} - \frac{\eta^*(\lambda^*-1)}{1+\eta^*} H(x) \tag{17}$$

Parallel to Theorem 1, we provide Theorem 4 below which is an integral condition of $(1+\gamma)$-SD^R.

Theorem 4 *Let X and Y be two random variables with cdfs F and G, respectively. Let R be a reference random variable with cdf H, which is independent to X and Y. Then $X \leq^{R,\eta^*,\lambda^*}_{(1+\gamma)-SD} Y$ if and only if for all $t \in \Re$,*

$$\int_{-\infty}^{t} \left(G(x) - F(x)\right)_+ f_{\eta^*,\lambda^*}(x;H) dx \leq \int_{-\infty}^{t} \gamma\left(F(x) - G(x)\right)_+ f_{\eta^*,\lambda^*}(x;H) dx \tag{18}$$

In the following, we will give an example to illustrate the use of Theorem 4.

Example 5 (Normal distribution). Let X and Y be two normal random variables with means μ_1, μ_2 and standard derivations σ_1, σ_2, respectively. Assume that $\mu_1 < \mu_2$ and the two cdfs are single-crossing at the point $x_0 = \frac{\mu_2\sigma_1 - \mu_1\sigma_2}{\sigma_1 - \sigma_2}$. For any stochastic reference point R with the cdf $H(x)$, let

$$A := \int_{-\infty}^{x_0} \left[\Phi\left(\frac{x-\mu_1}{\sigma_1}\right) - \Phi\left(\frac{x-\mu_2}{\sigma_2}\right) \right] f_{\eta^*,\lambda^*}(x;H)dx,$$

and

$$B := \int_{x_0}^{\infty} \left[\Phi\left(\frac{x-\mu_2}{\sigma_2}\right) - \Phi\left(\frac{x-\mu_1}{\sigma_1}\right) \right] f_{\eta^*,\lambda^*}(x;H)dx.$$

Then, $X \leq^{R,\lambda^*,\ \eta^*}_{(1+\gamma)-SD} Y$ if and only if

$$\frac{B}{A} \leq \gamma < 1.$$

Furthermore, we consider a question whether the stochastic multiple reference can be reduced to a single excepted non-stochastic reference point for the fractional degree reference dependent stochastic dominance rule, that is, whether the following result is true. For any $t \in \Re$ and $r = E(R)$,

$$\int_{-\infty}^{t} (G(x) - F(x))_+ f_{\eta^*,\lambda^*}(x,H)dx \leq \int_{-\infty}^{t} \gamma\left(F(x) - G(x)\right)_+ f_{\eta^*,\lambda^*}(x,H)dx$$

if and only if,

$$\int_{-\infty}^{t} (G(x) - F(x))_+ f_{\eta^*,\lambda^*}(x,r)dx \leq \int_{-\infty}^{t} \gamma\left(F(x) - G(x)\right)_+ f_{\eta^*,\lambda^*}(x,r)dx.$$

But such equivalence is not generally available. Let us see the following example.

Example 6 For given $\epsilon > 0$ and $x_0 = \gamma\epsilon\frac{1+\eta^*\lambda^*}{1+\eta^*}$, suppose that investors are asked to choose between the risky asset $X = (x_0, 0.5; -\epsilon, 0.5)$ and the risk-free asset $Y = 0$. If the dependent-reference point is a real number $r = 0$, then

$$\int_{-\infty}^{x} \left(\gamma(F(x) - G(x))_+ - (G(x) - F(x))_+\right) f_{\eta^*,\lambda^*}(x;r)dx$$
$$= \begin{cases} 0, & x \leq -\epsilon \\ \frac{\gamma(x+\epsilon)(1+\eta^*\lambda^*)}{2(1+\eta^*)}, & -\epsilon < x \leq 0 \\ \frac{\gamma\epsilon(1+\eta^*\lambda^*)}{2(1+\eta^*)} - \frac{x}{2}, & 0 < x \leq x_0 \\ \frac{\gamma\epsilon(1+\eta^*\lambda^*)}{2(1+\eta^*)} - \frac{x_0}{2}, & x > x_0 \end{cases}$$

Clearly, $X \preceq^{r,\eta^*,\lambda^*}_{(1+\gamma)-SD} Y$. Let $R = (-\epsilon, 50\%; \epsilon, 50\%)$ with $E(R) = 0$. Then when $\frac{\gamma(1+\eta^*\lambda^*)}{1+\eta^*} > 1$, and $\epsilon < x \leq x_0$,

$$\int_{-\infty}^{x} \left(\gamma(F(x) - G(x))_+ - (G(x) - F(x))_+\right) f_{\eta^*,\lambda^*}(x; R)dx$$
$$= \frac{\epsilon(\gamma - 1)}{2} \left(\frac{1 + \eta^*\lambda^*}{1 + \eta^*} - \frac{\eta^*(\lambda^* - 1)}{2(1 + \eta^*)} \right) - \frac{1}{2}(x - \epsilon) < 0,$$

that is, $X \not\preceq^{R,\eta^*,\lambda^*}_{(1+\gamma)-SD} Y$.

6 Applications

6.1 Allais-type anomalies

[27] provide two following risk choice problems with the Allais-type anomalies.

Example 7 (Common consequence effect)

Scenario 1. Choose between $X_1 = (0, 1\%; 2400, 66\%; 2500, 33\%)$ and $Y_1 \equiv 2400$

Scenario 2. Choose between $X_2 = (0, 67\%; 2500, 33\%)$ and $Y_2 = (0, 66\%; 2400, 34\%)$.

The choice set (X_2, Y_2) can be obtained by moving away the“common consequence” of “winning 2400 with probability 0.66” from the choice set (X_1, Y_1). For any utility function u, $Eu(X_1) \leq Eu(Y_1)$ if and only if $Eu(X_2) \leq Eu(Y_2)$. By the expected utility theory, choosing the lottery Y_1 in the Scenario 1 implies choosing Y_2 in Scenario 2. However, repeatedly confirmed experiments show that most subjects choose Y_1 in the Scenario 1 and X_2 in Scenario 2. This generates a Allais-type anomaly.

Example 8 (Common ratio effect)

Scenario 1. Choose between $X_1 = (0, 20\%; 4000, 80\%)$ and $Y_1 \equiv 3000$

Scenario 2. Choose between $X_2 = (0, 80\%; 4000, 20\%)$ and $Y_2 = (0, 75\%; 3000, 25\%)$.

The ratio of the winning probabilities is the same for both choice sets. Similar to Problem 1, for any utility function u, $Eu(X_1) \leq Eu(Y_1)$ if and only if $Eu(X_2) \leq Eu(Y_2)$. By the expected utility theory, choosing the lottery Y_1 in the Scenario 1 implies choosing Y_2 in Scenario 2. However, in experiments, most subjects choose Y_1 in the Scenario 1 and X_2 in Scenario 2. This also generates an Allais-type anomaly.

When an investor choose between two lotteries X and Y, he can also take X or Y as the reference point. This type of reference points is called endogenous reference point. [18] introduce the concept of personal equilibrium which is defined as the situation that the investor's choice between two risky prospects X and Y coincides with the endogenous reference point itself. If the investor's “consumption utility” in U_γ and “reference-dependent gain-loss utility” as 3, then his personal equilibrium is depicted by the following $(1 + \gamma)$-SDPE stochastic dominance rules.

Definition 5 Let X and Y be two risky prospects. Choosing Y is a $(1+\gamma)$-degree stochastic dominant personal equilibrium, denoted by $(1+\gamma)$-SDPE, if and only if

$$X \leq^{Y,\eta^*,\lambda^*}_{(1+\gamma)-SD} Y.$$

We can use $(1+\gamma)$-SDPE to explain the two previous Allais-type anomalies. [21] and [31] generalized these two choice problems in Example 7 and Example 8. Let F_1, F_2 and G_1, G_2 be the cdfs of X_1, X_2 and Y_1, Y_2. Assume that

(1). $F_1(x) - G_1(x) = k\,(F_2(x) - G_2(x))$ for some $k > 0$;
(2). X_1 is a simple spread of Y_1 with crossing point x_0;
(3). $G_2(x) \geq G_1(x)$ on $[-\infty, x_0)$ and $G_2(x) = G_1(x)$ on $[x_0, \infty)$.

Set

$$\gamma_0 = \frac{\int_{x_0}^{\infty} [G_1(x) - F_1(x)]\, f_{\eta^*,\lambda^*}(x, G_1) dx}{\int_{-\infty}^{x_0} [F_1(x) - G_1(x)]\, f_{\eta^*,\lambda^*}(x, G_1) dx}.$$

We have the next Proposition 3 to justify the existence of Allais-type behaviour in personal equilibrium.

Proposition 3 *Assume that X_1, X_2 and Y_1, Y_2 satisfy* (1)-(3). *If $\gamma_0 \leq 1$, then Y_1 is a $(1+\gamma_0)$-SDPE of the choice between X_1 and Y_1, while Y_2 is not $(1+\gamma_0)$-SDPE of the choice between X_2 and Y_2.*

In Example 7, let $\gamma_0 \frac{1+\eta^*\lambda^*}{1+\eta^*} = \frac{17}{12}$. If $0 < \gamma_0 \leq 1$, then $X_1 \leq^{Y_1,\lambda^*,\eta^*}_{(1+\gamma_0)-SD} Y_1$, but $X_2 \nleq^{Y_2,\lambda^*,\eta^*}_{(1+\gamma_0)-SD} Y_2$. That is, all investors with consumption utility in U_{γ_0} will choose Y_1 in personal equilibrium in the face of X_1 and Y_1, but may choose X_2 in personal equilibrium in the face of X_2 and Y_2.

6.2 The endowment effect for risk

[18] introduce the endowment effect for risk. It is a relatively new notion in the literature of behavioral economics, and it can be used to refer to the phenomenon "a person is less risk averse in eliminating a risk she expected to face than in taking on the same risk if she did not expect it". [32] supports the existence of the endowment effect for risk by conducting some experiments, but he provides no analytical insights. We can use the $(1+\gamma)$-degree stochastic dominance rules relative to stochastic reference point to analytically formulate the endowment effect for risk for the decision maker's consumption utility in U_γ. Let us see the following Example 9.

Example 9 For given $\epsilon > 0$ and $x_0 = \gamma\epsilon\frac{1+\eta^*\lambda^*}{1+\eta^*}$, suppose that investors are asked to choose between

$$X = (x_0, 50\%; -\epsilon, 50\%)\,, \qquad Y \equiv 0.$$

Taking the reference point $r \equiv 0$, since

$$\int_{-\infty}^{x} \left(\gamma(F(x)-G(x))_+ - (G(x)-F(x))_+\right) f_{\eta^*,\lambda^*}(x;r)dx$$
$$= \begin{cases} 0, & x \le -\epsilon \\ \frac{\gamma(x+\epsilon)(1+\eta^*\lambda^*)}{2(1+\eta^*)}, & -\epsilon < x \le 0 \\ \frac{1}{2}\left(\frac{\gamma\epsilon(1+\eta^*\lambda^*)}{1+\eta^*} - x\right), & 0 < x \le x_0 \\ \frac{1}{2}\left(\frac{\gamma\epsilon(1+\eta^*\lambda^*)}{1+\eta^*} - x_0\right), & x > x_0 \end{cases}$$

$X \le_{(1+\gamma)-SD}^{r,\eta^*,\lambda^*} Y$. If $R = (-\epsilon, 50\%; \epsilon, 50\%)$ and $\frac{\gamma(1+\eta^*\lambda^*)}{1+\eta^*} > 1$, then when $\epsilon < x \le x_0$,

$$\int_{-\infty}^{x} \left(\gamma(F(x)-G(x))_+ - (G(x)-F(x))_+\right) f_{\eta^*,\lambda^*}(x;R)dx$$
$$= \frac{\epsilon(\gamma-1)}{2}\left(\frac{1+\eta^*\lambda^*}{1+\eta^*} - \frac{\eta^*(\lambda^*-1)}{2(1+\eta^*)}\right) - \frac{1}{2}(x-\epsilon) < 0,$$

Thus, $X \not\le_{(1+\gamma)-SD}^{R,\eta^*,\lambda^*} Y$. That is, with a deterministic reference point, investors with consumption utility in U_γ dislike the mean-increasing spread. However, when the reference becomes R which is more disperse than r, the investors become less averse to the spread of risk X and can choose it.

We generalize Example 9 and provide a formal statement of the endowment effect for risk as Proposition 4.

Proposition 4 *Let X and Y be two lotteries with cdfs $F(x)$ and $G(x)$, and let R_1 and R_2 be two stochastic reference with cdfs $H_1(x)$ and $H_2(x)$. Assume that X is a simple spread of Y with crossing point x_0. Set*

$$\gamma_1 = \frac{\int_{x_0}^{\infty} [G(x) - F(x)] f_{\eta^*,\lambda^*}(x, H_1)dx}{\int_{-\infty}^{x_0} [F(x) - G(x)] f_{\eta^*,\lambda^*}(x, H_1)dx}.$$

[1]. *If $\gamma_1 \le 1$ and R_2 is a simple spread of R_1 with the same crossing point x_0, then $X \le_{(1+\gamma_1)-SD}^{R_1\eta^*,\lambda^*} Y$, but $X \not\le_{(1+\gamma_1)-SD}^{R_2,\eta^*,\lambda^*} Y$*

[2]. *If $\gamma_1 \le 1$ and R_1 is a simple spread of R_2 with the same crossing point x_0, then $X \le_{(1+\gamma_1)-SD}^{R_1\eta^*,\lambda^*} Y$, but $X \le_{(1+\gamma_1)-SD}^{R_2,\eta^*,\lambda^*} Y$.*

7 Conclusion

The fractional stochastic dominance introduced by [25] plays an important role in economic and financial researches. But it can not resolve Allais-type anomalies introduced by [27] and explain the existence of the endowment effect for risk introduced by [18]. In this paper fractional degree reference dependent stochastic dominance are developed. It can be used as a semi-parametric approach to compare risks relative to the reference point for decision makers whose utility function is not concave everywhere, and possesses abilities to resolve the Allais-type anomalies and to accommodate the endowment effect for risk. Our main contributions are listed as follows:

(1) The fractional degree stochastic dominance rules are related to some simple and tractable equivalent integral conditions (Theorem 1 and Theorem 4).
(2) The parameter γ is interpreted as an index of risk aversion relative to the reference point r, and some examples are provided to illustrate how to determine it(see Corollary 1 and Example 1-3).
(3) The proposition that the fractional degree stochastic dominance implies the fractional degree reference dependent stochastic dominance but its converse is not true is proved (Proposition 1).
(4) The effects of reference point on risk choice is analyzed (Proposition 2).
(5) The Allais-type anomalies and formulate the endowment effect for risk are resolved by using the new stochastic dominance rules(Proposition 3 and Proposition 4).

A topic for future research is to apply the fractional degree reference dependent stochastic dominance to stochastic optimization and to develop a new stochastic optimal model. The usefulness of $(1+\gamma)-SD$ in handling stochastic dominance constraints have been discussed recently [36], and the results developed here should be applicable in a similar way.

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8 Appendices

8.1 Proof of Theorem 1

Proof [1] $\Rightarrow$ [2]. Since U_γ^* is invariant under translations, any $u \in U_\gamma^*$ can be approximated by a sequence of functions $\{u_n \in U_\gamma, n = 1, 2, \cdots\}$ as in the proof of Theorem 2.1 in [6]. From this result it follows that [1] implies [2].

[2] $\Rightarrow$ [3]. For a fixed $t \in \Re$, we define the consumption utility function $u(x;t)$ with the following right derivative:

$$u'(x;t) = \begin{cases} \gamma, \ x \le t \text{ and } G(x) \le F(x), \\ 1, \ x \le t \text{ and } G(x) > F(x), \\ 0, \ x > t. \end{cases}$$

Obviously, $u(x;t) \in U_\gamma^*$. Due to the integration by part, it holds that

$$\begin{aligned} & E\left[v(Y, r, u(x;t))\right] - E\left[v(X, r, u(x;t)\right] \\ & \quad = \int_{-\infty}^{\infty} u'(x;t)\left(F(x) - G(x)\right) dx + \eta\lambda \int_{-\infty}^{r} u'(x;t)\left(F(x) - G(x)\right) dx \\ & \qquad + \eta \int_{r}^{\infty} u'(x;t)\left(F(x) - G(x)\right) dx \end{aligned}$$

$$= (1+\eta)\left[\int_{-\infty}^{\infty} u'(x;t)\left(F(x)-G(x)\right) f_{\eta,\lambda}(x;r)dx\right]$$

$$= (1+\eta)\left[\int_{-\infty}^{t} \left[\gamma\left(F(x)-G(x)\right)_+ - ((G(x)-F(x))_+\right] f_{\eta,\lambda}(x;r)dx\right].$$

Hence, for all $t \in \Re$, $\eta^* > 0$ and $\lambda^* > 1$, $E\left[v(Y,r,u(x;t))\right] \geq E\left[v(X,r,u(x;t))\right]$ implies that

$$\int_{-\infty}^{t} \left[\gamma\left(F(x)-G(x)\right)_+ - (G(x)-F(x))_+\right] f_{\eta^*,\lambda^*}(x;r)dx \geq 0.$$

[3] ⇒ [1]. Let $u \in U_\gamma$. Without loss of generality we can assume

$$R := \sup_{x\in\Re} u'(x) \in (0,\infty).$$

For any fixed $n \geq 2$, define $\varepsilon_n = 2^{-n}$ and K as the largest integer k for which

$$R(1-k\varepsilon_n) \geq \inf_{x\in\Re} u'(x),$$

and define a partition of a real line into intervals $[x_k, x_{k+1}]$ as follows: let $x_0 = -\infty$, $x_{K+1} = \infty$ and

$$x_k = \sup\left\{x : u'(x) \geq R(1-k\varepsilon_n)\right\}, \qquad k = 1, ..., K.$$

Then we define

$$m_k = \sup\left\{u'(x) : x_{k-1} < x \leq x_k\right\} = R\left[1-(k-1)\varepsilon_n\right].$$

It follows that

$$\gamma(m_k - R\varepsilon_n) \leq u'(x) \leq m_k, \text{ for } x \in (x_{k-1}, x_k],\ k = 1, \ldots, K+1.$$

This implies that for all $0 \leq k \leq K$,

$$\sum_{k=0}^{K}\int_{x_k}^{x_{k+1}} \left[F(x)-G(x)\right]_+ u'(x) f_{\eta,\lambda}(x;r)dx$$
$$\geq \gamma\sum_{k=0}^{K}(m_k - R\varepsilon_n)\int_{x_k}^{x_{k+1}} \left[F(x)-G(x)\right]_+ f_{\eta,\lambda}(x;r)dx.$$

and

$$\sum_{k=0}^{K}\int_{x_k}^{x_{k+1}} \left[G(x)-F(x)\right]_+ u'(x) f_{\eta,\lambda}(x;r)dx$$
$$\leq \sum_{k=0}^{K} m_k\int_{x_k}^{x_{k+1}} \left[G(x)-F(x)\right]_+ f_{\eta,\lambda}(x;r)dx.$$

Let

$$T_k = \int_{x_k}^{x_{k+1}} \left[\gamma\left(F(x) - G(x)\right)_+ - \left(G(x) - F(x)\right)_+\right] f_{\eta,\lambda}(x;r)dx,$$

and

$$c_k = \int_{x_k}^{x_{k+1}} \left(F(x) - G(x)\right)_+ f_{\eta,\lambda}(x;r)dx.$$

Thus,

$$\begin{aligned} E[v(Y,r,u)] - E[v(X,r,u)] &= \sum_{k=0}^{K} \int_{x_k}^{x_{k+1}} [F(x) - G(x)]_+ u'(x) f_{\eta,\lambda}(x;r)dx \\ &\quad - \sum_{k=0}^{K} \int_{x_k}^{x_{k+1}} [G(x) - F(x)]_+ u'(x) f_{\eta,\lambda}(x;r)dx \\ &\geq \sum_{k=0}^{K} m_k T_k - \gamma R \varepsilon_n \sum_{k=0}^{K} c_k. \end{aligned}$$

Set

$$A(x,r) = \frac{f_{\eta,\lambda}(x,r)}{f_{\eta^*,\lambda^*}(x,r)} = \begin{cases} \frac{\frac{1+\eta\lambda}{1+\eta}}{\frac{1+\eta^*\lambda^*}{1+\eta^*}}, & x \leq r \\ 1, & x > r. \end{cases}$$

Note that for all $k = 0, \ldots, K+1$,

$$\sum_{i=0}^{k} T_i = \int_{-\infty}^{x_{k+1}} \left[\gamma\left(F(x) - G(x)\right)_+ - \left(G(x) - F(x)\right)_+\right] f_{\eta^*,\lambda^*}(x,r)A(x,r)dx$$

and

$$\int_{-\infty}^{x_{k+1}} \left[\gamma\left(F(x) - G(x)\right)_+ - \left(G(x) - F(x)\right)_+\right] f_{\eta^*,\lambda^*}(x,r)dx \geq 0,$$

and $A(x,r)$ is positive and noningcreasing, which implies that $\sum_{i=0}^{k} T_i \geq 0$. Furthermore, since m_k is a decreasing non-negative sequences, $\sum_{i=0}^{k} m_i T_i \geq 0$. Therefore,

$$E[v(Y,r,u)] - E[v(X,r,u)] \geq -\gamma R \varepsilon_n \int_{-\infty}^{\infty} \left(F(x) - G(x)\right)_+ dx.$$

Letting $n \to \infty$ yields part [1] holds. This completes the proof of the theorem.

8.2 Proof of Proposition 1

Proof Let $F_1(x)$ and $F_2(x)$ be two cdfs of Z and W, respectively. The proof is main to construct cdfs $F_1(x)$ and $F_2(x)$. When $x_1 \leq r \leq x_2$, define $F_1(x)$ and $F_2(x)$ as

$$F_1(x) = \begin{cases} F(x), & x \geq x_2 \text{ or } x < x_1 \\ G(x) + \zeta_1^*, & x_1 \leq x < x_2, \end{cases} \tag{19}$$

and

$$F_2(x) = \begin{cases} G(x), & x \geq x_2 \text{ or } x < x_1 \\ F(x) - \zeta_1^*, & x_1 \leq x < x_2, \end{cases} \tag{20}$$

where ζ_1^* satisfies that

$$\gamma\zeta_1^* \left[\frac{1+\eta^*\lambda^*}{1+\eta^*}(r - x_1) + (x_2 - r)\right] = \zeta_2(x_4 - x_3) \tag{21}$$

Obviously, $0 \leq \zeta_1^* \leq \zeta_1$. Hence, $F_1(x) \leq F(x)$ and $F_2(x) \geq G(x)$ for all $x \in \Re$, that is, $X \leq_{1-SD} Z$ and $W \leq_{1-SD} Y$. From (13), (19), we have that Z is simple spread of Z. Combing with (21), it holds that $Z \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)-SD} Y$ but not $Z \leq_{(1+\gamma)-SD} Y$. And since X is also a simple spread of W based on (13), (20), $X \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)-SD} W$ but not $X \leq_{(1+\gamma)-SD} W$ by (21).

When $x_2 < r \leq x_3$, we also define cdfs $F_1(x)$ as (19) and $F_2(x)$ as (20). But ζ_1^* satisfies that

$$\gamma\zeta_1^*(x_2 - x_1)\frac{1+\eta^*\lambda^*}{1+\eta^*} = \zeta_2(x_4 - x_3) \tag{22}$$

Similarly, we can obtain $X \leq_{1-SD} Z$, $W \leq_{1-SD} Y$, $Z \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)-SD} Y$ but not $Z \leq_{(1+\gamma)-SD} Y$ and $X \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)-SD} W$ but not $X \leq_{(1+\gamma)-SD} W$.

When $x_3 < r \leq x_4$, we define

$$F_1(x) = \begin{cases} F(x), & x < x_3 \text{ or } x \geq x_4 \\ G(x) - \zeta_2^*, & x_3 \leq x < x_4 \end{cases} \tag{23}$$

and

$$F_2(x) = \begin{cases} G(x), & x < x_3 \text{ or } x \geq x_4 \\ F(x) - \zeta_2^*, & x_3 \leq x < x_4, \end{cases} \tag{24}$$

where ζ_2^* satisfies that

$$\gamma\zeta_1(x_2 - x_1)\frac{1+\eta^*\lambda^*}{1+\eta^*} = \zeta_2^* \left[(r - x_3)\frac{1+\eta^*\lambda^*}{1+\eta^*} + (x_4 - r)\right] \tag{25}$$

Clearly, $\zeta_2^* \geq \zeta_2$. Hence, $F_1(x) \leq F(x)$ and $F_2(x) \geq G(x)$ for all $x \in \Re$. That is, $X \leq_{1-SD} Z$ and $W \leq_{1-SD} Y$. Based on (13), (23), (25), we have $Z \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)-SD} Y$ but not $Z \leq_{(1+\gamma)-SD} Y$. And from (13), (24), (25), it holds that $X \leq^{r,\eta^*,\lambda^*}_{(1+\gamma)-SD} W$ but not $X \leq_{(1+\gamma)-SD} W$. Combing these three cases, we complete the proof.

8.3 Proof of Proposition 2

Proof Without loss of generality, we assume $r_1 < r_2$. Let X be a random variable with probability mass function $p_1(x)$. We take $x_1 < x_2 \leq x_3 < x_4$ with $x_2 = r_1$ and $x_4 = r_2$. Continue to take $\eta_1 > 0$, $\eta_2 > 0$, $0 < \gamma \leq 1$ such that $\gamma\eta_1 (x_2 - x_1) \frac{1+\eta^*\lambda^*}{1+\eta^*} = \eta_2(x_4 - x_3)$. Define random variable Y with probability mass function $p_2(x)$ as

$$\begin{aligned} p_2(x_1) &= p_1(x_1) - \eta_1, \\ p_2(x_2) &= p_1(x_2) + \eta_1, \\ p_2(x_3) &= p_1(x_3) + \eta_2, \\ p_2(x_4) &= p_1(x_4) - \eta_2, \\ p_2(x) &= p_1(x) \qquad \text{for all other values } x. \end{aligned}$$

Let $F(x)$ and $G(x)$ be the cdfs of X and Y, respectively. Obviously, we have

$$F(x) - G(x) = \begin{cases} \eta_1, & x_1 \leq x < x_2 \\ -\eta_2, & x_3 \leq x < x_4 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, X is a simple spread of Y with a single crossing point x_2. And since $\gamma\eta_1 (x_2 - x_1) \frac{1+\eta^*\lambda^*}{1+\eta^*} = \eta_2(x_4 - x_3)$, we have

$$\gamma \int_{-\infty}^{x_2} (F(x) - G(x)) f_{\eta^*,\lambda^*}(x, r_1) dx = \int_{x_2}^{\infty} (G(x) - F(x)) f_{\eta^*,\lambda^*}(x, r_1) dx$$

Thus, $X \preceq_{(1+\gamma)-SD}^{r_1,\eta^*,\lambda^*} Y$. And since

$$f_{\eta^*,\lambda^*}(x, r_1) < f_{\eta^*,\lambda^*}(x, r_2) \text{ on } [x_3, x_4]$$

and

$$f_{\eta^*,\lambda^*}(x, r_1) = f_{\eta^*,\lambda^*}(x, r_2) \text{ on otherwise,}$$

we have

$$\begin{aligned} \gamma \int_{-\infty}^{x_2} (F(x) - G(x)) f_{\eta^*,\lambda^*}(x, r_2) dx &= \gamma \int_{-\infty}^{x_2} (F(x) - G(x)) f_{\eta^*,\lambda^*}(x, r_1) dx \\ &= \int_{x_2}^{\infty} (G(x) - F(x)) f_{\eta^*,\lambda^*}(x, r_1) dx \\ &\leq \int_{x_2}^{\infty} (G(x) - F(x)) f_{\eta^*,\lambda^*}(x, r_2) dx. \end{aligned}$$

Thus, $X \npreceq_{(1+\gamma)-SD}^{r_2,\eta^*,\lambda^*} Y$. This completes the proof of proposition 2.

8.4 Proof of Theorem 2

To prove Theorem 2, we need the next lemma.

Lemma 1 *Given cdfs F and G, let $F^{-1}(s) = \inf\{x : F(x) \geq s\}$ and $G^{-1}(s) = \inf\{x : G(x) \geq s\}$. $F \leq^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} G$ if and only if, for all $p \in [0, 1]$,*

$$\int_0^p H_{\gamma,r}(G^{-1}(s))ds \geq \int_0^p H_{\gamma,r}(F^{-1}(s))ds \tag{26}$$

Proof We can use the similar proof of [25]. Define the function

$$t \to h(x) := \int_{-\infty}^x [F(t) - G(t)]\, dH_{\gamma,r}(t).$$

Then, $F \leq^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} G$ if and only if for all x, $h(x) \geq 0$. Since $H^{'}_{\gamma,r}(t) \geq 0$, the function h assumes local minima in the points b_k where the distribution functions F and G cross, going from $F \leq G$ left of b_k to $F > G$ right of b_k. If we define $p_k := G(b_k)$ and

$$\tilde{h}(p) := \int_0^p \left[H_{\gamma,r}(G^{-1}(s)) - H_{\gamma,r}(F^{-1}(s))\right] ds,$$

then we have $h(b_k) = \tilde{h}(p_k)$ and the function $\tilde{h}$ assumes its local minima in the points p_k. Therefore, $h \geq 0$ if and only if $\tilde{h} \geq 0$.

Proof Based on the idea of [25], define

$$A_1(p) := \int_0^p \left[H_{\gamma,r}(G^{-1}(s)) - H_{\gamma,r}(F^{-1}(s))\right]_+ ds$$

and

$$A_2(p) := \int_0^p \left[H_{\gamma,r}(F^{-1}(s)) - H_{\gamma,r}(G^{-1}(s))\right]_+ ds.$$

Without the loss of generality, we assume that $A_2(1) > 0$. Let $\alpha(a)$ and $\beta(a)$ be the smallest probabilities that solve

$$A_1(\alpha(a)) = a \quad \text{and} \quad A_2(\beta(a)) = a, \quad 0 < a < A_2(1).$$

It follows from Lemma 1 that $A_1(p) \geq A_2(p)$. Hence, $\alpha(a) \leq \beta(a)$ for all $0 < a < A_2(1)$. We set

$$x_1(a) := F^{-1}(\alpha(a)),\ x_2(a) := G^{-1}(\alpha(a))$$

and

$$x_3(a) := G^{-1}(\beta(a)),\ x_4(a) := F^{-1}(\beta(a)).$$

Since X and Y assume only finitely many values, there is a sequence $0 = a_1 < \cdots < a_k \leq A_2(1)$ such that $a \mapsto x_1(a), \cdots, x_4(a)$ are constant on (a_{i-1}, a_i). Denote the corresponding values of these functions as

$$x_{l,i} = x_l(a) \qquad \text{for } a \in (a_{i-1}, a_i),\ l = 1, \ldots, 4.$$

Moreover, for $i = 1, \ldots, k$, at the points $x_{1,i}$ and $x_{4,i}$ the function F has jumps of sizes at least ζ_{1i} and ζ_{2i}, and at the corresponding points $x_{2,i}$ and $x_{4,i}$ the function G has jumps of sizes at least ζ_{1i} and ζ_{2i}, where ζ_{1i} and ζ_{2i} are given by the equation

$$\zeta_{1i}\left(H_{\gamma,r}(x_{2,i}) - H_{\gamma,r}(x_{1,i})\right) = \zeta_{2i}\left(H_{\gamma,r}(x_{4,i}) - H_{\gamma,r}(x_{3,i})\right) = a_i - a_{i-1}$$

For $x > x_{4,k}$, we have $F(x) > G(x)$. Thus, G is obtained from F by a sequence of k (γ, r)-transfers described by the corresponding x's and ζ's above, plus a finite number of increasing transfers moving the mass from F to G right of $x_{4,k}$. This completes the proof.

8.5 Proof of Theorem 3

Proof Note that for random variables X_n, X with distribution functions F_n, F the convergence $X_n \Rightarrow X$ mentioned in the theorem holds if and only if

$$\int_{-\infty}^{\infty} |F_n(x) - F(x)| dx \to 0,$$

and since $0 < H'_{\gamma,r} \leq \frac{1+\eta^*\lambda^*}{1+\eta^*}$, it holds that

$$\int_{-\infty}^{\infty} |F_n(x) - F(x)| dH_{\gamma,r}(x) \to 0.$$

This implies that, for any $t \in \Re$,

$$\int_{-\infty}^{t} \left(F_n(x) - F(x)\right) dH_{\gamma,r}(x) \to 0.$$

The if-part thus follows from (14).

For the only-if-part, if X, Y are are bounded, then the proof is similar to [25]. We can define for any $n \in \mathbb{N}$

$$X_n = \frac{i}{n}, \text{ if } \frac{i}{n} \leq X < \frac{i+1}{n},\ i \in \mathbb{Z}$$

and

$$Y_n = \frac{i+1}{n}, \text{ if } \frac{i}{n} \leq Y < \frac{i+1}{n},\ i \in \mathbb{Z}.$$

Then X_n and Y_n have finite support with $X_n \leq_{1-SD} X$ and $Y \leq_{1-SD} Y_n$. Therefore,

$$X_n \leq^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} X \leq^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} Y \leq^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} Y_n$$

and $X_n \Rightarrow X$ and $Y_n \Rightarrow Y$.

If X and Y are unbounded, then we define

$$X_n := \begin{cases} x_n^*, \text{ if } X < -n, \\ X, \ \text{ if } -n \leq X \leq n, \\ n, \ \text{ if } n < X, \end{cases} \tag{27}$$

and

$$Y_n := \begin{cases} -n, \ \text{if } Y < -n, \\ Y, \ \ \text{if } -n \leq Y \leq n, \\ n, \ \ \text{if } n < Y, \end{cases} \tag{28}$$

where x_n^* satisfies that

$$H_{\gamma,r}(x_n^*) = H_{\gamma,r}(-n) - \frac{\int_{-\infty}^{-n} F(x) dH_{\gamma,r}(x)}{P(X < -n)}.$$

An easy calculation for the corresponding distribution functions F_n, G_n shows that

$$\begin{aligned} &\int_{-\infty}^{t} (F_n(x) - G_n(x)) \, dH_{\gamma,r}(x) \\ &= \begin{cases} 0, & t \leq x_n^*, \\ P(X < -n) \left(H_{\gamma,r}(t) - H_{\gamma,r}(x_n^*)\right), & x_n^* < t \leq -n, \\ \int_{-\infty}^{t} (F(x) - G(x)) \, dH_{\gamma,r}(x) + \int_{-\infty}^{-n} G(x) dH_{\gamma,r}(x), & -n < t \leq n, \\ 0, & t > n. \end{cases} \end{aligned}$$

Thus $X \leq^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} Y$, that is,

$$\int_{-\infty}^{t} (F(x) - G(x)) \, dH_{\gamma,r}(x) \geq 0, \ \text{for all} \ \ t \in \mathbb{R}$$

implies that

$$\int_{-\infty}^{t} (F_n(x) - G_n(x)) \, dH_{\gamma,r}(x) \geq 0 \ \ \text{for all} \ \ t \in \mathbb{R}.$$

Then, it follows that $X \leq^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} Y$ implies that $X_n \leq^{r,\lambda^*,\eta^*}_{(1+\gamma)-SD} Y_n$, and obviously X_n, Y_n are bounded and $X_n \Rightarrow X$, and $Y_n \Rightarrow Y$.

For each fixed n we can approximate X_n and Y_n by sequences $\{X_{nn}\}$ and $\{Y_{nn}\}$ as in (27) and (28). Then, the sequences $\{X_{nn}\}$ and $\{Y_{nn}\}$ fulfill the conditions of the theorem. This completes the proof.

8.6 Proof of Theorem 4

To prove Theorem 4, we need Lemma 2.

Lemma 2 *Define*

$$A(x;H) = \frac{f_{\eta,\lambda}(x;H)}{f_{\eta^*,\lambda^*}(x;H)} \tag{29}$$

Then, $A(x;H)$ is nonegative and nonincreasing.

Proof Obviously, $A(x;H)$ is greater than zero. Since

$$A(x;H) = \frac{\eta(\lambda-1)(1+\eta^*)}{\eta^*(\lambda^*-1)(1+\eta)} + \frac{\frac{1+\eta\lambda}{1+\eta} - \frac{(1+\eta^*\lambda^*)\eta(\lambda-1)}{(1+\eta)\eta^*(\lambda^*-1)}}{f_{\eta^*,\lambda^*}(x;H)}$$

and cdf $H(x)$ is nondecreasing in x, to prove $A(x;H)$ is nonincreasing is just to prove

$$\frac{1+\eta\lambda}{1+\eta} - \frac{(1+\eta^*\lambda^*)\eta(\lambda-1)}{(1+\eta)\eta^*(\lambda^*-1)}$$

is less than zero.

$$\begin{aligned}&\frac{1+\eta\lambda}{1+\eta} - \frac{(1+\eta^*\lambda^*)\eta(\lambda-1)}{(1+\eta)\eta^*(\lambda^*-1)}\\ &\quad= \frac{1}{\eta^*(1+\eta)(\lambda^*-1)}\left[\eta^*(\lambda^*-1) - \eta(\lambda-1) - \eta\eta^*(\lambda-\lambda^*)\right].\end{aligned}$$

From $\eta \geq \eta^* \geq 0$, $\lambda \geq \lambda^* \geq 1$, we have

$$\frac{1+\eta\lambda}{1+\eta} - \frac{(1+\eta^*\lambda^*)\eta(\lambda-1)}{(1+\eta)\eta^*(\lambda^*-1)} \leq 0.$$

This completes the proof of Lemma 2.

Proof Let $X \vee R = \max\{X,R\}$ and $X \wedge R = \min\{X,R\}$ with cdfs $F_{X\vee R}(x)$ and $F_{X\wedge R}(x)$, respectively. Then,

$$\begin{aligned}&E\left[v(X;R,u)\right]\\
&\quad= E\left[u(X) + \eta u(X\vee R) + \eta\lambda u(X\wedge R) - \eta(1+\lambda)u(R)\right]\\
&\quad= \int_{-\infty}^{\infty} u(x)dF(x) + \eta\int_{-\infty}^{\infty} u(x)dF_{X\vee R}(x) + \eta\lambda\int_{-\infty}^{\infty} u(x)dF_{X\wedge R}(x)\\
&\qquad - \eta(1+\lambda)\int_{-\infty}^{\infty} u(x)dH(x)\\
&\quad= \int_{-\infty}^{\infty} u(x)dF(x) + \eta\int_{-\infty}^{\infty} u(x)d\left[F(x)H(x)\right] - \eta(1+\lambda)\int_{-\infty}^{\infty} u(x)dH(x)\\
&\qquad + \eta\lambda\int_{-\infty}^{\infty} u(x)d\left[F(x)+H(x)-H(x)F(x)\right]\\
&\quad= \int_{-\infty}^{\infty} u(x)d\left[F(x)\left(1+\eta\lambda-\eta(\lambda-1)H(x)\right)\right] - \eta\int_{-\infty}^{\infty} u(x)dH(x)\\
&\quad= \int_{-\infty}^{\infty} u(x)d\left[F(x)(1+\eta)f_{\eta,\lambda}(x,H)\right] - \eta\int_{-\infty}^{\infty} u(x)dH(x).\end{aligned}$$

Thus,

$$E\left[v(X;R,u)\right]-E\left[v(Y;R,u)\right]=(1+\eta)\int_{-\infty}^{\infty}u(x)d\left[\left(F(x)-G(x)\right)f_{\eta,\lambda}(x,H)\right]$$

Combing with Lemma 2, the equivalent integral condition of

$$E\left[v(X;R,H)\right]\leq E\left[v(Y;R,H)\right]$$

for all $u\in U_{\gamma}$, and $\eta\geq\eta^{*},\lambda\geq\lambda^{*}$ follows in a similar manner of the proof of Theorem 1.

8.7 Proof of Proposition 3

Proof Since X_1 is a simple spread of Y_1 with crossing point x_0 and

$$\begin{aligned}\gamma_0\int_{-\infty}^{x_0}&\left[F_1(x)-G_1(x)\right]f_{\eta^*,\lambda^*}(x,G_1)dx\\&=\int_{x_0}^{\infty}\left[G_1(x)-F_1(x)\right]f_{\eta^*,\lambda^*}(x,G_1)dx\end{aligned}$$

it holds that $X_1\preceq_{(1+\gamma_0)-SD}^{Y_1,\eta^*,\lambda^*}Y_1$. But we have

$$\begin{aligned}\int_{x_0}^{\infty}&\left[G_2(x)-F_2(x)\right]f_{\eta^*,\lambda^*}(x,G_2)dx\\&=k\int_{x_0}^{\infty}\left[G_1(x)-F_1(x)\right]f_{\eta^*,\lambda^*}(x,G_1)dx\\&=k\gamma_0\int_{-\infty}^{x_0}\left[F_1(x)-G_1(x)\right]f_{\eta^*,\lambda^*}(x,G_1)dx\\&\geq\gamma_0\int_{-\infty}^{x_0}\left[F_2(x)-G_2(x)\right]f_{\eta^*,\lambda^*}(x,G_2)dx.\end{aligned}$$

That is, $X_2\not\preceq_{(1+\gamma_0)-SD}^{Y_2,\eta^*,\lambda^*}Y_2$. This completes the proof.

8.8 Proof of Proposition 4

Proof Since X is a simple spread of Y with crossing point x_0 and $\gamma_1\leq1$, we have $X\preceq_{(1+\gamma_1)-SD}^{R_1\eta^*,\lambda^*}Y$. For case [1], it holds that $H_2(x)\geq H_1(x)$ on $[-\infty,x_0)$ and $H_2(x)\leq H_1(x)$ on $[x_0,\infty)$. Thus, we have

$$\begin{aligned}\gamma_1\int_{-\infty}^{x_0}&\left[F(x)-G(x)\right]f_{\eta^*,\lambda^*}(x,H_2)dx\\&\leq\gamma_1\int_{-\infty}^{x_0}\left[F(x)-G(x)\right]f_{\eta^*,\lambda^*}(x,H_1)dx\end{aligned}$$

$$= \int_{x_0}^{\infty} [G(x) - F(x)]\, f_{\eta^*,\lambda^*}(x, H_1) dx \qquad (30)$$
$$\leq \int_{x_0}^{\infty} [G(x) - F(x)]\, f_{\eta^*,\lambda^*}(x, H_2) dx.$$

That is, $X \not\leq^{R_2,\eta^*,\lambda^*}_{(1+\gamma_1)-SD} Y$. For case [2], it holds that $H_2(x) \leq H_1(x)$ on $[-\infty, x_0)$ and $H_2(x) \geq H_1(x)$ on $[x_0, \infty)$. Thus, in contrast to (30), we have

$$\gamma_1 \int_{-\infty}^{x_0} [F(x) - G(x)]\, f_{\eta^*,\lambda^*}(x, H_2) dx \geq \int_{x_0}^{\infty} [G(x) - F(x)]\, f_{\eta^*,\lambda^*}(x, H_2) dx.$$

That is, $X \leq^{R_2,\eta^*,\lambda^*}_{(1+\gamma_1)-SD} Y$. This completes the proof.

References

1. Bawa, S.:Research bibliography¡ªstochastic dominance: A research bibliography. **Management Science 28**, 698-712(1982).
2. Bell, D.E.:Disappointment in Decision Making under Uncertainty. **Operations Research 33(1)**, 1-27(1985).
3. Bell, D. E.: The value of pre-decision side bets for utility maximizers. **Management Science 34**, 797-800(1988).
4. Bi, H.W. and Zhu, W.: The non-integer higher-order Stochastic dominance. **Operations Research Letters 47(2)**, 77-82(2019).
5. Chateauneuf, A., Cohen, M., and Meilijson, I.: More pessimism than greediness: a characterization of monotone risk aversion in the rank-dependent expected utility model. **Economic Theory 25**, 649-667(2005).
6. Denuit, M. and Müller, A.: Smooth generators of integral stochastic orders. **The Annals of Applied Probability 12**, 1174-1184(2002)
7. Delquié, P. and Cillo, A.: Disappointment without prior expectation: a unifying perspective on decision under risk. **Journal of Risk Uncertainty 33**, 197-215(2006).
8. Dentcheva, D. and Ruszczyński, A.: Optimization with stochastic dominance constraints. **SIAM Journal on optimization 14**, 548-566(2003).
9. Dentcheva, D. and Ruszczyński, A.: Portfolio optimization with stochastic dominance constraints. **Journal of Banking and Finance 30**, 433-451(2006).
10. Dittmann, I., Maug, E., and Spalt, O.: Sticks or carrots? Optimal CEO compensation when managers are loss averse. **Journal of Finance 65 (6)**, 2015-2050(2010).
11. Fishburn, P.C.:**Decision and Value Theory**, John Wiley & Sons, New York (1964).
12. Friedman, M. and Savage, L. J.: The utility analysis of choices involving risk. **Journal of Political Economy 56**, 279-304(1948).
13. Guo, D., Hu, Y., Wang, S. and Zhao, L.: Comparing risks with reference points: A stochastic dominance approach. **Insurance: Mathematics and Economics 70**, 105-116(2016).
14. Gollier, C.: **Stochastic volatility implies fourth-degree risk dominance: Applications to asset pricing**. TSE Working Papers(2016).
15. Hanoch, G. and Levy, H.: The efficiency analysis of choices involving risk. **Review of Economic Studies 36**, 335-346(1969).
16. Huang R., Tzeng L. and Zhao L.: Fractional Degree Stochastic Dominance. **Management Science** forthcoming(2019).
17. Kahnemann, D. and Tversky, A.: Prospect Theory: An Analysis of Decision under Risk. **Econometrica 47(2)**, 263-291(1979).
18. Köszegi, B. and Rabin, M.: A Model of Reference-Dependent Preferences. **Quarterly Journal of Economics 121(4)**, 1133-65(2006).
19. Köszegi, B. and Rabin, M.: Reference-dependent risk attitudes. **Amer. Econ. Rev 97**, 1047-1073(2007).

20. Levy, H.: Stochastic dominance and expected utility: Survey and analysis. **Management Science 38**, 555-593(1992).
21. Machina, M.: “Expected utility” analysis without the independence axiom. **Econometrica 50**, 277-323(1982).
22. Mao, T. and Hu, T.: Fractional stochastic dominance. Technical Report. University of Science and Technology of China. Hefei(2018).
23. Markowitz, H.: The utility of wealth. **Journal of Political Economy 60**, 151-158(1952).
24. Müller, A. and Stoyan, D.: **Comparison Methods for Stochastic Models and Risks**, John Wiley & Sons, Chichester, UK(2002).
25. Müller, A., Scarsini, M., Tsetlin I. and Winkler, R.: Between first-and second-Order stochatic dominance. **Management Science 63**, 2933-2974(2017).
26. Post, T. and Kopa, M.: Portfolio choice based on third-degree stochastic dominance. **Management Science 63(10)**, 3381-3392 (2016).
27. Quiggin, J.: **Generalized Expected Utility Theory: the Rank-Dependent Model**. Kluwer Academic, Boston(1993).
28. Rabin, M.: Risk Aversion and Expected-Utility Theory: A Calibration Theorem. **Econometrica 68(5)**, 1281-92(2000).
29. Richard T.H. and Johnson, E.J.: Gambling with the House Money and Trying to Break Even: The Effects of Prior Outcomes on Risky Choice. **Management Science 36(6)**, 643-60(1990).
30. Rothschild, M. and Stiglitz, J. E.: Increasing risk. I. A definition. **Journal of Economic Theory 2**, 225-243(1970).
31. Segal, U.: Some remarks on Quiggin¡¯s anticipated utility. **Journal of Economic Behavior & Organization 8**, 145-154(1987).
32. Sprenger, C.: An endowment effect for risk: Experimental tests of stochastic reference points. **Journal of Political Economy 123**, 1456-1499(2015).
33. Sugden, R.: Reference-Dependent Subjective Expected Utility. **Journal of Economic Theory 1l1(2)**, 172-191(2003).
34. Tversky, A., and Kahneman, D.: Loss aversion in risk-less choice: A reference dependent model. **Quarterly Journal of Economics 106**, 1039-1061(1991).
35. von Neumann, J., and Morgenstern, O.: **Theory of games and economic behavior**, 2nd ed. Princeton, NJ: Princeton University Press(1947).
36. Yang, J., Han, S., Zhang, X. and Zhou, D.: Optimization with fractional stochastic dominance constraints. Technical Report. Zhejiang Sci-Tech University. Hangzhou(2019).